Solutions for Calculus Early Transcendentals 3rd Edition by Briggs

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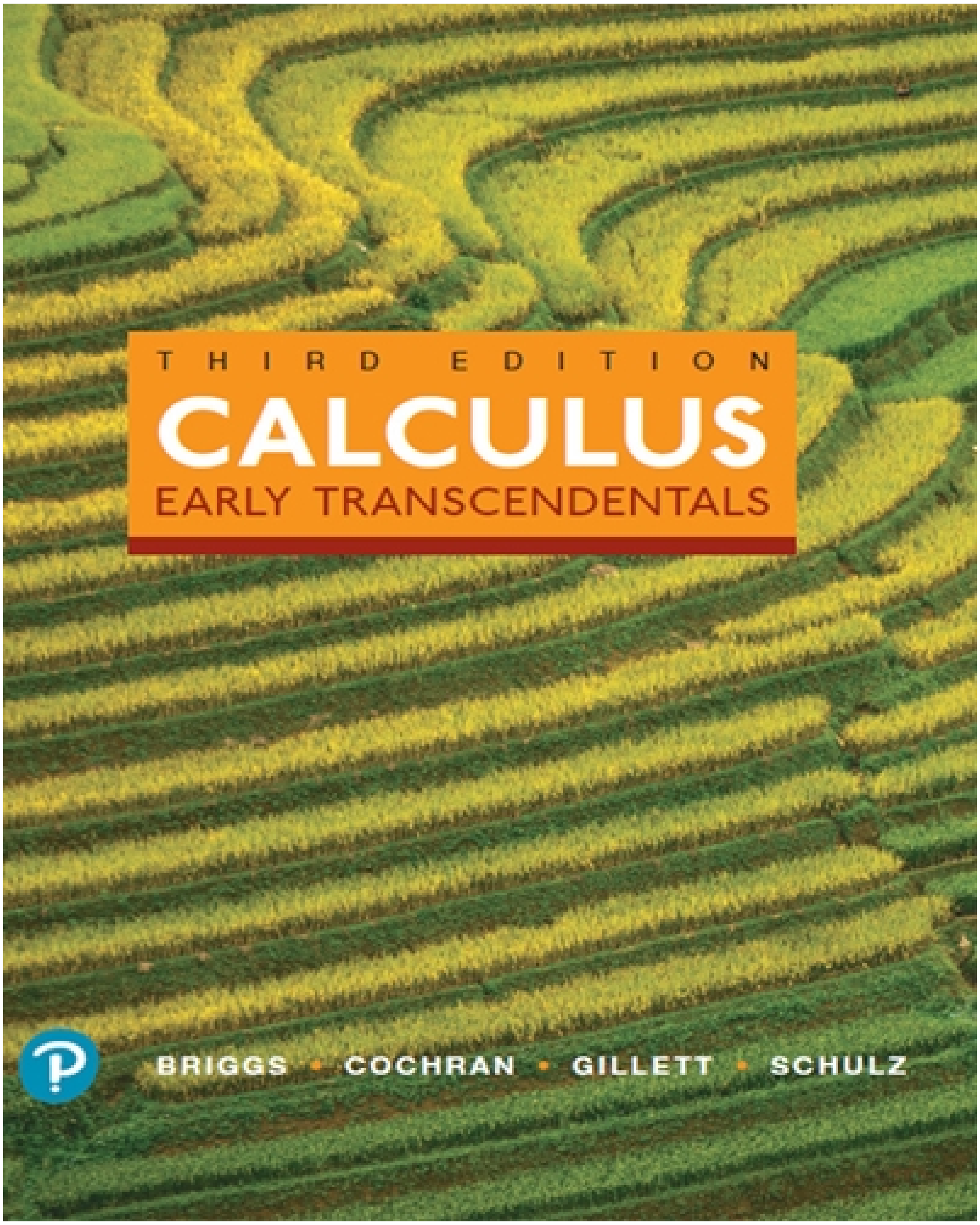


Solutions

Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time $t = a$ and $t = b$ is the change in position divided by the elapsed time: $v_{\text{av}} = \dfrac{s(b) - s(a)}{b - a}$.

2.1.2 In order to compute the instantaneous velocity of the object at time $t = a$, we compute the average velocity over smaller and smaller time intervals of the form $[a, t]$, using the formula: $v_{\text{av}} = \dfrac{s(t) - s(a)}{t - a}$. We let t approach a. If the quantity $\dfrac{s(t) - s(a)}{t - a}$ approaches a limit as $t \to a$, then that limit is called the instantaneous velocity of the object at time $t = a$.

2.1.3 The average velocity is $\dfrac{s(3) - s(2)}{3 - 2} = 156 - 136 = 20$.

2.1.4 The average velocity is $\dfrac{s(4) - s(1)}{4 - 1} = \dfrac{144 - 84}{3} = \dfrac{60}{3} = 20$.

2.1.5

a. $\dfrac{s(2) - s(0)}{2 - 0} = \dfrac{72 - 0}{2} = 36$.

b. $\dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{66 - 0}{1.5} = 44$.

c. $\dfrac{s(1) - s(0)}{1 - 0} = \dfrac{52 - 0}{1} = 52$.

d. $\dfrac{s(.5) - s(0)}{.5 - 0} = \dfrac{30 - 0}{.5} = 60$.

2.1.6

a. $\dfrac{s(2.5) - s(.5)}{2.5 - .5} = \dfrac{150 - 46}{2} = 52$.

b. $\dfrac{s(2) - s(.5)}{2 - .5} = \dfrac{136 - 46}{1.5} = 60$.

c. $\dfrac{s(1.5) - s(.5)}{1.5 - .5} = \dfrac{114 - 46}{1} = 68$.

d. $\dfrac{s(1) - s(.5)}{1 - .5} = \dfrac{84 - 46}{.5} = 76$.

2.1.7 $\dfrac{s(1.01)-s(1)}{.01} = 47.84$, while $\dfrac{s(1.001)-s(1)}{.001} = 47.984$ and $\dfrac{s(1.0001)-s(1)}{.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.8 $\dfrac{s(2.01)-s(2)}{.01} = -4.16$, while $\dfrac{s(2.001)-s(2)}{.001} = -4.016$ and $\dfrac{s(2.0001)-s(2)}{.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.9 The slope of the secant line between points $(a, f(a))$ and $(b, f(b))$ is the ratio of the differences $f(b) - f(a)$ and $b - a$. Thus $m_{\text{sec}} = \dfrac{f(b)-f(a)}{b-a}$.

2.1.10 In order to compute the slope of the tangent line to the graph of $y = f(t)$ at $(a, f(a))$, we compute the slope of the secant line over smaller and smaller time intervals of the form $[a, t]$. Thus we consider $\dfrac{f(t)-f(a)}{t-a}$ and let $t \to a$. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve $y = f(t)$ at $t = a$.

2.1.11 Both problems involve the same mathematics, namely finding the limit as $t \to a$ of a quotient of differences of the form $\dfrac{g(t)-g(a)}{t-a}$ for some function g.

2.1.12 Note that $f(2) = 64$.

a. i. $f(0.5) = 28$. So the slope of the secant line is $\frac{28-64}{0.5-2} = \frac{-36}{-3/2} = 24$.
   ii. $f(1.9) \approx 63.84$. So the slope of the secant line is about $\frac{63.84-64}{1.9-2} = 1.6$.
   iii. $f(1.99) \approx 63.9984$. So the slope of the secant line is about $\frac{63.9984-64}{1.99-2} = 0.16$.
   iv. $f(1.999) \approx 63.999984$. So the slope of the secant line is about $\frac{63.999984-64}{1.999-2} = 0.016$.
   v. $f(1.9999) \approx 63.99999984$. So the slope of the secant line is about $\frac{63.99999984-64}{1.9999-2} = 0.0016$.

b. A good guess is that the limit is 0.

c. The slope of the tangent line is the limit of the slopes of the secant lines, so it is also 0.

2.1.13

a. Over $[1, 4]$, we have $v_{\text{av}} = \dfrac{s(4)-s(1)}{4-1} = \dfrac{256-112}{3} = 48$.

b. Over $[1, 3]$, we have $v_{\text{av}} = \dfrac{s(3)-s(1)}{3-1} = \dfrac{240-112}{2} = 64$.

c. Over $[1, 2]$, we have $v_{\text{av}} = \dfrac{s(2)-s(1)}{2-1} = \dfrac{192-112}{1} = 80$.

d. Over $[1, 1+h]$, we have

$$v_{\text{av}} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{-16(1+h)^2 + 128(1+h) - (112)}{h} = \frac{-16h^2 - 32h + 128h}{h}$$
$$= \frac{h(-16h+96)}{h} = 96 - 16h = 16(6-h).$$

2.1.14

a. Over $[0, 3]$, we have $v_{\text{av}} = \dfrac{s(3)-s(0)}{3-0} = \dfrac{65.9-20}{3} = 15.3$.

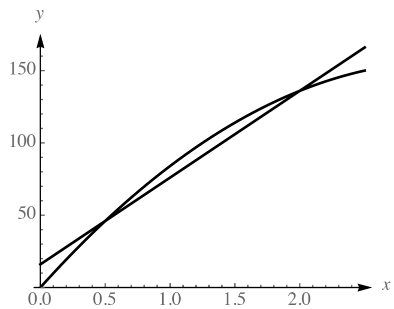
b. Over $[0, 2]$, we have $v_{\text{av}} = \dfrac{s(2)-s(0)}{2-0} = \dfrac{60.4-20}{2} = 20.2$.

c. Over $[0, 1]$, we have $v_{\text{av}} = \dfrac{s(1) - s(0)}{1 - 0} = \dfrac{45.1 - 20}{1} = 25.1$.

d. Over $[0, h]$, we have

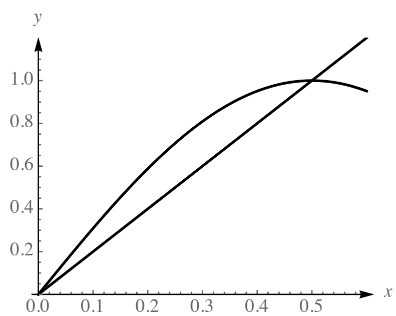
$$v_{\text{av}} = \frac{s(h) - s(0)}{h - 0} = \frac{-4.9h^2 + 30h + 20 - 20}{h}$$
$$= \frac{(h)(-4.9h + 30)}{h} = -4.9h + 30.$$

2.1.15



The slope of the secant line is given by $\dfrac{s(2) - s(0.5)}{2 - 0.5} = \dfrac{136 - 46}{1.5} = 60$. This represents the average velocity of the object over the time interval $[0.5, 2]$.

2.1.16



The slope of the secant line is given by $\dfrac{s(0.5) - s(0)}{0.5 - 0} = \dfrac{1}{0.5} = 2$. This represents the average velocity of the object over the time interval $[0, 0.5]$.

2.1.17

Time Interval	$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.18

Time Interval	$[2, 3]$	$[2, 2.25]$	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.19

Time Interval	$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[2.99999, 3]$
Average Velocity	20	5.6	4.16	4.016	4.002	4.0002

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	$[\pi/2, \pi]$	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-1.90986	-0.149875	-0.0149999	-0.0015	-0.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
Average Velocity	-17.6	-16.16	-16.016	-16.002

The instantaneous velocity appears to be -16 ft/s.

2.1.22

Time Interval	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-19.9667	-19.9997	-20.0000	-20.0000

The instantaneous velocity appears to be -20 ft/s.

2.1.23

Time Interval	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Average Velocity	79.468	79.995	80.000	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Average Velocity	-10	-18.1818	-19.802	-19.98

The instantaneous velocity appears to be -20 ft/s.

2.1.25

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x **Interval**	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Slope of Secant Line	-2.995	-2.99995	-3.0000	-3.0000

The slope of the tangent line appears to be -3.

2.1.27

x **Interval**	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Slope of the Secant Line	1.05171	1.00502	1.0005	1.00005

The slope of the tangent line appears to be 1.

2.1.28

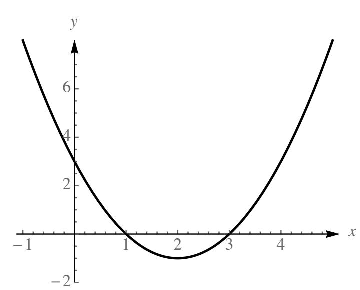
x **Interval**	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex $(2, -1)$.

b. At $(2, -1)$ the function has tangent line with slope 0.

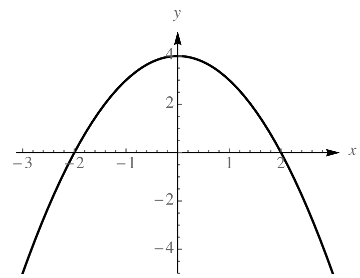


c.

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of the Secant Line	0.1	0.01	0.001	0.0001

The slope of the tangent line at $(2, -1)$ appears to be 0.

2.1.30

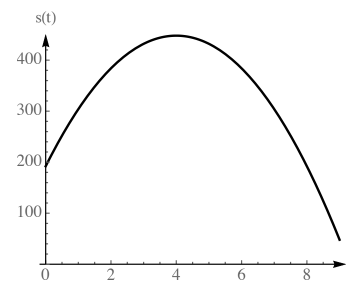


a. Note that the graph is a parabola with vertex $(0, 4)$.

b. At $(0, 4)$ the function has a tangent line with slope 0.

c. This is true for this function – because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus $f(0+h) = 4-(0+h)^2 = 4-(-h)^2 = f(0-h)$. So the slope of any such secant line is $\dfrac{4-h^2-(4-h^2)}{h-(-h)} = \dfrac{0}{2h} = 0$.

2.1.31



a. Note that the graph is a parabola with vertex $(4, 448)$.

b. At $(4, 448)$ the function has tangent line with slope 0, so $a = 4$.

c.

x **Interval**	$[4, 4.1]$	$[4, 4.01]$	$[4, 4.001]$	$[4, 4.0001]$
Slope of the Secant Line	-1.6	$-.16$	$-.016$	$-.0016$

The slopes of the secant lines appear to be approaching zero.

d. On the interval $[0, 4)$ the instantaneous velocity of the projectile is positive.

e. On the interval $(4, 9]$ the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when $s(t) = 96$. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

b.

t **Interval**	$[\sqrt{6}-.1, \sqrt{6}]$	$[\sqrt{6}-.01, \sqrt{6}]$	$[\sqrt{6}-.001, \sqrt{6}]$	$[\sqrt{6}-.0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx 0.63662.$$

For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + .5) - f(\pi/2)}{(\pi/2 + .5) - (\pi/2)} = -\frac{\cos(\pi/2 + .5)}{.5} \approx 0.958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + .05) - f(\pi/2)}{(\pi/2 + .05) - (\pi/2)} = -\frac{\cos(\pi/2 + .05)}{.05} \approx 0.999583.$$

Computing one more slope of a secant line:

$$m_{\sec} = \frac{f(\pi/2 + .01) - f(\pi/2)}{(\pi/2 + .01) - (\pi/2)} = -\frac{\cos(\pi/2 + .01)}{.01} \approx 0.999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definition of a Limit

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3

a. $h(2) = 5$.

b. $\lim_{x\to 2} h(x) = 3$.

c. $h(4)$ does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.4

a. $g(0) = 0$.

b. $\lim_{x\to 0} g(x) = 1$.

c. $g(1) = 2$.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.5

a. $f(1) = -1$.

b. $\lim_{x\to 1} f(x) = 1$.

c. $f(0) = 2$.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.6

a. $f(2) = 2$.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.7

a.

x	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
$f(x) = \frac{x^2 - 4}{x - 2}$	3.9	3.99	3.999	3.9999	4.1	4.01	4.001	4.0001

b. $\lim_{x\to 2} f(x) = 4$.

2.2.8

a.

x	0.9	0.99	0.999	0.9999	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	3.31	3.30301	3.003	3.0003

b. $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = 3$

2.2.9

a.

t	8.9	8.99	8.999	9.1	9.01	9.001
$g(t) = \dfrac{t-9}{\sqrt{t}-3}$	5.98329	5.99833	5.99983	6.01662	6.00167	6.00017

b. $\lim_{t\to 9} \dfrac{t-9}{\sqrt{t}-3} = 6.$

2.2.10

a.

x	0.01	0.001	0.0001	0.00001
$f(x) = (1+x)^{1/x}$	2.70481	2.71692	2.71815	2.71827

x	−0.01	−0.001	−0.0001	−0.00001
$f(x) = (1+x)^{1/x}$	2.732	2.71964	2.71842	2.71830

b. $\lim_{x\to 0}(1+x)^{1/x} \approx 2.718.$

c. $\lim_{x\to 0}(1+x)^{1/x} = e.$

2.2.11 Suppose the function f is defined for all x near a but greater than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then we write $\lim_{x\to a^+} f(x) = L$.

2.2.12 Suppose the function f is defined for all x near a but less than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then we write $\lim_{x\to a^-} f(x) = L$.

2.2.13 It must be true that $L = M$.

2.2.14

a.

x	1.9	1.99	1.999	1.9999
$g(x) = \frac{x^3-4x}{8\lvert x-2\rvert}$	-0.92625	-0.99251	-0.99925	-0.99993

x	2.1	2.01	2.001	2.0001
$g(x) = \frac{x^3-4x}{8\lvert x-2\rvert}$	1.07625	1.00751	1.00075	1.00008

b. $\lim_{x\to 2^-} g(x) = -1$. $\lim_{x\to 2^+} g(x) = 1$. $\lim_{x\to 2} g(x)$ does not exist.

2.2.15

a. $f(1) = 0.$

b. $\lim_{x\to 1^-} f(x) = 1.$

c. $\lim_{x\to 1^+} f(x) = 0.$

d. $\lim_{x\to 1} f(x)$ does not exist, since the two one-sided limits aren't equal.

2.2.16 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

2.2.17

a. $f(1) = 3$.

b. $\lim_{x\to 1^-} f(x) = 2$.

c. $\lim_{x\to 1^+} f(x) = 2$.

d. $\lim_{x\to 1} f(x) = 2$.

e. $f(3) = 2$.

f. $\lim_{x\to 3^-} f(x) = 4$.

g. $\lim_{x\to 3^+} f(x) = 1$.

h. $\lim_{x\to 3} f(x)$ does not exist.

i. $f(2) = 3$.

j. $\lim_{x\to 2^-} f(x) = 3$.

k. $\lim_{x\to 2^+} f(x) = 3$.

l. $\lim_{x\to 2} f(x) = 3$.

2.2.18

a. $g(2) = 3$.

b. $\lim_{x\to 2^-} g(x) = 2$.

c. $\lim_{x\to 2^+} g(x) = 3$.

d. $\lim_{x\to 2} g(x)$ does not exist.

e. $g(3) = 2$.

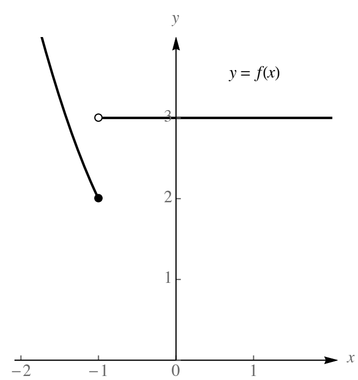
f. $\lim_{x\to 3^-} g(x) = 3$.

g. $\lim_{x\to 3^+} g(x) = 2$.

h. $g(4) = 3$.

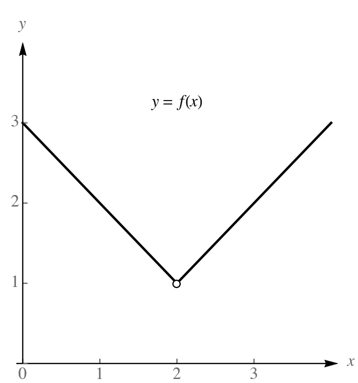
i. $\lim_{x\to 4} g(x) = 3$.

2.2.19



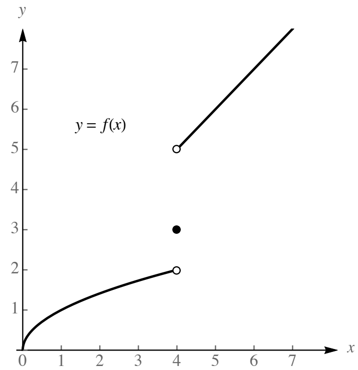
$f(-1) = 2$, $\lim_{x\to -1^-} f(x) = 2$, $\lim_{x\to -1^+} f(x) = 3$, $\lim_{x\to -1} f(x)$ does not exist.

2.2.20



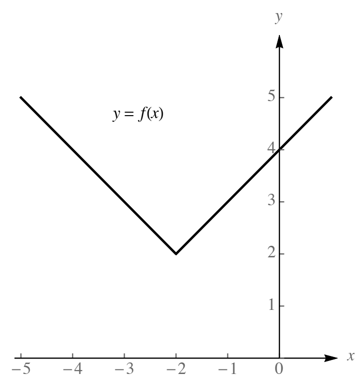
$f(2)$ is undefined. $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 1$, and $\lim_{x\to 2} f(x) = 1$.

2.2.21



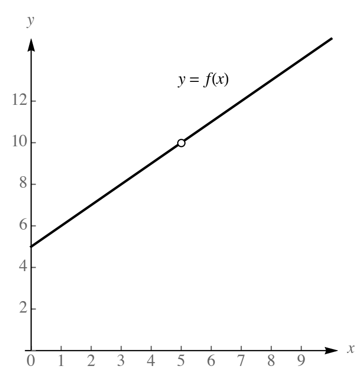
$f(4) = 3$, $\lim_{x\to 4^-} f(x) = 2$, $\lim_{x\to 4^+} f(x) = 5$, $\lim_{x\to 4} f(x)$ does not exist.

2.2.22



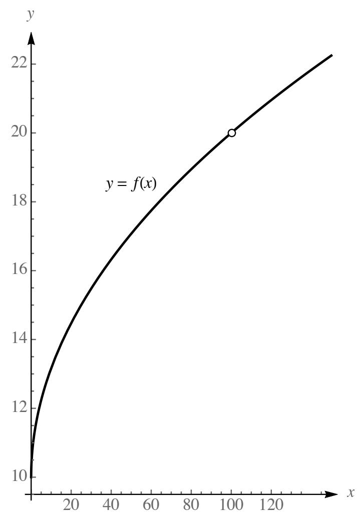
$f(-2) = 2$, $\lim_{x\to -2^-} f(x) = 2$, $\lim_{x\to -2^+} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$.

2.2.23



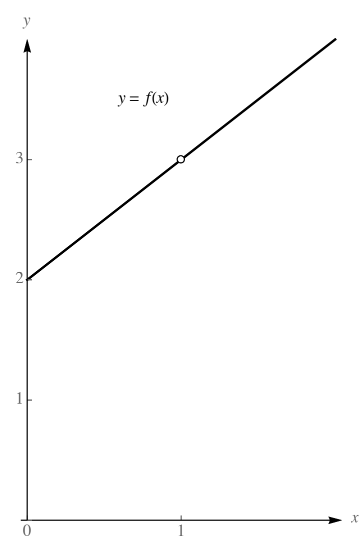
$f(5)$ does not exist. $\lim_{x\to 5^-} f(x) = 10$, $\lim_{x\to 5^+} f(x) = 10$, $\lim_{x\to 5} f(x) = 10$.

2.2.24



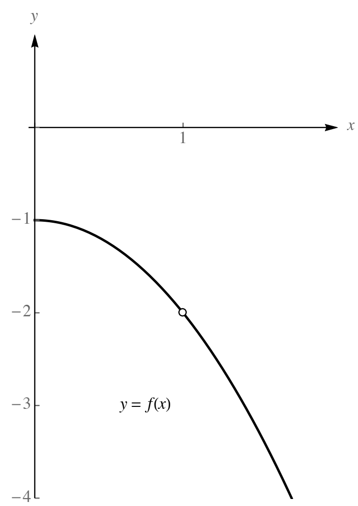
$f(100)$ does not exist. $\lim_{x\to 100^-} f(x) = 20$, $\lim_{x\to 100^+} f(x) = 20$, $\lim_{x\to 100} f(x) = 20$.

2.2.25



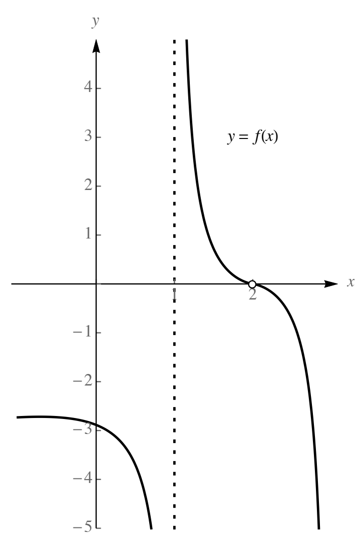
$f(1)$ does not exist. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 3$, $\lim_{x\to 1} f(x) = 3$.

2.2.26



$f(1)$ does not exist. $\lim_{x\to1^-} f(x) = -2$, $\lim_{x\to1^+} f(x) = -2$, $\lim_{x\to1} f(x) = -2$.

2.2.27

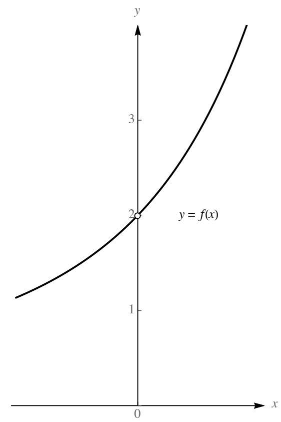


x	1.99	1.999	1.9999
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	0.0021715	0.00014476	0.000010857

x	2.0001	2.001	2.01
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	−0.000010857	−0.00014476	−0.0021715

From the graph and the table, the limit appears to be 0.

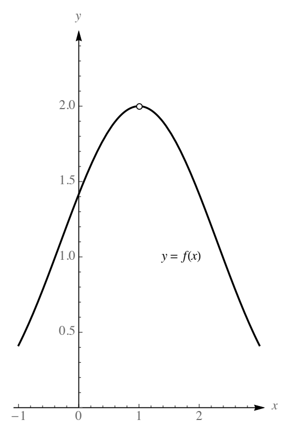
2.2.28



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.8731	1.98673	1.9987	2.0013	2.0134	2.1403

From both the graph and the table, the limit appears to be 2.

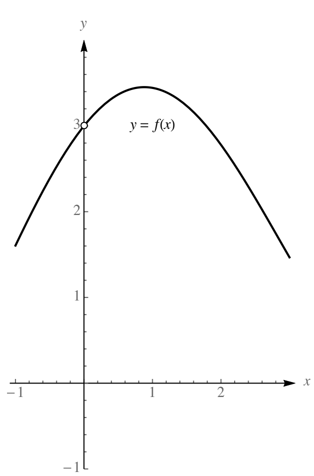
2.2.29



x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.993342	1.999933	1.999999	1.999999	1.999933	1.993342

From both the graph and the table, the limit appears to be 2.

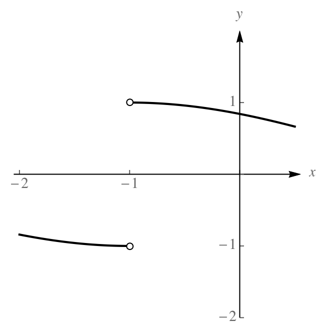
2.2.30



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.8951	2.99	2.999	3.001	3.0099	3.0949

From both the graph and the table, the limit appears to be 3.

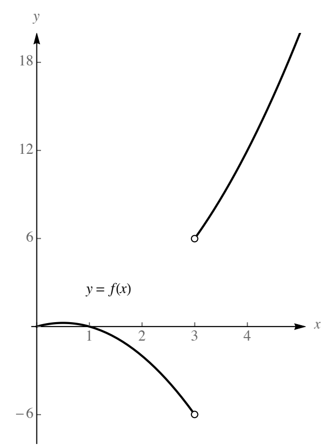
2.2.31



x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$	-0.9983342	-0.9999833	-0.9999998	0.9999998	0.9999833	0.9983342

From both the graph and the table, it appears that the limit does not exist.

2.2.32



x	2.9	2.99	2.999	3.001	3.01	3.1
$g(x)$	-5.51	-5.9501	-5.995001	6.005001	6.0501	6.51

From both the graph and the table, it appears that the limit does not exist.

2.2.33

a. False. In fact $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6$.

b. False. For example, if $f(x) = \begin{cases} x^2 & \text{if } x \neq 0; \\ 5 & \text{if } x = 0 \end{cases}$ and if $a = 0$ then $f(a) = 5$ but $\displaystyle\lim_{x\to a} f(x) = 0$.

c. False. For example, the limit in part a of this problem exists, even though the corresponding function is undefined at $a = 3$.

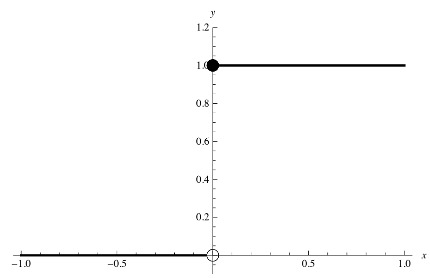
d. False. It is true that the limit of $\sqrt{x}$ as x approaches zero from the right is zero, but because the domain of $\sqrt{x}$ does not include any numbers to the left of zero, the two-sided limit doesn't exist.

e. True. Note that $\lim_{x\to\pi/2} \cos x = 0$ and $\lim_{x\to\pi/2} \sin x = 1$, so $\lim_{x\to\pi/2} \frac{\cos x}{\sin x} = \frac{0}{1} = 0$.

2.2.34

a. Note that H is piecewise constant.

b. $\lim_{x\to 0^-} H(x) = 0$, $\lim_{x\to 0^+} H(x) = 1$, and so $\lim_{x\to 0} H(x)$ does not exist.

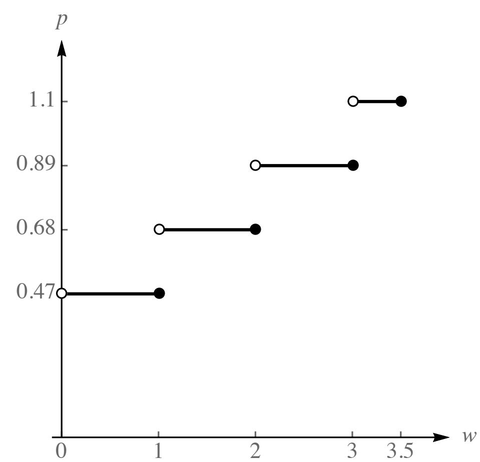


2.2.35

a. Note that the function is piecewise constant.

b. $\lim_{w\to 2.3} f(w) = .89$.

c. $\lim_{w\to 3^+} f(w) = 1.1$ corresponds to the fact that for any piece of mail that weighs slightly over 3 ounces, the postage will cost \$1.1 cents. $\lim_{w\to 3^-} f(w) = \$0.89$ corresponds to the fact that for any piece of mail that weighs slightly less than 3 ounces, the postage will cost 89 cents. Because the two one-sided limits are not equal, $\lim_{w\to 3} f(w)$ does not exist.



2.2.36

h	0.01	0.001	0.0001	−0.0001	−0.001	−0.01
$\frac{(1+2h)^{1/h}}{2e^{2+h}}$	0.48535	0.498504	0.49985	0.50015	0.501504	0.515367

$\lim_{h\to 0} \frac{(1+2h)^{1/h}}{2e^{2+h}} = \frac{1}{2}$.

2.2.37

x	1.37	1.47	1.57	1.58	1.68	1.78
$\frac{\cot 3x}{\cos x}$	3.44773	3.10016	3.00001	3.0008	3.11834	3.49316

$\lim_{x\to\pi/2} \frac{\cot 3x}{\cos x} = 3$.

2.2.38

x	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}}$	2.29222	2.02691	2.00267	1.99734	1.97357	1.75541

$\lim_{x\to 1} \frac{18(\sqrt[3]{x}-1)}{x^3-1} = 2$.

2.2.39

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$\frac{18(\sqrt[3]{x}-1)}{x^3-1}$	15.5803	15.9574	15.9957	16.0043	16.0427	16.4339

$$\lim_{x\to 1} \frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}} = 16.$$

2.2.40

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\frac{6^x-3^x}{x\ln 16}$	0.24614	0.249639	0.249964	0.250036	0.250362	0.25364

$$\lim_{x\to 0} \frac{6^x-3^x}{x\ln 16} = \frac{1}{4}.$$

2.2.41

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{\ln(1+h)}{h}$	0.995033	0.9995	0.99995	1.00005	1.0005	1.00503

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$$

2.2.42

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{4^h-1}{h\ln(h+2)}$	1.99954	1.99994	1.99999	2.00001	2.00006	2.00067

$$\lim_{h\to 0} \frac{4^h-1}{h\ln(h+2)} = 2.$$

2.2.43

a.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$
$f(x)=\sin(1/x)$	1	-1	1	-1	1	-1

If $x_n = \frac{2}{(2n+1)\pi}$, then $f(x_n) = (-1)^n$ where n is a non-negative integer.

b. As $x \to 0$, $1/x \to \infty$. So the values of $f(x)$ oscillate dramatically between -1 and 1.

c. $\lim_{x\to 0} \sin(1/x)$ does not exist.

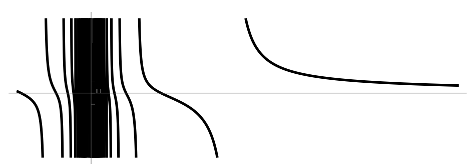
2.2.44

a.

x	$\frac{12}{\pi}$	$\frac{12}{3\pi}$	$\frac{12}{5\pi}$	$\frac{12}{7\pi}$	$\frac{12}{9\pi}$	$\frac{12}{11\pi}$
$f(x)=\tan(3/x)$	1	-1	1	-1	1	-1

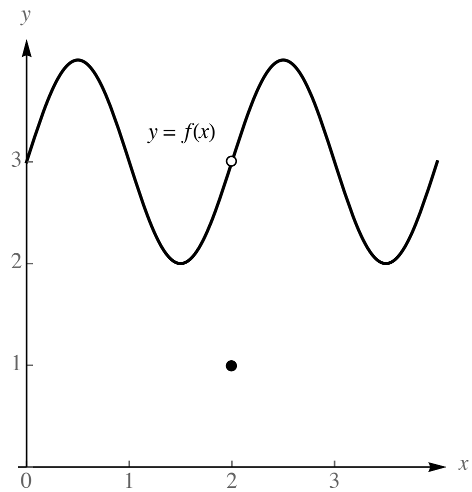
We have alternating 1's and -1's.

b. $\tan 3x$ alternates between 1 and -1 infinitely many times on $(0, h)$ for any $h > 0$.

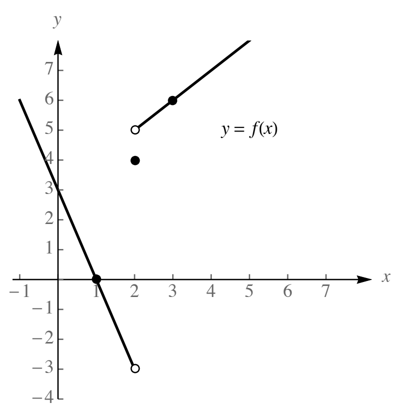


c. $\lim_{x \to 0} \tan(3/x)$ does not exist.

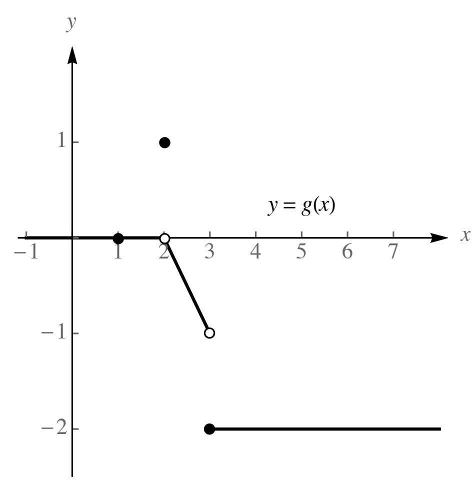
2.2.45



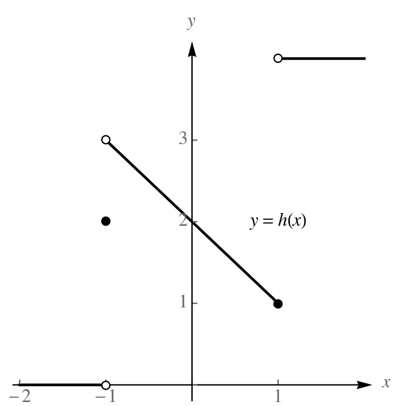
2.2.46



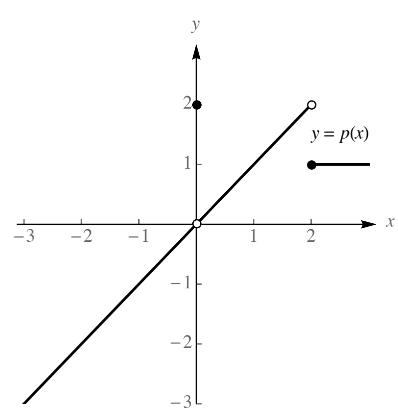
2.2.47



2.2.48



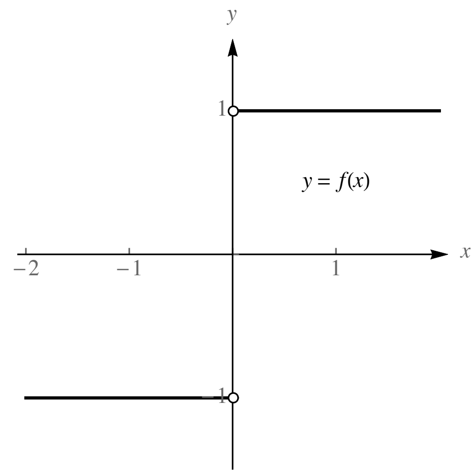
2.2.49



2.2.50

a. Note that $f(x) = \frac{\lfloor x \rfloor}{x}$ is undefined at 0, and $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$.

b. $\lim_{x\to 0} f(x)$ does not exist, since the two one-side limits aren't equal.



2.2.51

a. $\lim_{x\to -1^-} \lfloor x \rfloor = -2$, $\lim_{x\to -1^+} \lfloor x \rfloor = -1$, $\lim_{x\to 2^-} \lfloor x \rfloor = 1$, $\lim_{x\to 2^+} \lfloor x \rfloor = 2$.

b. $\lim_{x\to 2.3^-} \lfloor x \rfloor = 2$, $\lim_{x\to 2.3^+} \lfloor x \rfloor = 2$, $\lim_{x\to 2.3} \lfloor x \rfloor = 2$.

c. In general, for an integer a, $\lim_{x\to a^-} \lfloor x \rfloor = a - 1$ and $\lim_{x\to a^+} \lfloor x \rfloor = a$.

d. In general, if a is not an integer, $\lim_{x\to a^-} \lfloor x \rfloor = \lim_{x\to a^+} \lfloor x \rfloor = \lfloor a \rfloor$.

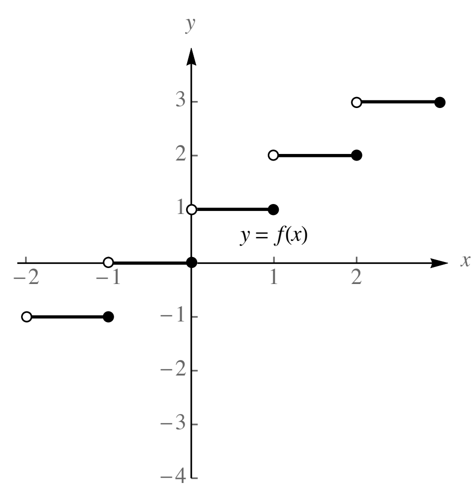
e. $\lim_{x\to a} \lfloor x \rfloor$ exists and is equal to $\lfloor a \rfloor$ for non-integers a.

2.2.52

a. Note that the graph is piecewise constant.

b. $\lim_{x\to 2^-} \lceil x \rceil = 2$, $\lim_{x\to 1^+} \lceil x \rceil = 2$, $\lim_{x\to 1.5} \lceil x \rceil = 2$.

c. $\lim_{x\to a} \lceil x \rceil$ exists and is equal to $\lceil a \rceil$ for non-integers a.



2.2.53

a. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^+} f(x) = 8$.

b. Because of the symmetry about the y axis, we must have $\lim_{x\to -2^-} f(x) = 5$.

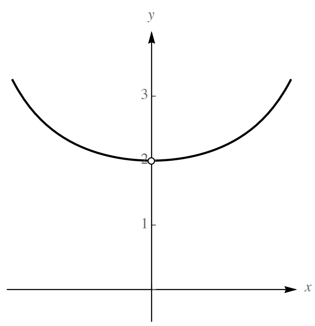
2.2.54

a. Because of the symmetry about the origin, we must have $\lim_{x\to -2^+} g(x) = -8$.

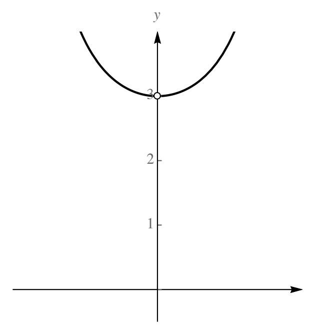
b. Because of the symmetry about the origin, we must have $\lim_{x\to -2^-} g(x) = -5$.

2.2.55

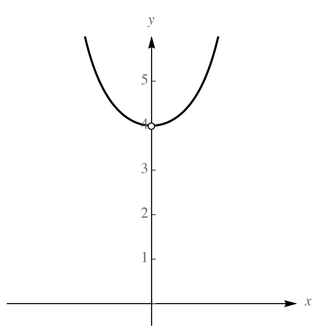
a.



$$\lim_{x\to 0} \frac{\tan 2x}{\sin x} = 2.$$



$$\lim_{x\to 0} \frac{\tan 3x}{\sin x} = 3.$$

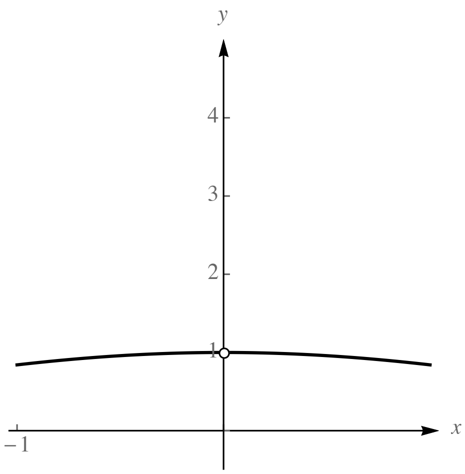


$$\lim_{x\to 0} \frac{\tan 4x}{\sin x} = 4.$$

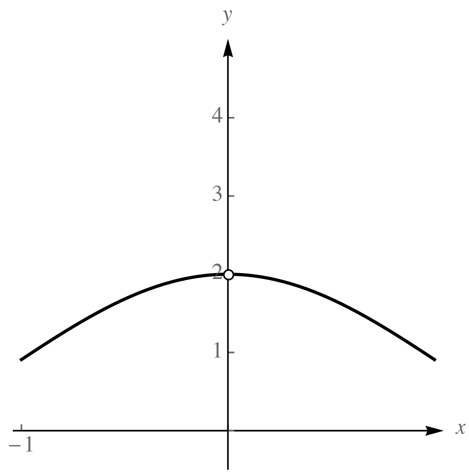
b. It appears that $\lim_{x\to 0} \frac{\tan(px)}{\sin x} = p$.

2.2.56

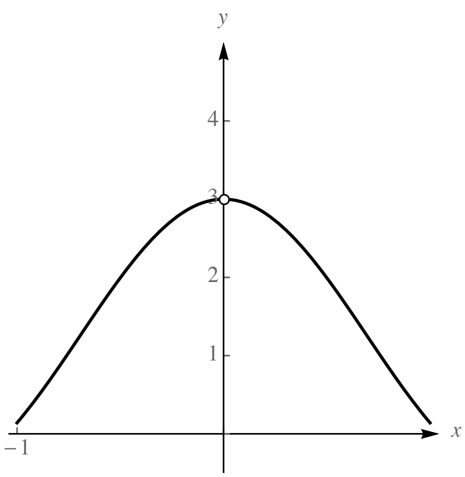
a.



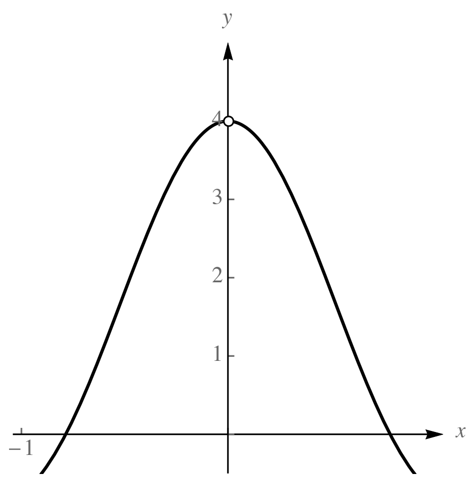
$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$



$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2.$$



$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3.$$

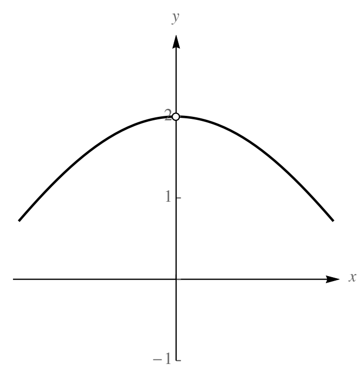


$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$$

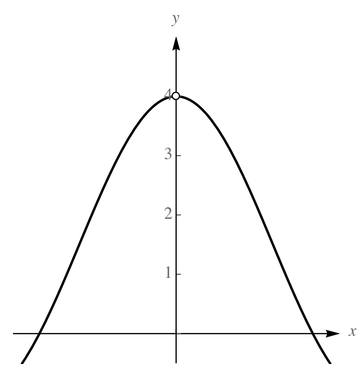
b. It appears that $\lim_{x\to 0} \frac{\sin(px)}{x} = p$.

2.2.57

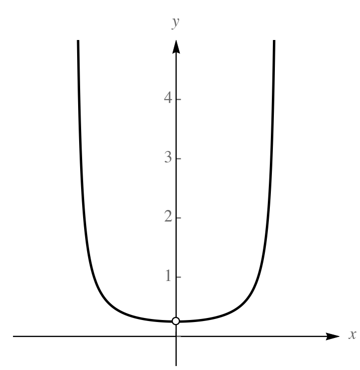
For $p = 8$ and $q = 2$, it appears that the limit is 4.



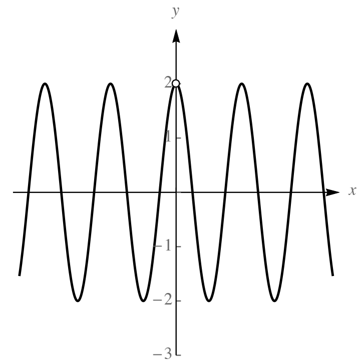
For $p = 12$ and $q = 3$, it appears that the limit is 4.



For $p = 4$ and $q = 16$, it appears that the limit is $1/4$.



For $p = 100$ and $q = 50$, it appears that the limit is 2.



Conjecture: $\lim_{x\to 0} \dfrac{\sin px}{\sin qx} = \dfrac{p}{q}$.

2.3 Techniques for Computing Limits

2.3.1 If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to a} p(x) = \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$
$= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1 \lim_{x\to a} x + \lim_{x\to a} a_0$
$= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0. = p(a)$.

2.3.2 $\lim_{x\to 1}(x^3 + 3x^2 - 3x + 1) = 1 + 3 - 3 + 1 = 2$.

2.3.3 For a rational function $r(x)$, we have $\lim_{x\to a} r(x) = r(a)$ exactly for those numbers a which are in the domain of r. (Which are those for which the denominator isn't zero.)

2.3.4 $\lim_{x\to 4}\left(\dfrac{x^2 - 4x - 1}{3x - 1}\right) = \dfrac{16 - 16 - 1}{12 - 1} = -\dfrac{1}{11}$.

2.3.5 Because $\frac{x^2-7x+12}{x-3} = \frac{(x-3)(x-4)}{x-3} = x - 4$ (for $x \neq 3$), we can see that the graphs of these two functions are the same except that one is undefined at $x = 3$ and the other is a straight line that is defined everywhere. Thus the function $\frac{x^2-7x+12}{x-3}$ is a straight line except that it has a "hole" at $(3, -1)$. The two functions have the same limit as $x \to 3$, namely $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = \lim_{x\to 3}(x - 4) = -1$.

2.3.6 $\lim_{x\to 5} \dfrac{4x^2 - 100}{x - 5} = \lim_{x\to 5} \dfrac{4(x-5)(x+5)}{x - 5} = \lim_{x\to 5} 4(x + 5) = 40$.

2.3.7 $\lim_{x\to 1} 4f(x) = 4\lim_{x\to 1} f(x) = 4 \cdot 8 = 32$. This follows from the Constant Multiple Law.

2.3.8 $\lim_{x\to 1} \dfrac{f(x)}{h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} h(x)} = \dfrac{8}{2} = 4$. This follows from the Quotient Law.

2.3.9 $\lim_{x\to 1}(f(x) - g(x)) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 8 - 3 = 5$. This follows from the Difference Law.

2.3.10 $\lim_{x\to 1} f(x)h(x) = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} h(x) = 8 \cdot 2 = 16$. This follows from the Product Law.

2.3.11 $\lim_{x\to 1} \dfrac{f(x)}{g(x) - h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1}[g(x) - h(x)]} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - \lim_{x\to 1} h(x)} = \dfrac{8}{3 - 2} = 8$. This follows from the Quotient and Difference Laws.

2.3.12 $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3} = \sqrt[3]{\lim_{x\to 1}(f(x)g(x)+3)} = \sqrt[3]{\lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x) + \lim_{x\to 1} 3} = \sqrt[3]{8\cdot 3+3} = \sqrt[3]{27} = 3$. This follows form the Root, Product, Sum and Constant Laws.

2.3.13 $\lim_{x\to 1} f(x)^{2/3} = \left(\lim_{x\to 1} f(x)\right)^{2/3} = 8^{2/3} = 2^2 = 4$. This follows from the Root and Power Laws.

2.3.14 If $p(x)$ is a polynomial, then $\lim_{x\to a^-} p(x) = \lim_{x\to a^+} p(x) = p(a)$.

2.3.15 $\lim_{x\to 0} g(x) = \lim_{x\to 0}(2x+1) = 1$, while $g(0) = 5$.

2.3.16 $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 4 = 4$, and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x+2) = 5$. Because the two one-sided limits differ, the two-sided limit doesn't exist.

2.3.17 If p and q are polynomials then $\lim_{x\to 0} \frac{p(x)}{q(x)} = \frac{\lim_{x\to 0} p(x)}{\lim_{x\to 0} q(x)} = \frac{p(0)}{q(0)}$. Because this quantity is given to be equal to 10, we have $\frac{p(0)}{2} = 10$, so $p(0) = 20$.

2.3.18 By a direct application of the squeeze theorem, $\lim_{x\to 2} g(x) = 5$.

2.3.19 $\lim_{x\to 4}(3x-7) = 3\lim_{x\to 4} x - 7 = 3\cdot 4 - 7 = 5$.

2.3.20 $\lim_{x\to 1}(-2x+5) = -2\lim_{x\to 1} x + 5 = -2\cdot 1 + 5 = 3$.

2.3.21 $\lim_{x\to -9}(5x) = 5\lim_{x\to -9} x = 5\cdot -9 = -45$.

2.3.22 $\lim_{x\to 6} 4 = 4$.

2.3.23 $\lim_{x\to 1}(2x^3-3x^2+4x+5) = \lim_{x\to 1} 2x^3 - \lim_{x\to 1} 3x^2 + \lim_{x\to 1} 4x + \lim_{x\to 1} 5 = 2(\lim_{x\to 1} x)^3 - 3(\lim_{x\to 1} x)^2 + 4(\lim_{x\to 1} x) + 5 = 2(1)^3 - 3(1)^2 + 4\cdot 1 + 5 = 8$.

2.3.24 $\lim_{t\to -2}(t^2+5t+7) = \lim_{t\to -2} t^2 + \lim_{t\to -2} 5t + \lim_{t\to -2} 7 = \left(\lim_{t\to -2} t\right)^2 + 5\lim_{t\to -2} t + 7 = (-2)^2 + 5\cdot(-2) + 7 = 1$.

2.3.25 $\lim_{x\to 1} \frac{5x^2+6x+1}{8x-4} = \frac{\lim_{x\to 1}(5x^2+6x+1)}{\lim_{x\to 1}(8x-4)} = \frac{5(\lim_{x\to 1} x)^2 + 6\lim_{x\to 1} x + \lim_{x\to 1} 1}{8\lim_{x\to 1} x - \lim_{x\to 1} 4} = \frac{5(1)^2+6\cdot 1+1}{8\cdot 1-4} = 3$.

2.3.26 $\lim_{t\to 3} \sqrt[3]{t^2-10} = \sqrt[3]{\lim_{t\to 3}(t^2-10)} = \sqrt[3]{\lim_{t\to 3} t^2 - \lim_{t\to 3} 10} = \sqrt[3]{\left(\lim_{t\to 3} t\right)^2 - 10} = \sqrt[3]{(3)^2-10} = -1$.

2.3.27 $\lim_{p\to 2} \frac{3p}{\sqrt{4p+1}-1} = \frac{\lim_{p\to 2} 3p}{\lim_{p\to 2}(\sqrt{4p+1}-1)} = \frac{3\lim_{p\to 2} p}{\lim_{p\to 2}\sqrt{4p+1} - \lim_{p\to 2} 1} = \frac{3\cdot 2}{\sqrt{\lim_{p\to 2}(4p+1)}-1} = \frac{6}{3-1} = 3$.

2.3.28 $\lim_{x\to 2}(x^2-x)^5 = \left(\lim_{x\to 2}(x^2-x)\right)^5 = \left(\lim_{x\to 2} x^2 - \lim_{x\to 2} x\right)^5 = (4-2)^5 = 32$.

2.3.29 $\lim_{x\to 3} -\frac{5x}{\sqrt{4x-3}} = \frac{\lim_{x\to 3} -5x}{\lim_{x\to 3}\sqrt{4x-3}} = \frac{-5\lim_{x\to 3} x}{\sqrt{\lim_{x\to 3}(4x-3)}} = -\frac{5\cdot 3}{\sqrt{4\lim_{x\to 3} x - \lim_{x\to 3} 3}} = -\frac{15}{\sqrt{4\cdot 3-3}} = -5$.

2.3.30

$$\lim_{h\to 0}\frac{3}{\sqrt{16+3h}+4} = \frac{\lim_{h\to 0} 3}{\lim_{h\to 0}(\sqrt{16+3h}+4)} = \frac{3}{\sqrt{\lim_{h\to 0}(16+3h)}+\lim_{h\to 0} 4} = \frac{3}{\sqrt{\lim_{h\to 0} 16+\lim_{h\to 0} 3h}+4}$$
$$= \frac{3}{\sqrt{16+3\cdot 0}+4} = \frac{3}{4+4} = \frac{3}{8}.$$

2.3.31 $\lim_{x\to 2}(5x-6)^{3/2} = (5\cdot 2-6)^{3/2} = 4^{3/2} = 2^3 = 8.$

2.3.32 $\lim_{h\to 0}\frac{100}{(10h-1)^{11}+2} = \frac{100}{(-1)^{11}+2} = \frac{100}{1} = 100.$

2.3.33 $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 2.$

2.3.34 $\lim_{x\to 3}\frac{x^2-2x-3}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+1)}{x-3} = \lim_{x\to 3}(x+1) = 4.$

2.3.35 $\lim_{x\to 4}\frac{x^2-16}{4-x} = \lim_{x\to 4}\frac{(x+4)(x-4)}{-(x-4)} = \lim_{x\to 4}[-(x+4)] = -8.$

2.3.36 $\lim_{t\to 2}\frac{3t^2-7t+2}{2-t} = \lim_{t\to 2}\frac{(t-2)(3t-1)}{-(t-2)} = \lim_{t\to 2}[-(3t-1)] = -5.$

2.3.37 $\lim_{x\to b}\frac{(x-b)^{50}-x+b}{x-b} = \lim_{x\to b}\frac{(x-b)^{50}-(x-b)}{x-b} = \lim_{x\to b}\frac{(x-b)((x-b)^{49}-1)}{x-b} =$
$\lim_{x\to b}[(x-b)^{49}-1] = -1.$

2.3.38 $\lim_{x\to -b}\frac{(x+b)^7+(x+b)^{10}}{4(x+b)} = \lim_{x\to -b}\frac{(x+b)((x+b)^6+(x+b)^9)}{4(x+b)} = \lim_{x\to -b}\frac{(x+b)^6+(x+b)^9}{4} = \frac{0}{4} = 0.$

2.3.39

$$\lim_{x\to -1}\frac{(2x-1)^2-9}{x+1} = \lim_{x\to -1}\frac{(2x-1-3)(2x-1+3)}{x+1}$$
$$= \lim_{x\to -1}\frac{2(x-2)2(x+1)}{x+1} = \lim_{x\to -1} 4(x-2) = 4\cdot(-3) = -12.$$

2.3.40

$$\lim_{h\to 0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h} = \lim_{h\to 0}\frac{\left(\frac{1}{5+h}-\frac{1}{5}\right)\cdot 5\cdot(5+h)}{h\cdot 5\cdot(5+h)}$$
$$= \lim_{h\to 0}\frac{5-(5+h)}{5h(5+h)} = \lim_{h\to 0} -\frac{h}{5h(5+h)} = \lim_{h\to 0} -\frac{1}{5(5+h)} = -\frac{1}{25}.$$

2.3.41 $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{6}.$

2.3.42 $\lim_{w\to 1}\left(\frac{1}{w^2-w}-\frac{1}{w-1}\right) = \lim_{w\to 1}\left(\frac{1}{w(w-1)}-\frac{w}{w(w-1)}\right) = \lim_{w\to 1}\frac{1-w}{w(w-1)} = -\lim_{w\to 1}\frac{1}{w} = -1.$

2.3.43

$$\lim_{t\to 5}\left(\frac{1}{t^2-4t-5}-\frac{1}{6(t-5)}\right) = \lim_{t\to 5}\left(\frac{1}{(t-5)(t+1)}-\frac{1}{6(t-5)}\right)$$
$$= \lim_{t\to 5}\left(\frac{6}{6(t-5)(t+1)}-\frac{t+1}{6(t-5)(t+1)}\right)$$
$$= \lim_{t\to 5}\frac{5-t}{6(t-5)(t+1)} = -\lim_{t\to 5}\frac{1}{6(t+1)} = -\frac{1}{36}.$$

2.3.44 Expanding gives

$$\lim_{t\to 3}\left(\left(4t - \frac{2}{t-3}\right)(6+t-t^2)\right) = \lim_{t\to 3}\left(4t(6+t-t^2) - \frac{2(6+t-t^2)}{t-3}\right)$$
$$= \lim_{t\to 3}\left(4t(6+t-t^2) - \frac{2(3-t)(2+t)}{t-3}\right).$$

Now because $t-3 = -(3-t)$, we have

$$\lim_{t\to 3}\left(4t(6+t-t^2) + 2(2+t)\right) = 12(6+3-9) + 2(2+3) = 10.$$

2.3.45 $\displaystyle\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a} = \lim_{x\to a}(\sqrt{x}+\sqrt{a}) = 2\sqrt{a}.$

2.3.46 $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a}\frac{(x-a)(x+a)(\sqrt{x}+\sqrt{a})}{x-a} =$
$(a+a)(\sqrt{a}+\sqrt{a}) = 4a^{3/2}$.

2.3.47

$$\lim_{h\to 0}\frac{\sqrt{16+h}-4}{h} = \lim_{h\to 0}\frac{(\sqrt{16+h}-4)(\sqrt{16+h}+4)}{h(\sqrt{16+h}+4)} = \lim_{h\to 0}\frac{(16+h)-16}{h(\sqrt{16+h}+4)}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{16+h}+4)} = \lim_{h\to 0}\frac{1}{(\sqrt{16+h}+4)} = \frac{1}{8}.$$

2.3.48 $\displaystyle\lim_{x\to c}\frac{x^2-2cx+c^2}{x-c} = \lim_{x\to c}\frac{(x-c)^2}{x-c} = \lim_{x\to c} x-c = c-c = 0.$

2.3.49 $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4} = \lim_{x\to 4}\frac{\frac{4-x}{4x}}{x-4} = \lim_{x\to 4}\frac{4-x}{4x(x-4)} = -\lim_{x\to 4}\frac{1}{4x} = -\frac{1}{16}.$

2.3.50

$$\lim_{x\to 3}\frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3} = \lim_{x\to 3}\frac{\frac{15-(x^2+2x)}{15(x^2+2x)}}{x-3} = \lim_{x\to 3}\frac{15-(x^2+2x)}{15(x^2+2x)(x-3)} = \lim_{x\to 3}\frac{15-2x-x^2}{15(x^2+2x)(x-3)}$$
$$= \lim_{x\to 3}\frac{(3-x)(5+x)}{15(x^2+2x)(x-3)} = \lim_{x\to 3} -\frac{(5+x)}{15(x^2+2x)} = -\frac{8}{225}.$$

2.3.51

$$\lim_{x\to 1}\frac{\sqrt{10x-9}-1}{x-1} = \lim_{x\to 1}\frac{(\sqrt{10x-9}-1)(\sqrt{10x-9}+1)}{(x-1)(\sqrt{10x-9}+1)} = \lim_{x\to 1}\frac{(10x-9)-1}{(x-1)(\sqrt{10x-9}+1)}$$
$$= \lim_{x\to 1}\frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)} = \lim_{x\to 1}\frac{10}{(\sqrt{10x-9}+1)} = \frac{10}{2} = 5.$$

2.3.52 $\displaystyle\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{2}{x^2-2x}\right) = \lim_{x\to 2}\left(\frac{x}{x(x-2)}-\frac{2}{x(x-2)}\right) = \lim_{x\to 2}\left(\frac{x-2}{x(x-2)}\right) = \lim_{x\to 2}\frac{1}{x} = \frac{1}{2}.$

2.3.53 $\displaystyle\lim_{h\to 0}\frac{(5+h)^2-25}{h} = \lim_{h\to 0}\frac{25+10h+h^2-25}{h} = \lim_{h\to 0}\frac{h(10+h)}{h} = \lim_{h\to 0}(10+h) = 10.$

2.3.54 We have

$$\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw} = \lim_{w\to -k}\frac{(w+4k)(w+k)}{(w)(w+k)} = \lim_{w\to -k}\frac{w+4k}{w} = \frac{-k+4k}{-k} = -3.$$

If $k=0$, we have $\displaystyle\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw} = \lim_{w\to 0}\frac{w^2}{w^2} = 1.$

2.3.55 $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}=\lim_{x\to 1}(\sqrt{x}+1)=2.$

2.3.56

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{4x+5}-3}&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{(\sqrt{4x+5}-3)(\sqrt{4x+5}+3)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9}\\&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)}=\lim_{x\to 1}\frac{(\sqrt{4x+5}+3)}{4}=\frac{6}{4}=\frac{3}{2}.\end{aligned}$$

2.3.57

$$\begin{aligned}\lim_{x\to 4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{(3-\sqrt{x+5})(3+\sqrt{x+5})}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{9-(x+5)}\\&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{-(x-4)}\\&=\lim_{x\to 4}[-3(\sqrt{x+5})(3+\sqrt{x+5})]=(-3)(3)(3+3)=-54\end{aligned}$$

2.3.58 Assume $c\neq 0$.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{cx+1}-1}&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(\sqrt{cx+1}-1)(\sqrt{cx+1}+1)}=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(cx+1)-1}\\&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{cx}=\lim_{x\to 0}\frac{(\sqrt{cx+1}+1)}{c}=\frac{2}{c}.\end{aligned}$$

2.3.59 $\lim_{x\to 0}x\cos x=0\cdot 1=0.$

2.3.60 $\lim_{x\to 0}\frac{\sin 2x}{\sin x}=\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}=\lim_{x\to 0}2\cos x=2.$

2.3.61 $\lim_{x\to 0}\frac{1-\cos x}{\cos^2 x-3\cos x+2}=\lim_{x\to 0}\frac{1-\cos x}{(\cos x-2)(\cos x-1)}=-\lim_{x\to 0}\frac{1}{\cos x-2}=-\frac{1}{1-2}=1.$

2.3.62 $\lim_{x\to 0}\frac{\cos x-1}{\cos^2 x-1}=\lim_{x\to 0}\frac{\cos x-1}{(\cos x-1)(\cos x+1)}=\lim_{x\to 0}\frac{1}{\cos x+1}=\frac{1}{2}.$

2.3.63 $\lim_{x\to 0^-}\frac{x^2-x}{|x|}=\lim_{x\to 0^-}\frac{x(x-1)}{-x}=-\lim_{x\to 0^-}(x-1)=1.$

2.3.64 $\lim_{w\to 3^-}\frac{|w-3|}{w^2-7w+12}=\lim_{w\to 3^-}\frac{3-w}{(w-3)(w-4)}=-\lim_{w\to 3^-}\frac{1}{w-4}=1.$

2.3.65 $\lim_{t\to 2^+}\frac{|2t-4|}{t^2-4}=\lim_{t\to 2^+}\frac{2(t-2)}{(t-2)(t+2)}=\frac{2}{4}=\frac{1}{2}.$

2.3.66 $\lim_{x\to -1^-}g(x)=\lim_{x\to -1^-}\frac{x^2-1}{x+1}=\lim_{x\to -1^-}\frac{(x+1)(x-1)}{x+1}=\lim_{x\to -1^-}(x-1)=-2$. Also, $\lim_{x\to -1^+}g(x)=\lim_{x\to -1^+}(-2)=-2$. Therefore, $\lim_{x\to -1}g(x)=-2$.

2.3.67 $\lim_{x\to 3^+}\frac{x-3}{|x-3|}=\lim_{x\to 3^+}\frac{x-3}{x-3}=\lim_{x\to 3^+}1=1$. On the other hand, $\lim_{x\to 3^-}\frac{x-3}{|x-3|}=\lim_{x\to 3^-}\frac{x-3}{3-x}=\lim_{x\to 3^-}(-1)=-1$. Therefore, $\lim_{x\to 3}\frac{x-3}{|x-3|}$ does not exist.

2.3.68 $\lim_{x\to 5^+} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$. On the other hand, $\lim_{x\to 5^-} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^-} \frac{5-x}{(x-5)(x+5)} = -\lim_{x\to 5^+} \frac{1}{x+5} = -\frac{1}{10}$. Therefore, $\lim_{x\to 5} \frac{|x-5|}{x^2-25}$ does not exist.

2.3.69 Because the domain of $f(x) = \frac{x^3+1}{\sqrt{x-1}}$ is the interval $(1,\infty)$, the limit doesn't exist.

2.3.70 $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim_{x\to 1^+} \frac{x-1}{(x-1)^{1/2}(x+1)^{1/2}} = \lim_{x\to 1^+} \frac{(x-1)^{1/2}}{(x+1)^{1/2}} = \frac{0}{\sqrt{2}} = 0.$

2.3.71

a. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ then $\lim_{x\to 1} f(x) = 1$ but $f(1) = 4$.

b. False. For example, if $f(x) = \begin{cases} x+1 & \text{if } x \leq 1; \\ x-6 & \text{if } x > 1, \end{cases}$ then $\lim_{x\to 1^-} f(x) = 2$ but $\lim_{x\to 1^+} f(x) = -5$.

c. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ and $g(x) = 1$, then f and g both have limit 1 as $x \to 1$, but $f(1) = 4 \neq g(1)$.

d. False. For example $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exists and is equal to 4.

e. False. For example, it would be possible for the domain of f to be $[1,\infty)$, so that the one-sided limit exists but the two-sided limit doesn't even make sense. This would be true, for example, if $f(x) = x-1$.

2.3.72

a. $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (5x-15) = 5.$

b. $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \sqrt{6x+1} = 5.$

c. Because the two one-sided limits both exist and are equal to 5, $\lim_{x\to 4} g(x) = 5$.

2.3.73

a. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+1) = (-1)^2 + 1 = 2.$

b. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = \sqrt{-1+1} = 0.$

c. Because the two one-sided limits differ, $\lim_{x\to -1} f(x)$ does not exist.

2.3.74

a. $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} 0 = 0.$

b. $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \sqrt{25-x^2} = \sqrt{25-25} = 0.$

c. $\lim_{x\to -5} f(x) = 0.$

d. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{25-x^2} = \sqrt{25-25} = 0.$

e. $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 3x = 15.$

f. $\lim_{x\to 5} f(x)$ does not exist.

2.3.75

a. $\lim_{x\to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0.$

b. The domain of $f(x) = \sqrt{x-2}$ is $[2,\infty)$. Thus, any question about this function that involves numbers less than 2 doesn't make any sense, because those numbers aren't in the domain of f.

2.3.76

a. Note that the domain of $f(x) = \sqrt{\frac{x-3}{2-x}}$ is $(2,3]$. $\lim_{x\to 3^-} \sqrt{\frac{x-3}{2-x}} = 0.$

b. Because the numbers to the right of 3 aren't in the domain of this function, the limit as $x \to 3^+$ of this function doesn't make any sense.

2.3.77 $\lim_{x\to 10} E(x) = \lim_{x\to 10} \frac{4.35}{x\sqrt{x^2+0.01}} = \frac{4.35}{10\sqrt{100.01}} \approx 0.0435$ N/C.

2.3.78 $\lim_{t\to 200^-} d(t) = \lim_{t\to 200^-} (3-0.015t)^2 = (3-(0.015)(200))^2 = (3-3)^2 = 0$. As time approaches 200 seconds, the depth of the water in the tank is approaching 0.

2.3.79 $\lim_{S\to 0^+} r(S) = \lim_{S\to 0^+} (1/2)\left(\sqrt{100+\frac{2S}{\pi}} - 10\right) = 0.$

The radius of the circular cylinder approaches zero as the surface area approaches zero.

2.3.80

a. $L(c/2) = L_0\sqrt{1-\frac{(c/2)^2}{c^2}} = L_0\sqrt{1-(1/4)} = \sqrt{3}L_0/2.$

b. $L(3c/4) = L_0\sqrt{1-(1/c^2)(3c/4)^2} = L_0\sqrt{1-(9/16)} = \sqrt{7}L_0/4.$

c. It appears that that the observed length L of the ship decreases as the ship speed increases.

d. $\lim_{x\to c^-} L_0\sqrt{1-(\nu^2/c^2)} = L_0 \cdot 0 = 0$. As the speed of the ship approaches the speed of light, the observed length of the ship shrinks to 0.

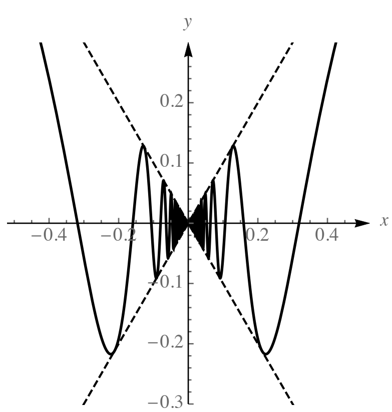
2.3.81

a. The statement we are trying to prove can be stated in cases as follows: For $x > 0$, $-x \le x\sin(1/x) \le x$, and for $x < 0$, $x \le x\sin(1/x) \le -x$.

Now for all $x \ne 0$, note that $-1 \le \sin(1/x) \le 1$ (because the range of the sine function is $[-1,1]$). We will consider the two cases $x > 0$ and $x < 0$ separately, but in each case, we will multiply this inequality through by x, switching the inequalities for the $x < 0$ case.

For $x > 0$ we have $-x \le x\sin(1/x) \le x$, and for $x < 0$ we have $-x \ge x\sin(1/x) \ge x$, which are exactly the statements we are trying to prove.

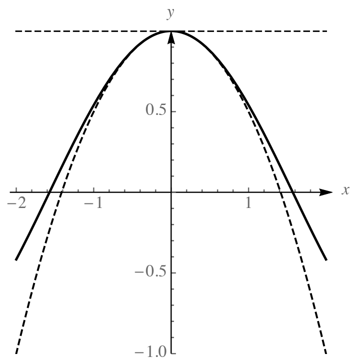
b.



c. Because $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$, and because $-|x| \le x\sin(1/x) \le |x|$, the Squeeze Theorem assures us that $\lim_{x\to 0}[x\sin(1/x)] = 0$ as well.

2.3.82

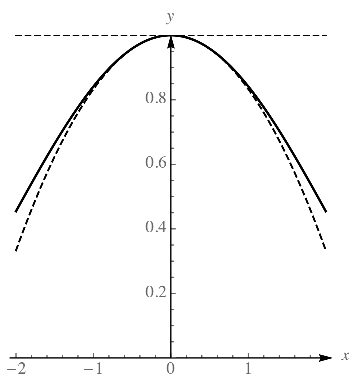
a.



b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{2}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{2} \le \cos x \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\cos x = 1$ as well.

2.3.83

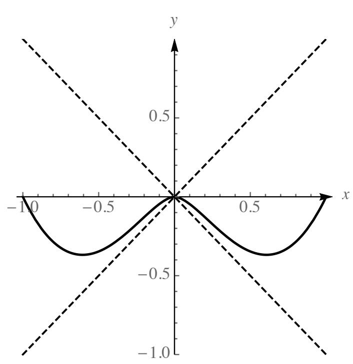
a.



b. Note that $\lim_{x\to 0}\left[1-\frac{x^2}{6}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ as well.

2.3.84

a.



b. Note that $\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|$. So because $-|x| \le x^2\ln x^2 \le |x|$, the squeeze theorem assures us that $\lim_{x\to 0}(x^2\ln x^2) = 0$ as well.

2.3.85 Using the definition of $|x|$ given, we have $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = -0 = 0$. Also, $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$. Because the two one-sided limits are both 0, we also have $\lim_{x\to 0}|x| = 0$.

2.3.86

If $a > 0$, then for x near a, $|x| = x$. So in this case, $\lim_{x\to a} |x| = \lim_{x\to a} x = a = |a|$.

If $a < 0$, then for x near a, $|x| = -x$. So in this case, $\lim_{x\to a} |x| = \lim_{x\to a}(-x) = -a = |a|$, (because $a < 0$).

If $a = 0$, we have already seen in a previous problem that $\lim_{x\to 0} |x| = 0 = |0|$.

Thus in all cases, $\lim_{x\to a} |x| = |a|$.

2.3.87 $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2-5x+6}{x-3} = \lim_{x\to 3} \dfrac{(x-3)(x-2)}{x-3} = \lim_{x\to 3}(x-2) = 1$. So $a = 1$.

2.3.88 In order for $\lim_{x\to 2} f(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x+b) = 6+b$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2) = 0$. So we need $6+b = 0$, so we require that $b = -6$. Then $\lim_{x\to 2} f(x) = 0$.

2.3.89 In order for $\lim_{x\to -1} g(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} (x^2-5x) = 6$, and $\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} (ax^3-7) = -a-7$. So we need $-a-7 = 6$, so we require that $a = -13$. Then $\lim_{x\to -1} f(x) = 6$.

2.3.90 $\lim_{x\to 2} \dfrac{x^5-32}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x^4+2x^3+4x^2+8x+16)}{x-2} = \lim_{x\to 2}(x^4+2x^3+4x^2+8x+16) = 16+16+16+16+16 = 80$.

2.3.91 $\lim_{x\to 1} \dfrac{x^6-1}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(x^5+x^4+x^3+x^2+x+1)}{x-1} = \lim_{x\to 1}(x^5+x^4+x^3+x^2+x+1) = 6$.

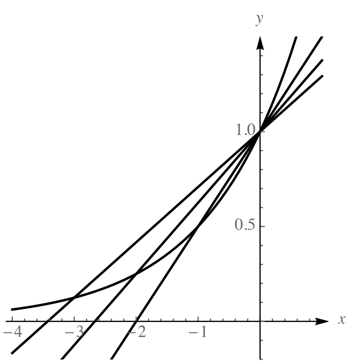
2.3.92 $\lim_{x\to -1} \dfrac{x^7+1}{x+1} = \lim_{x\to -1} \dfrac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1} = \lim_{x\to -1}(x^6-x^5+x^4-x^3+x^2-x+1) = 7$.

2.3.93 $\lim_{x\to a} \dfrac{x^5-a^5}{x-a} = \lim_{x\to a} \dfrac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a} = \lim_{x\to a}(x^4+ax^3+a^2x^2+a^3x+a^4) = 5a^4$.

2.3.94 $\lim_{x\to a} \dfrac{x^n-a^n}{x-a} = \lim_{x\to a} \dfrac{(x-a)(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1})}{x-a} = \lim_{x\to a}(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1}) = na^{n-1}$.

2.3.95

a.



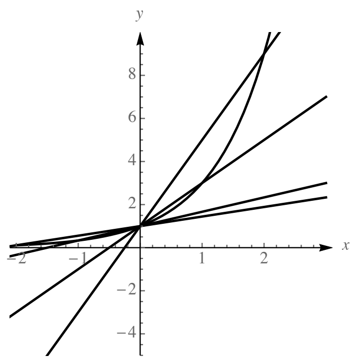
b. The slope of the secant line between $(0,1)$ and $(x, 2^x)$ is $\frac{2^x-1}{x}$.

c.

x	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
$\frac{2^x-1}{x}$	0.5	0.66967	0.69075	0.692907	0.693123	0.693145

It appears that $\lim_{x\to 0^-} \frac{2^x-1}{x} \approx 0.693$.

2.3.96



a.

b. The slope of the secant line between $(0,1)$ and $(x, 3^x)$ is $\frac{3^x-1}{x}$.

c.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$\frac{3^x-1}{x}$	1.04042	1.0926	1.09801	1.09855	1.09867	1.09922	1.10467	1.16123

It appears that $\lim_{x\to 0} \frac{3^x - 1}{x} \approx 1.099$.

2.3.97 $\lim_{x\to -1^-} f(x) = \lim_{x\to 1^+} f(x) = 6$. $\lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = 5$.

2.3.98 $\lim_{x\to -1^-} g(x) = -\lim_{x\to 1^+} g(x) = -6$. $\lim_{x\to -1^+} g(x) = -\lim_{x\to 1^-} g(x) = -5$.

2.3.99 $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1} = \frac{1}{3}$.

2.3.100 $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16} = \lim_{x\to 16} \frac{\sqrt[4]{x}-2}{(\sqrt[4]{x}-2)(\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8)} = \lim_{x\to 16} \frac{1}{\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8} = \frac{1}{32}$.

2.3.101 Let $f(x) = x-1$ and $g(x) = \frac{5}{x-1}$. Then $\lim_{x\to 1} f(x) = 0$, $\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} \frac{5(x-1)}{x-1} = \lim_{x\to 1} 5 = 5$.

2.3.102 Let $f(x) = x^2 - 1$. Then $\lim_{x\to 1} \frac{f(x)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$.

2.3.103 Let $p(x) = x^2 + 2x - 8$. Then $\lim_{x\to 2} \frac{p(x)}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+4)}{x-2} = \lim_{x\to 2}(x+4) = 6$.

The constants are unique. We know that 2 must be a root of p (otherwise the given limit couldn't exist), so it must have the form $p(x) = (x-2)q(x)$, and q must be a degree 1 polynomial with leading coefficient 1 (otherwise p wouldn't have leading coefficient 1.) So we have $p(x) = (x-2)(x+d)$, but because $\lim_{x\to 2} \frac{p(x)}{x-2} = \lim_{x\to 2}(x+d) = 2+d = 6$, we are forced to realize that $d = 4$. Therefore, we have deduced that the only possibility for p is $p(x) = (x-2)(x+4) = x^2 + 2x - 8$.

2.3.104 Because $\lim_{x\to 1} f(x) = 4$, we know that f is near 4 when x is near 1 (but not equal to 1). It follows that $\lim_{x\to -1} f(x^2) = 4$ as well, because when x is near but not equal to -1, x^2 is near 1 but not equal to 1. Thus $f(x^2)$ is near 4 when x is near -1.

2.3.105 As $x \to 0^+$, $(1-x) \to 1^-$. So $\lim_{x\to 0^+} g(x) = \lim_{(1-x)\to 1^-} f(1-x) = \lim_{z\to 1^-} f(z) = 6$. (Where $z = 1-x$.)

As $x \to 0^-$, $(1-x) \to 1^+$. So $\lim_{x\to 0^-} g(x) = \lim_{(1-x)\to 1^+} f(1-x) = \lim_{z\to 1^+} f(z) = 4$. (Where $z = 1-x$.)

2.3.106

a. Suppose $0 < \theta < \pi/2$. Note that $\sin\theta > 0$, so $|\sin\theta| = \sin\theta$. Also, $\sin\theta = \frac{|AC|}{1}$, so $|AC| = |\sin\theta|$.

Now suppose that $-\pi/2 < \theta < 0$. Then $\sin\theta$ is negative, so $|\sin\theta| = -\sin\theta$. We have $\sin\theta = \frac{-|AC|}{1}$, so $|AC| = -\sin\theta = |\sin\theta|$.

b. Suppose $0 < \theta < \pi/2$. Because AB is the hypotenuse of triangle ABC, we know that $|AB| > |AC|$. We have $|\sin\theta| = |AC| < |AB| <$ the length of arc $AB = \theta = |\theta|$.

If $-\pi/2 < \theta < 0$, we can make a similar argument. We have

$$|\sin\theta| = |AC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

c. If $0 < \theta < \pi/2$, we have $\sin\theta = |\sin\theta| < |\theta|$, and because $\sin\theta$ is positive, we have $-|\theta| \le 0 < \sin\theta$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

If $-\pi/2 < \theta < 0$, then $|\sin\theta| = -\sin\theta$. From the previous part, we have $|\sin\theta| = -\sin\theta < |\theta|$. Therefore, $-|\theta| < \sin\theta$. Now because $\sin\theta$ is negative on this interval, we have $\sin\theta < 0 \le |\theta|$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

d. If $0 < \theta < \pi/2$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = \theta = |\theta|.$$

For $-\pi/2 < \theta < 0$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

e. Using the result of part d, we multiply through by -1 to obtain $-|\theta| \le \cos\theta - 1 \le 0$, and then add 1 to all parts, obtaining $1 - |\theta| \le \cos\theta \le 1$, as desired.

2.3.107

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\
&= \lim_{x\to a}(a_n x^n) + \lim_{x\to a}(a_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(a_1 x) + \lim_{x\to a} a_0 \\
&= a_n \lim_{x\to a} x^n + a_{n-1}\lim_{x\to a} x^{n-1} + \cdots + a_1 \lim_{x\to a} x + a_0 \\
&= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1(\lim_{x\to a} x) + a_0 \\
&= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0 = p(a).
\end{aligned}$$

2.4 Infinite Limits

2.4.1 As x approaches a from the right, the values of $f(x)$ are negative and become arbitrarily large in magnitude.

2.4.2 As x approaches a (from either side), the values of $f(x)$ are positive and become arbitrarily large in magnitude.

2.4.3 A vertical asymptote for a function f is a vertical line $x = a$ so that one or more of the following are true: $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$.

2.4.4 No. For example, if $f(x) = x^2 - 4$ and $g(x) = x - 2$ and $a = 2$, we would have $\lim_{x\to 2} \frac{f(x)}{g(x)} = 4$, even though $g(2) = 0$.

2.4.5

x	$\frac{x+1}{(x-1)^2}$	x	$\frac{x+1}{(x-1)^2}$
1.1	210	.9	190
1.01	20,100	.99	19,900
1.001	2,001,000	.999	1,999,000
1.0001	200,010,000	.9999	199,990,000

From the data given, it appears that $\lim_{x\to 1} f(x) = \infty$.

2.4.6 $\lim_{x\to 3} f(x) = \infty$, and $\lim_{x\to -1} f(x) = -\infty$.

2.4.7

a. $\lim_{x\to 1^-} f(x) = \infty$.
b. $\lim_{x\to 1^+} f(x) = \infty$.
c. $\lim_{x\to 1} f(x) = \infty$.
d. $\lim_{x\to 2^-} f(x) = \infty$.
e. $\lim_{x\to 2^+} f(x) = -\infty$.
f. $\lim_{x\to 2} f(x)$ does not exist.

2.4.8

a. $\lim_{x\to 2^-} g(x) = \infty$.
b. $\lim_{x\to 2^+} g(x) = -\infty$.
c. $\lim_{x\to 2} g(x)$ does not exist.
d. $\lim_{x\to 4^-} g(x) = -\infty$.
e. $\lim_{x\to 4^+} g(x) = -\infty$.
f. $\lim_{x\to 4} g(x) = -\infty$.

2.4.9

a. $\lim_{x\to -2^-} h(x) = -\infty$.
b. $\lim_{x\to -2^+} h(x) = -\infty$.
c. $\lim_{x\to -2} h(x) = -\infty$.
d. $\lim_{x\to 3^-} h(x) = \infty$.
e. $\lim_{x\to 3^+} h(x) = -\infty$.
f. $\lim_{x\to 3} h(x)$ does not exist.

2.4.10

a. $\lim_{x\to -2^-} p(x) = -\infty$.
b. $\lim_{x\to -2^+} p(x) = -\infty$.
c. $\lim_{x\to -2} p(x) = -\infty$.
d. $\lim_{x\to 3^-} p(x) = -\infty$.
e. $\lim_{x\to 3^+} p(x) = -\infty$.
f. $\lim_{x\to 3} p(x) = -\infty$.

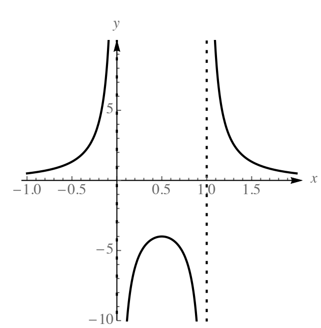
2.4.11

a. $\lim_{x\to 0^-} \frac{1}{x^2 - x} = \infty$.

b. $\lim_{x\to 0^+} \frac{1}{x^2 - x} = -\infty$.

c. $\lim_{x\to 1^-} \frac{1}{x^2 - x} = -\infty$.

d. $\lim_{x\to 1^+} \frac{1}{x^2 - x} = \infty$.



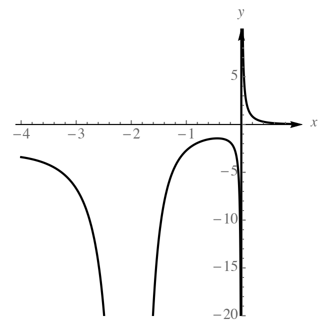
2.4.12

a. $\displaystyle\lim_{x\to -2^+} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

b. $\displaystyle\lim_{x\to -2} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

c. $\displaystyle\lim_{x\to 0^-} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

d. $\displaystyle\lim_{x\to 0^+} \frac{e^{-x}}{x(x+2)^2} = \infty.$



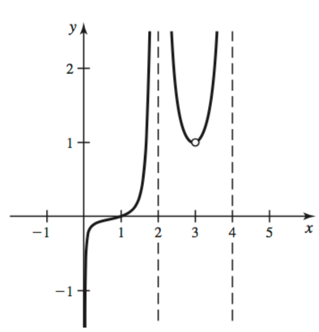
2.4.13 Because the numerator is approaching a non-zero constant while the denominator is approaching zero, the quotient of these numbers is getting big – at least the absolute value of the quotient is getting big. The quotient is actually always negative, because a number near 100 divided by a negative number is always negative. Thus $\displaystyle\lim_{x\to 2} \frac{f(x)}{g(x)} = -\infty$.

2.4.14 Using the same sort of reasoning as in the last problem – as $x \to 3$ the numerator is fixed at 1, but the denominator is getting small, so the quotient is getting big. It remains to investigate the sign of the quotient. As $x \to 3^-$, the quantity $x-3$ is negative, so the quotient of the positive number 1 and this small negative number is negative. On the other hand, as $x \to 3^+$, the quantity $x-3$ is positive, so the quotient of 1 and this number is positive. Thus: $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$, and $\displaystyle\lim_{x\to 3^+} \frac{1}{x-3} = \infty$.

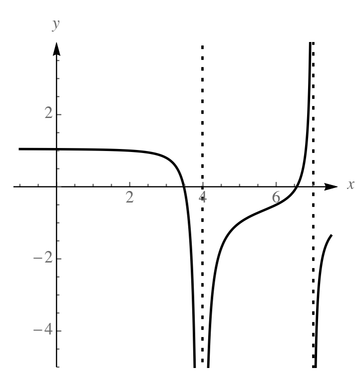
2.4.15 Note that $\displaystyle\lim_{x\to 1} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 1} \frac{x-3}{x-2} = \frac{-2}{-1} = 2$. So there is *not* a vertical asymptote at $x=1$. On the other hand, $\displaystyle\lim_{x\to 2^+} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 2^+} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 2^+} \frac{x-3}{x-2} = -\infty$, so there is a vertical asymptote at $x=2$.

2.4.16 Note the at $x \to 0$ the numerator has limit 1 while the denominator has limit 0, so the quotient is growing without bound. Note also that the denominator is always positive, because for all x, $\cos x \le 1$ so $1 - \cos x \ge 0$.

2.4.17



2.4.18



2.4.19 Both a and b are true statements.

2.4.20 Both a and c are true statements.

2.4.21

a. $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$.

b. $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$.

c. $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.

2.4.22

a. $\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \infty$.

b. $\lim_{x\to 3^-} \frac{2}{(x-3)^3} = -\infty$.

c. $\lim_{x\to 3} \frac{2}{(x-3)^3}$ does not exist.

2.4.23

a. $\lim_{x\to 4^+} \frac{x-5}{(x-4)^2} = -\infty$.

b. $\lim_{x\to 4^-} \frac{x-5}{(x-4)^2} = -\infty$.

c. $\lim_{x\to 4} \frac{x-5}{(x-4)^2} = -\infty$.

2.4.24

a. $\lim_{x\to 1^+} \frac{x}{|x-1|} = \infty$.

b. $\lim_{x\to 1^-} \frac{x}{|x-1|} = \infty$.

c. $\lim_{x\to 1} \frac{x}{|x-1|} = \infty$.

2.4.25

a. $\lim_{x\to 3^+} \frac{(x-1)(x-2)}{(x-3)} = \infty$.

b. $\lim_{x\to 3^-} \frac{(x-1)(x-2)}{(x-3)} = -\infty$.

c. $\lim_{x\to 3} \frac{(x-1)(x-2)}{(x-3)}$ does not exist.

2.4.26

a. $\lim_{x\to -2^+} \frac{(x-4)}{x(x+2)} = \infty$.

b. $\lim_{x\to -2^-} \frac{(x-4)}{x(x+2)} = -\infty$.

c. $\lim_{x\to -2} \frac{(x-4)}{x(x+2)}$ does not exist.

2.4.27

a. $\lim_{x\to 2^+} \frac{x^2-4x+3}{(x-2)^2} = -\infty.$

b. $\lim_{x\to 2^-} \frac{x^2-4x+3}{(x-2)^2} = -\infty.$

c. $\lim_{x\to 2} \frac{x^2-4x+3}{(x-2)^2} = -\infty.$

2.4.28

a. $\lim_{x\to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^+} \frac{x-3}{x(x+2)} = \infty.$

b. $\lim_{x\to -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^-} \frac{x-3}{x(x+2)} = -\infty.$

c. Because the two one-sided limits differ, $\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$ does not exist.

d. $\lim_{x\to 2} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to 2} \frac{x-3}{x(x+2)} = \frac{-1}{8}.$

2.4.29

a. $\lim_{x\to 2^+} \frac{1}{\sqrt{x(x-2)}} = \infty.$

b. $\lim_{x\to 2^-} \frac{1}{\sqrt{x(x-2)}}$ does not exist. Note that the domain of the function is $(-\infty,0)\cup(2,\infty)$.

c. $\lim_{x\to 2} \frac{1}{\sqrt{x(x-2)}}$ does not exist.

2.4.30

a. $\lim_{x\to 1^+} \frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist. Note that $x^2-5x+4=(x-4)(x-1)$ so the domain of the function is $(-\infty,1)\cup(4,\infty)$.

b.] $\lim_{x\to 1^-} \frac{x-3}{\sqrt{x^2-5x+4}} = -\infty.$

c. $\lim_{x\to 1} \frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist.

2.4.31

a. $\lim_{x\to 0} \frac{x-3}{x^4-9x^2} = \lim_{x\to 0} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 0} \frac{1}{x^2(x+3)} = \infty.$

b. $\lim_{x\to 3} \frac{x-3}{x^4-9x^2} = \lim_{x\to 3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 3} \frac{1}{x^2(x+3)} = \frac{1}{54}.$

c. $\lim_{x\to -3} \frac{x-3}{x^4-9x^2} = \lim_{x\to -3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to -3} \frac{1}{x^2(x+3)}$, which does not exist.

2.4.32

a. $\lim_{x\to 0} \frac{x-2}{x^5-4x^3} = \lim_{x\to 0} \frac{x-2}{x^3(x-2)(x+2)} = \lim_{x\to 0} \frac{1}{x^3(x+2)}$, which does not exist.

b. $\displaystyle\lim_{x\to 2}\frac{x-2}{x^5-4x^3}=\lim_{x\to 2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x^3(x+2)}=\frac{1}{32}.$

c. $\displaystyle\lim_{x\to -2}\frac{x-2}{x^5-4x^3}=\lim_{x\to -2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to -2}\frac{1}{x^3(x+2)}$, which does not exist.

2.4.33 $\displaystyle\lim_{x\to 0}\frac{x^3-5x^2}{x^2}=\lim_{x\to 0}\frac{x^2(x-5)}{x^2}=\lim_{x\to 0}(x-5)=-5.$

2.4.34 $\displaystyle\lim_{t\to 5}\frac{4t^2-100}{t-5}=\lim_{t\to 5}\frac{4(t-5)(t+5)}{t-5}=\lim_{t\to 5}[4(t+5)]=40.$

2.4.35 $\displaystyle\lim_{x\to 1^+}\frac{x^2-5x+6}{x-1}=\lim_{x\to 1^+}\frac{(x-2)(x-3)}{x-1}=\infty.$ (Note that as $x\to 1^+$, the numerator is near 2, while the denominator is near zero, but is positive. So the quotient is positive and large.)

2.4.36 $\displaystyle\lim_{z\to 4}\frac{z-5}{(z^2-10z+24)^2}=\lim_{z\to 4}\frac{z-5}{(z-4)^2(z-6)^2}=-\infty.$ (Note that as $z\to 4$, the numerator is near -1 while the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.37 $\displaystyle\lim_{x\to 6^+}\frac{x-7}{\sqrt{x-6}}=-\infty.$ (Note that as $x\to 6^+$ the numerator is near -1 and the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.38 $\displaystyle\lim_{x\to 2^-}\frac{x-1}{\sqrt{(x-3)(x-2)}}=\infty.$ Note that as $x\to 2^-$ the numerator is near 1 and the denominator is near zero but is positive. So the quotient is positive with large absolute value.)

2.4.39 $\displaystyle\lim_{\theta\to 0^+}\csc\theta=\lim_{\theta\to 0^+}\frac{1}{\sin\theta}=\infty.$

2.4.40 $\displaystyle\lim_{x\to 0^-}\csc x=\lim_{x\to 0^-}\frac{1}{\sin x}=-\infty.$

2.4.41 $\displaystyle\lim_{x\to 0^+}-10\cot x=\lim_{x\to 0^+}\frac{-10\cos x}{\sin x}=-\infty.$ (Note that as $x\to 0^+$, the numerator is near -10 and the denominator is near zero, but is positive. Thus the quotient is a negative number whose absolute value is large.)

2.4.42 $\displaystyle\lim_{\theta\to(\pi/2)^+}\frac{1}{3}\tan\theta=\lim_{\theta\to(\pi/2)^+}\frac{\sin\theta}{3\cos\theta}=-\infty.$ (Note that as $\theta\to(\pi/2)^+$, the numerator is near 1 and the denominator is near 0, but is negative. Thus the quotient is a negative number whose absolute value is large.)

2.4.43 $\displaystyle\lim_{\theta\to 0}\frac{2+\sin\theta}{1-\cos^2\theta}=\infty.$ (Note that as $\theta\to 0$, the numerator is near 2 and the denominator is near 0, but is positive. Thus the quotient is a positive number whose absolute value is large.)

2.4.44 $\displaystyle\lim_{\theta\to 0^-}\frac{\sin\theta}{\cos^2\theta-1}=\lim_{\theta\to 0^-}\frac{\sin\theta}{-\sin^2\theta}=-\lim_{\theta\to 0^-}\frac{1}{\sin\theta}=\infty.$

2.4.45

a. $\displaystyle\lim_{x\to 5}\frac{x-5}{x^2-25}=\lim_{x\to 5}\frac{1}{x+5}=\frac{1}{10}$, so there isn't a vertical asymptote at $x=5$.

b. $\displaystyle\lim_{x\to -5^-}\frac{x-5}{x^2-25}=\lim_{x\to -5^-}\frac{1}{x+5}=-\infty$, so there is a vertical asymptote at $x=-5$.

c. $\displaystyle\lim_{x\to -5^+}\frac{x-5}{x^2-25}=\lim_{x\to -5^+}\frac{1}{x+5}=\infty.$ This also implies that $x=-5$ is a vertical asymptote, as we already noted in part b.

2.4.46

a. $\lim_{x\to 7^-} \frac{x+7}{x^4-49x^2} = \lim_{x\to 7^-} \frac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^-} \frac{1}{x^2(x-7)} = -\infty$, so there is a vertical asymptote at $x = 7$.

b. $\lim_{x\to 7^+} \frac{x+7}{x^4-49x^2} = \lim_{x\to 7^+} \frac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 7^+} \frac{1}{x^2(x-7)} = \infty$. This also implies that there is a vertical asymptote at $x = 7$, as we already noted in part a.

c. $\lim_{x\to -7} \frac{x+7}{x^4-49x^2} = \lim_{x\to -7} \frac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to -7} \frac{1}{x^2(x-7)} = \frac{1}{-686}$. So there is not a vertical asymptote at $x = 7$.

d. $\lim_{x\to 0} \frac{x+7}{x^4-49x^2} = \lim_{x\to 0} \frac{x+7}{x^2(x+7)(x-7)} = \lim_{x\to 0} \frac{1}{x^2(x-7)} = -\infty$. So there is a vertical asymptote at $x = 0$.

2.4.47 $f(x) = \frac{x^2-9x+14}{x^2-5x+6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$. Note that $x = 3$ is a vertical asymptote, while $x = 2$ appears to be a candidate but isn't one. We have $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x-7}{x-3} = -\infty$ and $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x-7}{x-3} = \infty$, and thus $\lim_{x\to 3} f(x)$ doesn't exist. Note that $\lim_{x\to 2} f(x) = 5$.

2.4.48 $f(x) = \frac{\cos x}{x(x+2)}$ has vertical asymptotes at $x = 0$ and at $x = -2$. Note that $\cos x$ is near 1 when x is near 0 , and $\cos x$ is near -0.4 when x is near -2. Thus, $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = \infty$, and $\lim_{x\to -2^-} f(x) = -\infty$.

2.4.49 $f(x) = \frac{x+1}{x^3-4x^2+4x} = \frac{x+1}{x(x-2)^2}$. There are vertical asymptotes at $x = 0$ and $x = 2$. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x+1}{x(x-2)^2} = -\infty$, while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 0} f(x)$ doesn't exist.

Also we have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x+1}{x(x-2)^2} = \infty$, while $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x+1}{x(x-2)^2} = \infty$, and thus $\lim_{x\to 2} f(x) = \infty$ as well.

2.4.50 $g(x) = \frac{x^3-10x^2+16x}{x^2-8x} = \frac{x(x-2)(x-8)}{x(x-8)}$. This function has no vertical asymptotes.

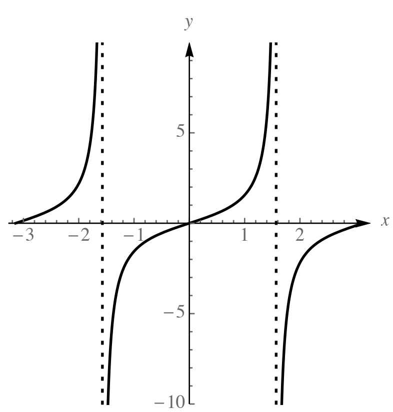
2.4.51

a. $\lim_{x\to(\pi/2)^+} \tan x = -\infty$.

b. $\lim_{x\to(\pi/2)^-} \tan x = \infty$.

c. $\lim_{x\to(-\pi/2)^+} \tan x = -\infty$.

d. $\lim_{x\to(-\pi/2)^-} \tan x = \infty$.



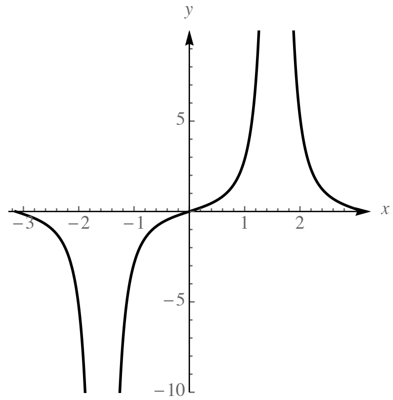
2.4.52

a. $\lim_{x\to(\pi/2)^+} \sec x \tan x = \infty$.

b. $\lim_{x\to(\pi/2)^-} \sec x \tan x = \infty$.

c. $\lim_{x\to(-\pi/2)^+} \sec x \tan x = -\infty$.

d. $\lim_{x\to(-\pi/2)^-} \sec x \tan x = -\infty$.



2.4.53

a. False. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = -\dfrac{5}{2}$.

b. True. For example, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \dfrac{(x-1)(x-6)}{(x-1)(x+1)} = -\infty$.

c. False. For example $g(x) = \frac{1}{x-1}$ has $\lim_{x\to 1^+} g(x) = \infty$, but $\lim_{x\to 1^-} g(x) = -\infty$.

2.4.54

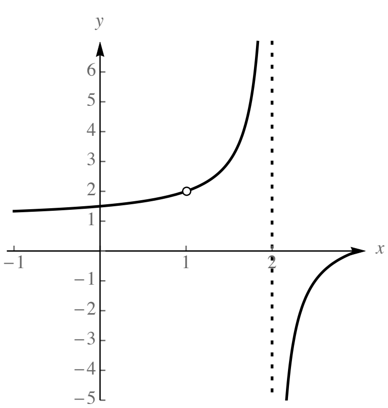
function	a	b	c	d	e	f
graph	D	C	F	B	A	E

2.4.55 We are seeking a function with a factor of $x-1$ in the denominator, but there should be more factors of $x-1$ in the numerator, and there should be a factor of $(x-2)^2$ in the denominator. This will accomplish the desired results. So

$$r(x) = \frac{(x-1)^2}{(x-1)(x-2)^2}.$$

2.4.56

One such function is $f(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-1)(x-3)}{(x-1)(x-2)}$.



2.4.57 One example is $f(x) = \frac{1}{x-6}$.

2.4.58 $f(x) = \dfrac{x^2-1}{(x^2-1)(x^2+1)} = \dfrac{1}{x^2+1}$ (for $x \neq \pm 1$). There are no vertical asymptotes, because for all a, $\lim_{x\to a} f(x) = \dfrac{1}{a^2+1}$.

2.4.59 $f(x) = \dfrac{x^2-3x+2}{x^{10}-x^9} = \dfrac{(x-2)(x-1)}{x^9(x-1)}$. f has a vertical asymptote at $x=0$, because $\lim_{x\to 0^+} f(x) = -\infty$ (and $\lim_{x\to 0^-} f(x) = \infty$.) Note that $\lim_{x\to 1} f(x) = -1$, so there isn't a vertical asymptote at $x=1$.

2.4.60 $g(x) = 2 - \ln x^2$ has a vertical asymptote at $x=0$, because $\lim_{x\to 0}(2-\ln x^2) = \infty$.

2.4.61 $h(x) = \dfrac{e^x}{(x+1)^3}$ has a vertical asymptote at $x=-1$, because

$$\lim_{x\to -1^+} \frac{e^x}{(x+1)^3} = \infty \quad \text{and} \quad \lim_{x\to -1^-} h(x) = -\infty.$$

2.4.62 $p(x) = \sec(\pi x/2) = \dfrac{1}{\cos(\pi x/2)}$ has a vertical asymptote on $(-2,2)$ at $x = \pm 1$.

2.4.63 $g(\theta) = \tan(\pi\theta/10) = \dfrac{\sin(\pi\theta/10)}{\cos(\pi\theta/10)}$ has a vertical asymptote at each $\theta = 10n+5$ where n is an integer. This is due to the fact that $\cos(\pi\theta/10) = 0$ when $\pi\theta/10 = \pi/2 + n\pi$ where n is an integer, which is the same as $\{\theta\colon \theta = 10n+5, n \text{ an integer}\}$. Note that at all of these numbers which make the denominator zero, the numerator isn't zero.

2.4.64 $q(s) = \dfrac{\pi}{s - \sin s}$ has a vertical asymptote at $s=0$. Note that this is the only number where $\sin s = s$.

2.4.65 $f(x) = \frac{1}{\sqrt{x}\sec x} = \frac{\cos x}{\sqrt{x}}$ has a vertical asymptote at $x=0$.

2.4.66 $g(x) = e^{1/x}$ has a vertical asymptote at $x=0$, because $\lim_{x\to 0^+} e^{1/x} = \infty$. (Note that as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty$ as well.)

2.4.67

a. Note that the numerator of the given expression factors as $(x-3)(x-4)$. So if $a=3$ or if $a=4$ the limit would be a finite number. In fact, $\lim_{x\to 3} \dfrac{(x-3)(x-4)}{x-3} = -1$ and $\lim_{x\to 4} \dfrac{(x-3)(x-4)}{x-4} = 1$.

b. For any number other than 3 or 4, the limit would be either $\pm\infty$. Because $x-a$ is always positive as $x \to a^+$, the limit would be $+\infty$ exactly when the numerator is positive, which is for a in the set $(-\infty, 3) \cup (4, \infty)$.

c. The limit would be $-\infty$ for a in the set $(3,4)$.

2.4.68

a. The slope of the secant line is given by $\dfrac{f(h)-f(0)}{h} = \dfrac{h^{1/3}}{h} = h^{-2/3}$.

b. $\lim_{h\to 0} \dfrac{1}{\sqrt[3]{h^2}} = \infty$. This tells us that the slope of the tangent line is infinite – which means that the tangent line at $(0,0)$ is vertical.

2.4.69

a. The slope of the secant line is $\dfrac{f(h)-f(0)}{h} = \dfrac{h^{2/3}}{h} = h^{-1/3}$.

b. $\lim_{h\to 0^+} \dfrac{1}{h^{1/3}} = \infty$, and $\lim_{h\to 0^-} \dfrac{1}{h^{1/3}} = -\infty$. The tangent line is infinitely steep at the origin (i.e., it is a vertical line.)

2.5 Limits at Infinity

2.5.1 As $x < 0$ becomes arbitrarily large in absolute value, the corresponding values of f approach 10.

2.5.2 $\lim_{x\to\infty} f(x) = -2$ and $\lim_{x\to-\infty} f(x) = 4$.

2.5.3 $\lim_{x\to\infty} x^{12} = \infty$. Note that x^{12} is positive when $x > 0$.

2.5.4 $\lim_{x\to-\infty} 3x^{11} = -\infty$. Note that x^{11} is negative when $x < 0$.

2.5.5 $\lim_{x\to\infty} x^{-6} = \lim_{x\to\infty} \dfrac{1}{x^6} = 0$.

2.5.6 $\lim_{x\to-\infty} x^{-11} = \lim_{x\to-\infty} \dfrac{1}{x^{11}} = 0$.

2.5.7 $\lim_{t\to\infty} (-12t^{-5}) = \lim_{t\to\infty} -\dfrac{12}{t^5} = 0$.

2.5.8 $\lim_{x\to-\infty} 2x^{-8} = \lim_{x\to-\infty} \dfrac{2}{x^8} = 0$.

2.5.9 $\lim_{x\to\infty} (3 + 10/x^2) = 3 + \lim_{x\to\infty} (10/x^2) = 3 + 0 = 3$.

2.5.10 $\lim_{x\to\infty} (5 + 1/x + 10/x^2) = 5 + \lim_{x\to\infty} (1/x) + \lim_{x\to\infty} (10/x^2) = 5 + 0 + 0 = 5$.

2.5.11 If $f(x) \to 100{,}000$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, then the ratio $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. (Because *eventually* the values of f are small compared to the values of g.)

2.5.12 $\lim_{x\to\infty} \dfrac{3 + 2x + 4x^2}{x^2} = \lim_{x\to\infty} \dfrac{3}{x^2} + \lim_{x\to\infty} \dfrac{2x}{x^2} + \lim_{x\to\infty} \dfrac{4x^2}{x^2} = 0 + \lim_{x\to\infty} \dfrac{2}{x} + \lim_{x\to\infty} 4 = 0 + 0 + 4 = 4$.

2.5.13 $\lim_{t\to\infty} e^t = \infty$, $\lim_{t\to-\infty} e^t = 0$, and $\lim_{t\to\infty} e^{-t} = 0$.

2.5.14 As $x \to \infty$, we note that $e^{-2x} \to 0$, while as $x \to -\infty$, we have $e^{-2x} \to \infty$.

2.5.15 Because $\lim_{x\to\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim_{x\to\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to\infty} g(x) = 3$.

Similarly, because $\lim_{x\to-\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim_{x\to-\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to-\infty} g(x) = 3$.

2.5.16 $\lim_{x\to-\infty} g(x) = 3$, $\lim_{x\to\infty} g(x) = -1$, $\lim_{x\to-2^-} g(x) = \infty$, $\lim_{x\to 2^+} g(x) = -\infty$.

2.5.17 $\lim_{\theta\to\infty} \dfrac{\cos\theta}{\theta^2} = 0$. Note that $-1 \le \cos\theta \le 1$, so $-\dfrac{1}{\theta^2} \le \dfrac{\cos\theta}{\theta^2} \le \dfrac{1}{\theta^2}$. The result now follows from the Squeeze Theorem.

2.5.18 Note that $\dfrac{5t^2 + t\sin t}{t^2}$ can be written as $5 + \dfrac{\sin t}{t}$. Also, note that because $-1 \le \sin t \le 1$, we have $-\dfrac{1}{t} \le \dfrac{\sin t}{t} \le \dfrac{1}{t}$, so $\dfrac{\sin t}{t} \to 0$ as $t \to \infty$ by the Squeeze Theorem. Therefore,

$$\lim_{t\to\infty} \frac{5t^2 + t\sin t}{t^2} = \lim_{t\to\infty} \left(5 + \frac{\sin t}{t}\right) = 5 + 0 = 5.$$

2.5.19 $\lim_{x\to\infty} \frac{\cos x^5}{\sqrt{x}} = 0$. Note that $-1 \le \cos x^5 \le 1$, so $\frac{-1}{\sqrt{x}} \le \frac{\cos x^5}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$. Because $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = \lim_{x\to\infty} \frac{-1}{\sqrt{x}} = 0$, we have $\lim_{x\to\infty} \frac{\cos x^5}{\sqrt{x}} = 0$ by the Squeeze Theorem.

2.5.20 $\lim_{x\to-\infty} \left(5 + \frac{100}{x} + \frac{\sin^4(x^3)}{x^2}\right) = 5 + 0 + 0 = 5$. For this last limit, note that $0 \le \sin^4(x^3) \le 1$, so $0 \le \frac{\sin^4(x^3)}{x^2} \le \frac{1}{x^2}$. The result now follows from the Squeeze Theorem.

2.5.21 $\lim_{x\to\infty} (3x^{12} - 9x^7) = \infty$.

2.5.22 $\lim_{x\to-\infty} (3x^7 + x^2) = -\infty$.

2.5.23 $\lim_{x\to-\infty} (-3x^{16} + 2) = -\infty$.

2.5.24 $\lim_{x\to-\infty} (2x^{-8} + 4x^3) = 0 + \lim_{x\to-\infty} 4x^3 = -\infty$.

2.5.25 $\lim_{x\to\infty} \frac{(14x^3 + 3x^2 - 2x)}{(21x^3 + x^2 + 2x + 1)} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{14 + (3/x) - (2/x^2)}{21 + (1/x) + (2/x^2) + (1/x^3)} = \frac{14}{21} = \frac{2}{3}$.

2.5.26 $\lim_{x\to\infty} \frac{(9x^3 + x^2 - 5)}{(3x^4 + 4x^2)} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{(9/x) + (1/x^2) - (5/x^4)}{3 + (4/x^2)} = \frac{0}{3} = 0$.

2.5.27 $\lim_{x\to-\infty} \frac{(3x^2 + 3x)}{(x+1)} \cdot \frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{3x + 3}{1 + (1/x)} = -\infty$.

2.5.28 $\lim_{x\to\infty} \frac{(x^4 + 7)}{(x^5 + x^2 - x)} \cdot \frac{1/x^5}{1/x^5} = \lim_{x\to\infty} \frac{(1/x) + (7/x^5)}{1 + (1/x^3) - (1/x^4)} = \frac{0 + 0}{1 + 0 - 0} = 0$.

2.5.29 Note that for $w > 0$, $w^2 = \sqrt{w^4}$. We have

$$\lim_{w\to\infty} \frac{(15w^2 + 3w + 1)}{\sqrt{9w^4 + w^3}} \cdot \frac{1/w^2}{1/\sqrt{w^4}} = \lim_{w\to\infty} \frac{15 + (3/w) + (1/w^2)}{\sqrt{9 + (1/w)}} = \frac{15}{\sqrt{9}} = 5.$$

2.5.30 Note that $\sqrt{x^8} = x^4$ (even for $x < 0$). We have

$$\lim_{x\to-\infty} \frac{(40x^4 + x^2 + 5x)}{\sqrt{64x^8 + x^6}} \cdot \frac{1/x^4}{1/\sqrt{x^8}} = \lim_{x\to-\infty} \frac{40 + (1/x^2) + (5/x^3)}{\sqrt{64 + (1/x^2)}} = \frac{40}{\sqrt{64}} = \frac{40}{8} = 5.$$

2.5.31 Note that for $x < 0$, $\sqrt{x^2} = -x$. We have

$$\lim_{x\to-\infty} \frac{\sqrt{16x^2 + x}}{x} \cdot \frac{\sqrt{1/x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{16 + (1/x)}}{-1} = -\sqrt{16} = -4.$$

2.5.32 Note that $x^2 = \sqrt{x^4}$ for all x. We have

$$\lim_{x\to\infty} \frac{6x^2}{(4x^2 + \sqrt{16x^4 + x^2})} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{6}{(4 + \sqrt{16 + (1/x^2)})} = \frac{6}{4 + \sqrt{16}} = \frac{3}{4}.$$

2.5.33 $\lim_{x\to\infty} \frac{(x^2 - \sqrt{x^4 + 3x^2})}{1} \cdot \frac{(x^2 + \sqrt{x^4 + 3x^2})}{(x^2 + \sqrt{x^4 + 3x^2})} = \lim_{x\to\infty} \frac{x^4 - (x^4 + 3x^2)}{x^2 + \sqrt{x^4 + 3x^2}} = \lim_{x\to\infty} \frac{-3x^2}{x^2 + \sqrt{x^4 + 3x^2}}$. Now divide the numerator and denominator by x^2 to give

$$\lim_{x\to\infty} \frac{-3x^2}{x^2 + \sqrt{x^4 + 3x^2}} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{-3}{1 + \sqrt{1 + (3/x^2)}} = -\frac{3}{2}.$$

2.5.34 $\lim_{x\to-\infty} \frac{(x+\sqrt{x^2-5x})}{1}\cdot\frac{(x-\sqrt{x^2-5x})}{(x-\sqrt{x^2-5x})} = \lim_{x\to-\infty} \frac{x^2-(x^2-5x)}{x-\sqrt{x^2-5x}} = \lim_{x\to-\infty} \frac{5x}{x-\sqrt{x^2-5x}}$. Now divide the numerator and denominator by x (and recall that for $x<0$ we have $-\sqrt{x^2}=x$) giving

$$\lim_{x\to-\infty} \frac{5x}{(x-\sqrt{x^2-5x})}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{5}{1+\sqrt{1-(5/x)}} = \frac{5}{2}.$$

2.5.35 Note that because $-1\le \sin x\le 1$, we have $-\frac{1}{e^x}\le\frac{\sin x}{e^x}\le\frac{1}{e^x}$. Then because $\lim_{x\to\infty}\frac{\pm 1}{e^x}=0$, the Squeeze Theorem tells us that $\lim_{x\to\infty}\frac{\sin x}{e^x}=0$.

2.5.36 Note that because $-1\le\cos x\le 1$, we have $-e^x\le e^x\cos x\le e^x$. Then $3-e^x\le e^x\cos x+3\le e^x+3$. Because $\lim_{x\to-\infty} 3-e^x=3$ and $\lim_{x\to-\infty} e^x+3=3$, the Squeeze Theorem tells us that $\lim_{x\to-\infty} e^x\cos x+3=3$.

2.5.37 $\lim_{x\to\infty}\frac{4x}{20x+1} = \lim_{x\to\infty}\frac{4x}{20x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{4}{20+1/x} = \frac{4}{20} = \frac{1}{5}$. Thus, the line $y=\frac{1}{5}$ is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{4x}{20x+1} = \lim_{x\to-\infty}\frac{4x}{20x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty}\frac{4}{20+1/x} = \frac{4}{20} = \frac{1}{5}$. This shows that the curve is also asymptotic to the asymptote in the negative direction.

2.5.38 $\lim_{x\to\infty}\frac{3x^2-7}{x^2+5x} = \lim_{x\to\infty}\frac{3x^2-7}{x^2+5x}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{3-(7/x^2)}{1+(5/x)} = \frac{3-0}{1+0} = 3$. Thus, the line $y=3$ is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to-\infty}\frac{3-(7/x^2)}{1+(5/x)} = \frac{3-0}{1+0} = 3$. Thus, the curve is also asymptotic to the asymptote in the negative direction.

2.5.39 $\lim_{x\to\infty}\frac{(6x^2-9x+8)}{(3x^2+2)}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{6-9/x+8/x^2}{3+2/x^2} = \frac{6-0+0}{3+0} = 2$. Similarly $\lim_{x\to-\infty} f(x)=2$. The line $y=2$ is a horizontal asymptote.

2.5.40 $\lim_{x\to\infty}\frac{(12x^8-3)}{(3x^8-2x^7)}\cdot\frac{1/x^8}{1/x^8} = \lim_{x\to\infty}\frac{12-3/x^8}{3-2/x} = \frac{12-0}{3-0} = 4$. Similarly $\lim_{x\to-\infty} f(x)=4$. The line $y=4$ is a horizontal asymptote.

2.5.41 $\lim_{x\to\infty}\frac{3x^3-7}{x^4+5x^2} = \lim_{x\to\infty}\frac{3x^3-7}{x^4+5x^2}\cdot\frac{3/x^4}{1/x^4} = \lim_{x\to\infty}\frac{1/x-(7/x^4)}{1+(5/x^2)} = \frac{0-0}{1+0} = 0$. Thus, the line $y=0$ (the x-axis) is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{3x^3-7}{x^4+5x^2} = \lim_{x\to-\infty}\frac{3x^3-7}{x^4+5x^2}\cdot\frac{3/x^4}{1/x^4} = \lim_{x\to-\infty}\frac{1/x-(7/x^4)}{1+(5/x^2)} = \frac{0-0}{1+0} = 0$. Thus, the curve is asymptotic to the x-axis in the negative direction as well.

2.5.42 $\lim_{x\to\infty}\frac{(2x+1)}{(3x^4-2)}\cdot\frac{1/x^4}{1/x^4} = \lim_{x\to\infty}\frac{2/x^3+1/x^4}{3-2/x^4} = \frac{0+0}{3-0} = 0$. Similarly $\lim_{x\to-\infty} f(x)=0$. The line $y=0$ is a horizontal asymptote.

2.5.43 $\lim_{x\to\infty}\frac{(40x^5+x^2)}{(16x^4-2x)}\cdot\frac{1/x^4}{1/x^4} = \lim_{x\to\infty}\frac{40x+1/x^2}{16-2/x^3} = \infty$. Similarly $\lim_{x\to-\infty} f(x) = -\infty$. There are no horizontal asymptotes.

2.5.44 Note that for all x, $\sqrt{x^4}=x^2$. Then

$$\lim_{x\to\pm\infty}\frac{(6x^2+1)}{\sqrt{4x^4+3x+1}}\cdot\frac{1/x^2}{\sqrt{1/x^4}} = \lim_{x\to\pm\infty}\frac{6+(1/x^2)}{\sqrt{4+(3/x^3)+(1/x^4)}} = \frac{6}{\sqrt{4}} = 3.$$

So $y=3$ is the only horizontal asymptote.

2.5.45 Note that for all x, $\sqrt{x^8} = x^4$. Then $\lim_{x\to\pm\infty} \frac{1}{(2x^4 - \sqrt{4x^8 - 9x^4}} \cdot \frac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(2x^4 + \sqrt{4x^8 - 9x^4})}$
$= \lim_{x\to\pm\infty} \frac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(4x^8 - (4x^8 - 9x^4))} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\pm\infty} \frac{2 + \sqrt{4 - (9/x^4)}}{9} = \frac{4}{9}$.
So $y = \frac{4}{9}$ is the only horizontal asymptote.

2.5.46 First note that $\sqrt{x^2} = x$ for $x > 0$, while $\sqrt{x^2} = -x$ for $x < 0$. Then $\lim_{x\to\infty} f(x)$ can be written as

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{1/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2+1/x} = \frac{1}{2}.$$

However, $\lim_{x\to-\infty} f(x)$ can be written as

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{1+1/x^2}}{-2-1/x} = -\frac{1}{2}.$$

2.5.47 First note that $\sqrt{x^6} = x^3$ if $x > 0$, but $\sqrt{x^6} = -x^3$ if $x < 0$. We have $\lim_{x\to\infty} \frac{4x^3+1}{(2x^3 + \sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} =$
$\lim_{x\to\infty} \frac{4 + 1/x^3}{2 + \sqrt{16 + 1/x^6}} = \frac{4+0}{2+\sqrt{16+0}} = \frac{2}{3}$.

However, $\lim_{x\to-\infty} \frac{4x^3+1}{(2x^3 + \sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to-\infty} \frac{4 + 1/x^3}{2 - \sqrt{16+1/x^6}} = \frac{4+0}{2-\sqrt{16+0}} = \frac{4}{-2} = -2$.

So $y = \frac{2}{3}$ is a horizontal asymptote (as $x \to \infty$) and $y = -2$ is a horizontal asymptote (as $x \to -\infty$).

2.5.48 First note that for $x > 0$ we have $\sqrt{x^2} = x$, but for $x < 0$, we have $-x = \sqrt{x^2}$. Then we have $\lim_{x\to\infty} x - \sqrt{x^2 - 9x} = \lim_{x\to\infty} \frac{(x - \sqrt{x^2-9x})(x + \sqrt{x^2-9x})}{x + \sqrt{x^2-9x}} = \lim_{x\to\infty} \frac{9x}{(x+\sqrt{x^2-9x})} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{9}{1+\sqrt{1-9/x}} = \frac{9}{2}$.
On the other hand, $\lim_{x\to-\infty} x - \sqrt{x^2-9x} = \lim_{x\to-\infty} \frac{(x-\sqrt{x^2-9x})(x+\sqrt{x^2-9x})}{x+\sqrt{x^2-9x}} =$
$\lim_{x\to-\infty} \frac{9x}{(x+\sqrt{x^2-9x})} \cdot \frac{-1/x}{-1/x} = \lim_{x\to-\infty} \frac{-9}{-1+\sqrt{1-9/x}} = -\infty$. The last equal sign follows because $\sqrt{1-9/x} > 1$ but is approaching 1 as $x \to -\infty$. We can therefore conclude that $y = \frac{9}{2}$ is the only horizontal asymptote, and is an asymptote as $x \to \infty$.

2.5.49 First note that $\sqrt[3]{x^6} = x^2$ and $\sqrt{x^4} = x^2$ for all x (even when $x < 0$.) We have $\lim_{x\to\infty} \frac{\sqrt[3]{x^6+8}}{(4x^2+\sqrt{3x^4+1})} \cdot$
$\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt[3]{1+8/x^6}}{4+\sqrt{3+1/x^4}} = \frac{1}{4+\sqrt{3+0}} = \frac{1}{4\sqrt{3}}$.

The calculation as $x \to -\infty$ is similar. So $y = \frac{1}{4\sqrt{3}}$ is a horizontal asymptote.

2.5.50 First note that $\sqrt{x^2} = x$ for $x > 0$ and $\sqrt{x^2} = -x$ for $x < 0$.

We have

$$\begin{aligned}\lim_{x\to\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} -\frac{4}{3+\sqrt{9+1/x^2}} = -\frac{4}{6} = -\frac{2}{3}.\end{aligned}$$

Moreover, as $x \to -\infty$ we have

$$\begin{aligned}\lim_{x\to-\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to-\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to-\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to-\infty} -\frac{4}{3-\sqrt{9+1/x^2}} = \infty.\end{aligned}$$

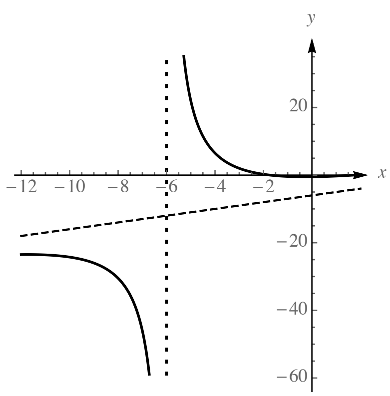
Note that this last equality is due to the fact that the numerator is the constant -4 and the denominator is approaching zero (from the left) so the quotient is positive and is getting large.

So $y = -\frac{2}{3}$ is the only horizontal asymptote.

2.5.51

a. $f(x) = \dfrac{x^2-3}{x+6} = x - 6 + \dfrac{33}{x+6}$. The oblique asymptote of f is $y = x - 6$.

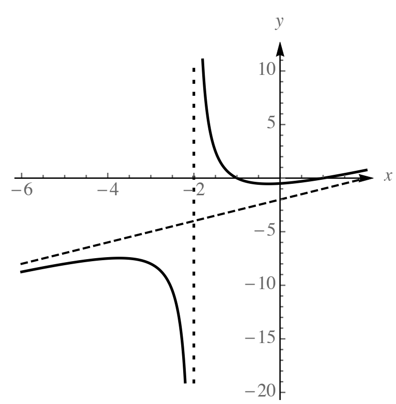
b. Because $\lim_{x\to-6^+} f(x) = \infty$, there is a vertical asymptote at $x = -6$. Note also that $\lim_{x\to-6^-} f(x) = -\infty$.

c. 

2.5.52

a. $f(x) = \dfrac{x^2-1}{x+2} = x - 2 + \dfrac{3}{x+2}$. The oblique asymptote of f is $y = x - 2$.

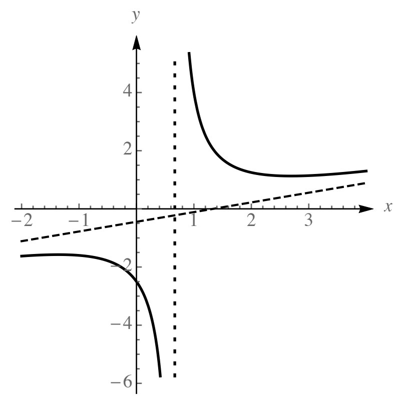
b. Because $\lim_{x\to-2^+} f(x) = \infty$, there is a vertical asymptote at $x = -2$. Note also that $\lim_{x\to-2^-} f(x) = -\infty$.

c. 

2.5.53

a. $f(x) = \dfrac{x^2-2x+5}{3x-2} = (1/3)x - 4/9 + \dfrac{37}{9(3x-2)}$. The oblique asymptote of f is $y = (1/3)x - 4/9$.

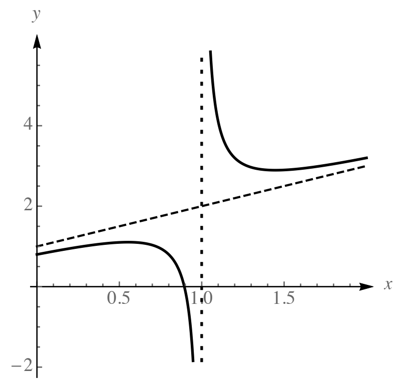
b. Because $\lim_{x\to(2/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = 2/3$. Note also that $\lim_{x\to(2/3)^-} f(x) = -\infty$.

c. 

2.5.54

a. $f(x) = \dfrac{5x^2 - 4}{5x - 5} = x + 1 + \dfrac{1}{5x - 5}$. The oblique asymptote of f is $y = x + 1$.

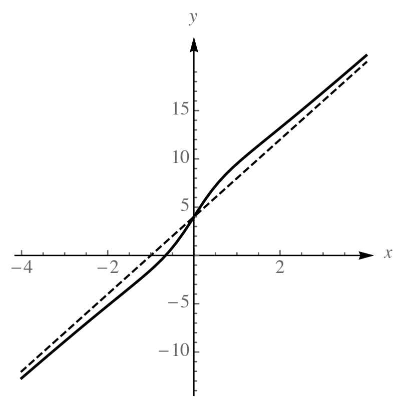
b. Because $\lim_{x\to 1^+} f(x) = \infty$, there is a vertical asymptote at $x = 1$. Note also that $\lim_{x\to 1^-} f(x) = -\infty$.

c. 

2.5.55

a. $f(x) = \dfrac{4x^3 + 4x^2 + 7x + 4}{1 + x^2} = 4x + 4 + \dfrac{3x}{1 + x^2}$. The oblique asymptote of f is $y = 4x + 4$.

b. There are no vertical asymptotes.

c. 

2.5.56

a. $f(x) = \dfrac{3x^2 - 2x + 5}{3x + 4} = x - 2 + \dfrac{13}{3x + 4}$. The oblique asymptote of f is $y = x - 2$.

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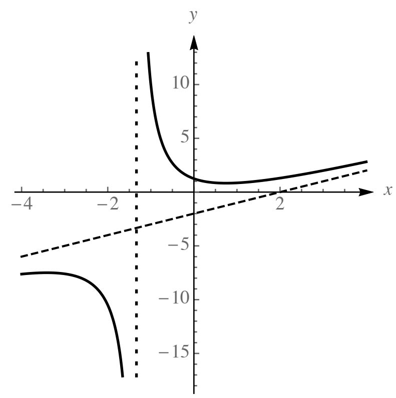
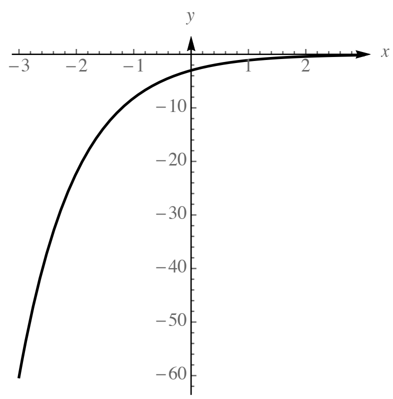
b. Because $\lim_{x\to(-4/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = -4/3$. Note also that $\lim_{x\to(-4/3)^-} f(x) = -\infty$.

c.



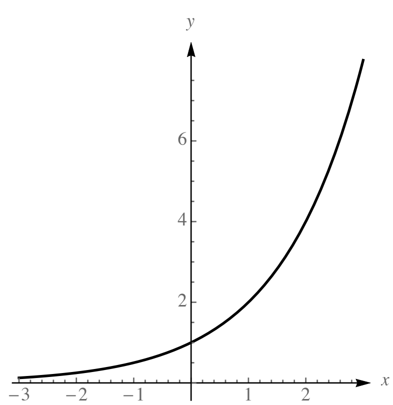
2.5.57

$\lim_{x\to\infty}(-3e^{-x}) = -3\cdot 0 = 0$. $\lim_{x\to-\infty}(-3e^{-x}) = -\infty$.



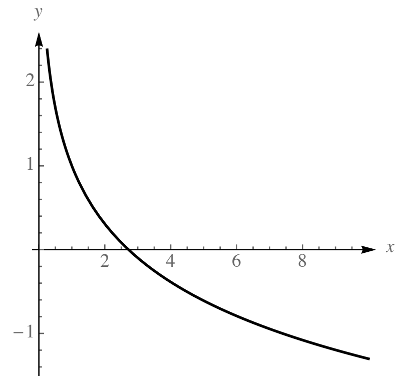
2.5.58

$\lim_{x\to\infty} 2^x = \infty$. $\lim_{x\to-\infty} 2^x = 0$.



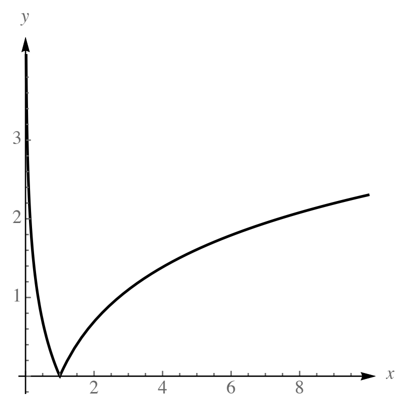
2.5.59

$\lim_{x\to\infty}(1-\ln x) = -\infty$. $\lim_{x\to 0^+}(1-\ln x) = \infty$.



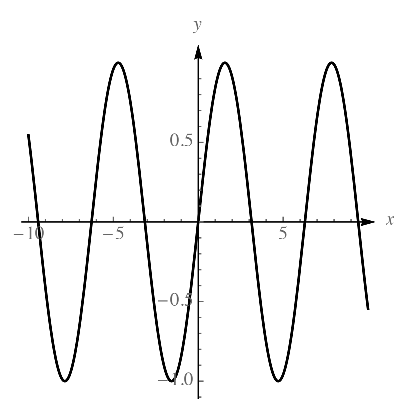
2.5.60

$\lim_{x\to\infty}|\ln x| = \infty$. $\lim_{x\to 0^+}|\ln x| = \infty$.



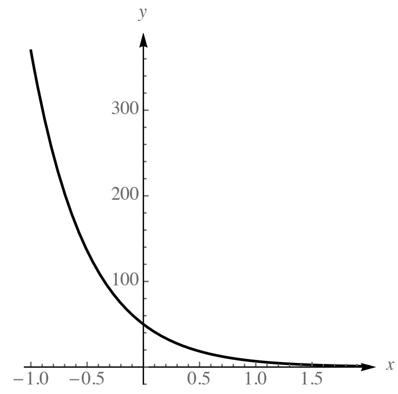
2.5.61

$y = \sin x$ has no asymptotes. $\lim_{x\to\infty}\sin x$ and $\lim_{x\to-\infty}\sin x$ do not exist.



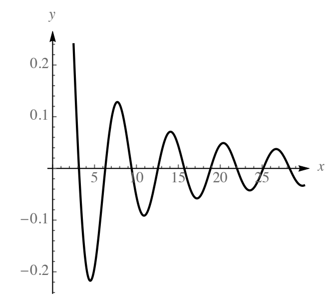
2.5.62

$\lim_{x\to\infty} \frac{50}{e^{2x}} = 0$. $\lim_{x\to-\infty} \frac{50}{e^{2x}} = \infty$.



2.5.63

a. False. For example, the function $y = \frac{\sin x}{x}$ on the domain $[1, \infty)$ has a horizontal asymptote of $y = 0$, and it crosses the x-axis infinitely many times.



b. False. If f is a rational function, and if $\lim_{x\to\infty} f(x) = L \neq 0$, then the degree of the polynomial in the numerator must equal the degree of the polynomial in the denominator. In this case, both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x) = \frac{a_n}{b_n}$ where a_n is the leading coefficient of the polynomial in the numerator and b_n is the leading coefficient of the polynomial in the denominator. In the case where $\lim_{x\to\infty} f(x) = 0$, then the degree of the numerator is strictly less than the degree of the denominator. This case holds for $\lim_{x\to-\infty} f(x) = 0$ as well.

c. True. There are only two directions which might lead to horizontal asymptotes: there could be one as $x \to \infty$ and there could be one as $x \to -\infty$, and those are the only possibilities.

d. False. The limit of the difference of two functions can be written as the difference of the limits only when both limits exist. It is the case that $\lim_{x\to\infty} (x^3 - x) = \infty$.

2.5.64 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{2500}{t+1} = 0$. The steady state exists. The steady state value is 0.

2.5.65 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{3500t}{t+1} = 3500$. The steady state exists. The steady state value is 3500.

2.5.66 $\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 200(1 - 2^{-t}) = 200$. The steady state exists. The steady state value is 200.

2.5.67 $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 1000e^{0.065t} = \infty$. The steady state does not exist.

2.5.68 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1500}{3 + 2e^{-.1t}} = \frac{1500}{3} = 500$. The steady state exists. The steady state value is 500.

2.5.69 $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} 2\left(\frac{t+\sin t}{t}\right) = \lim_{t\to\infty} 2\left(1+\frac{\sin t}{t}\right) = 2$. The steady state exists. The steady state value is 2.

2.5.70

a. $\lim_{x\to\infty} \frac{x^2-4x+3}{x-1} = \infty$, and $\lim_{x\to-\infty} \frac{x^2-4x+3}{x-1} = -\infty$. There are no horizontal asymptotes.

b. It appears that $x=1$ is a candidate to be a vertical asymptote, but note that $f(x) = \frac{x^2-4x+3}{x-1} = \frac{(x-1)(x-3)}{x-1}$. Thus $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x-3) = -2$. So f has no vertical asymptotes.

2.5.71

a. $\lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2}\cdot\frac{(1/x^3)}{(1/x^3)} = \lim_{x\to\infty}\frac{2+10/x+12/x^2}{1+2/x} = 2$. Similarly, $\lim_{x\to-\infty} f(x) = 2$. Thus, $y=2$ is a horizontal asymptote.

b. Note that $f(x) = \frac{2x(x+2)(x+3)}{x^2(x+2)}$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{2(x+3)}{x} = \infty$, and similarly, $\lim_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x=0$. Note that there is no asymptote at $x=-2$ because $\lim_{x\to -2} f(x) = -1$.

2.5.72

a. We have $\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}\cdot\frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty}\frac{\sqrt{16+64/x^2}+1}{2-4/x^2} = \frac{5}{2}$. Similarly, $\lim_{x\to-\infty} f(x) = \frac{5}{2}$. So $y=\frac{5}{2}$ is a horizontal asymptote.

b. $\lim_{x\to\sqrt{2}^+} f(x) = \lim_{x\to-\sqrt{2}^-} f(x) = \infty$, and $\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to-\sqrt{2}^+} f(x) = -\infty$ so there are vertical asymptotes at $x=\pm\sqrt{2}$.

2.5.73

a. We have $\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}\cdot\frac{(1/x^4)}{(1/x^4)} = \lim_{x\to\infty}\frac{3+3/x-36/x^2}{1-25/x^2+144/x^4} = 3$. Similarly, $\lim_{x\to-\infty} f(x) = 3$. So $y=3$ is a horizontal asymptote.

b. Note that $f(x) = \frac{3x^2(x+4)(x-3)}{(x+4)(x-4)(x+3)(x-3)}$. Thus, $\lim_{x\to-3^+} f(x) = -\infty$ and $\lim_{x\to-3^-} f(x) = \infty$. Also, $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$. Thus there are vertical asymptotes at $x=-3$ and $x=4$.

2.5.74

a. First note that

$$f(x) = x^2(4x^2-\sqrt{16x^4+1})\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}} = -\frac{x^2}{4x^2+\sqrt{16x^4+1}}.$$

We have $\lim_{x\to\infty} -\frac{x^2}{4x^2+\sqrt{16x^4+1}}\cdot\frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} -\frac{1}{4+\sqrt{16+1/x^4}} = -\frac{1}{8}$. Similarly, the limit as $x\to-\infty$ of $f(x)$ is $-\frac{1}{8}$ as well. So $y=-\frac{1}{8}$ is a horizontal asymptote.

b. f has no vertical asymptotes.

2.5.75

a. $\lim_{x\to\infty} \frac{x^2-9}{x^2-3x} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} \frac{1-9/x^2}{1-3/x} = 1$. A similar result holds as $x \to -\infty$. So $y = 1$ is a horizontal asymptote.

b. Because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+3}{x} = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$, there is a vertical asymptote at $x = 0$.

2.5.76

a. $\lim_{x\to\pm\infty} \frac{x^4-1}{x^2-1} = \lim_{x\to\pm\infty} \frac{(x^2-1)(x^2+1)}{x^2-1} = \lim_{x\to\pm\infty} x^2+1 = \infty$. There are no horizontal or slant asymptotes.

b. It appears that $x = \pm 1$ may be candidates for vertical asymptotes, but because

$$\frac{x^4-1}{x^2-1} = \frac{(x^2-1)(x^2+1)}{x-1} = x^2+1$$

for $x \neq \pm 1$ there are no vertical asymptotes either.

2.5.77

a. First note that $f(x)$ can be written s

$$\frac{\sqrt{x^2+2x+6}-3}{x-1} \cdot \frac{\sqrt{x^2+2x+6}+3}{\sqrt{x^2+2x+6}+3} = \frac{x^2+2x+6-9}{(x-1)(\sqrt{x^2+2x+6}+3)} = \frac{(x-1)(x+3)}{(x-1)(\sqrt{x^2+2x+6}+3)}.$$

Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+2x+6}+3} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1+3/x}{\sqrt{1+2/x+6/x^2}+3/x} = 1.$$

Using the fact that $\sqrt{x^2} = -x$ for $x < 0$, we have $\lim_{x\to-\infty} f(x) = -1$. Thus the lines $y = 1$ and $y = -1$ are horizontal asymptotes.

b. f has no vertical asymptotes.

2.5.78

a. Note that when x is large $|1-x^2| = x^2-1$. We have $\lim_{x\to\infty} \frac{|1-x^2|}{x^2+x} = \lim_{x\to\infty} \frac{x^2-1}{x^2+x} = 1$. Likewise $\lim_{x\to-\infty} \frac{|1-x^2|}{x^2+x} = \lim_{x\to-\infty} \frac{x^2-1}{x^2+x} = 1$. So there is a horizontal asymptote at $y = 1$.

b. Note that when x is near 0, we have $|1-x^2| = 1-x^2 = (1-x)(1+x)$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-x}{x} = \infty$. Similarly, $\lim_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$.

2.5.79

a. Note that when $x > 1$, we have $|x| = x$ and $|x-1| = x-1$. Thus

$$f(x) = (\sqrt{x}-\sqrt{x-1}) \cdot \frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}+\sqrt{x-1}}.$$

Thus $\lim_{x\to\infty} f(x) = 0$.

When $x < 0$, we have $|x| = -x$ and $|x-1| = 1-x$. Thus

$$f(x) = (\sqrt{-x}-\sqrt{1-x}) \cdot \frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}+\sqrt{1-x}} = -\frac{1}{\sqrt{-x}+\sqrt{1-x}}.$$

Thus, $\lim_{x\to-\infty} f(x) = 0$. There is a horizontal asymptote at $y = 0$.

b. f has no vertical asymptotes.

2.5.80

a. $\lim_{x\to\infty} \frac{(3e^x+10)}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{3+(10/e^x)}{1} = 3$. On the other hand, $\lim_{x\to-\infty} \frac{3e^x+10}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{3+(10e^{-x})}{1} = \infty$. $y=3$ is a horizontal asymptote as $x\to\infty$.

b. f has no vertical asymptotes.

2.5.81

a. $\lim_{x\to\infty} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \lim_{x\to\infty} \left(2 + \frac{\cos x}{\sqrt{x}}\right) = 2$. $y=2$ is a horizontal asymptote.

b. $\lim_{x\to 0^+} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \infty$. and $\lim_{x\to 0^-} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}}$ does not exist. $x=0$ is a vertical asymptote.

2.5.82

a. $\lim_{x\to\infty} \cot^{-1} x = 0$.

b. $\lim_{x\to-\infty} \cot^{-1} x = \pi$.

2.5.83

a. $\lim_{x\to\infty} \sec^{-1} x = \pi/2$.

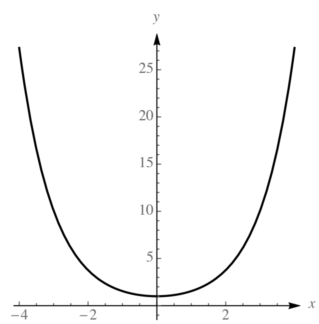
b. $\lim_{x\to-\infty} \sec^{-1} x = \pi/2$.

2.5.84

a. $\lim_{x\to\infty} \frac{e^x+e^{-x}}{2} = \infty$.

$\lim_{x\to-\infty} \frac{e^x+e^{-x}}{2} = \infty$.

b. $\cosh(0) = \frac{e^0+e^0}{2} = \frac{1+1}{2} = 1$.

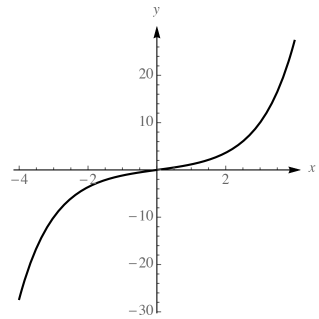


2.5.85

a. $\lim_{x\to\infty} \frac{e^x-e^{-x}}{2} = \infty$.

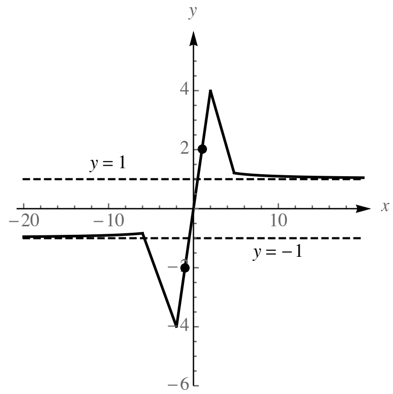
$\lim_{x\to-\infty} \frac{e^x-e^{-x}}{2} = -\infty$.

b. $\sinh(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0.$



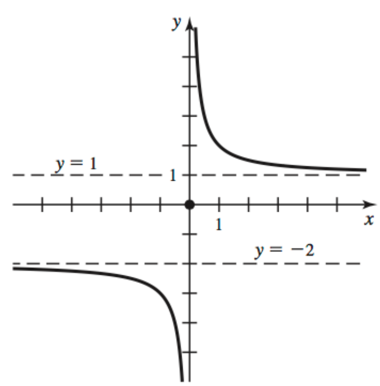
2.5.86

One possible such graph is:



2.5.87

One possible such graph is:



2.5.88 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{4}{n} = 0.$

2.5.89 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} [1 - (1/n)] = 1.$

2.5.90 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+1/n} = \infty$, so the limit does not exist.

2.5.91 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} [1/n + 1/n^2] = 0.$

2.5.92

a. Suppose $m = n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{a_n}{b_n}.\end{aligned}$$

b. Suppose $m < n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n/x^{n-m} + a_{n-1}/x^{n-m+1} + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{0}{b_n} = 0.\end{aligned}$$

2.5.93

a. No. If $m = n$, there will be a horizontal asymptote, and if $m = n+1$, there will be a slant asymptote.

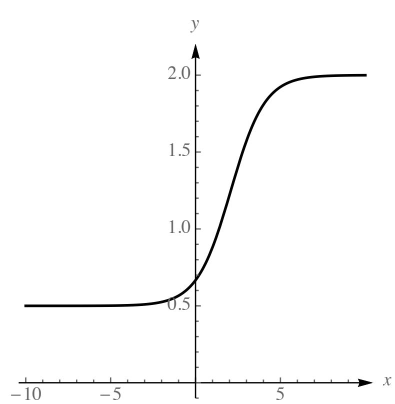
b. Yes. For example, $f(x) = \frac{x^4}{\sqrt{x^6+1}}$ has slant asymptote $y = x$ as $x \to \infty$ and slant asymptote $y = -x$ as $x \to -\infty$.

2.5.94

a. $\lim_{x\to\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^{2x}}{1/e^{2x}} = \lim_{x\to\infty} \frac{(4/e^x) + 2}{(8/e^x) + 1} = 2.$

b. $\lim_{x\to-\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to-\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{4 + 2e^x}{8 + e^x} = \frac{1}{2}.$

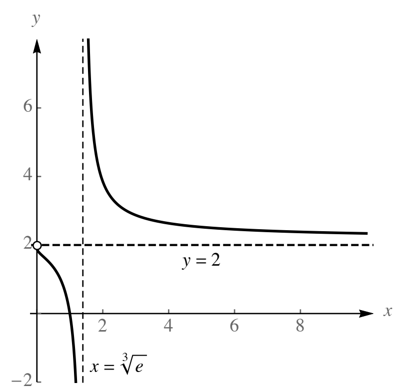
c. The lines $y = 2$ and $y = \frac{1}{2}$ are horizontal asymptotes.

d. 

2.5.95 $\lim_{x\to\infty} \frac{2e^x+3}{e^x+1} = \lim_{x\to\infty} \frac{(2e^x+3)}{(e^x+1)} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{2+3/e^x}{1+1/e^x} = \frac{2+0}{1+0} = 2$. Thus the line $y = 2$ is a horizontal asymptote. Also $\lim_{x\to-\infty} \frac{2e^x+3}{e^x+1} = \frac{0+3}{0+1} = 3$, so $y = 3$ is a horizontal asymptote.

2.5.96 $\lim_{x\to\infty} \frac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim_{x\to\infty} \frac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})} \cdot \frac{1/e^{6x}}{1/e^{6x}} = \lim_{x\to\infty} \frac{3e^{-x}+7}{9e^{-x}+14} = \frac{7}{14} = \frac{1}{2}$. So $y = \frac{1}{2}$ is a horizontal asymptote. Also, $\lim_{x\to-\infty} \frac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim_{x\to-\infty} \frac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})} \cdot \frac{1/e^{5x}}{1/e^{5x}} = \lim_{x\to-\infty} \frac{3+7e^x}{9+14e^x} = \frac{3}{9} = \frac{1}{3}$. So $y = \frac{1}{3}$ is a horizontal asymptote.

2.5.97 Using the rules of logarithms, $f(x) = \frac{6\ln x}{3\ln x - 1}$. The domain of f is $(0, \sqrt[3]{e}) \cup (\sqrt[3]{e}, \infty)$. We first examine the end behavior of the function. Observe that $\lim_{x\to\infty} \frac{6\ln x}{3\ln x - 1} = \lim_{x\to\infty} \frac{6}{3-(1/\ln x)} = \frac{6}{3} = 2$ and $\lim_{x\to 0^+} \frac{6\ln x}{3\ln x - 1} = \lim_{x\to 0^+} \frac{6}{3-(1/\ln x)} = \frac{6}{3} = 2$. So the function has a horizontal asymptote of $y = 2$ and it is undefined at $x = 0$ but has limit 2 as x approaches 0 from the right. Notice also that as $x \to \sqrt[3]{e}^+$, $6\ln x \to 2$ and $3\ln x - 1$ is positive and approaches 0. Therefore, $\lim_{x\to\sqrt[3]{e}^+} \frac{6\ln x}{3\ln x - 1} = \infty$ and by a similar argument, $\lim_{x\to\sqrt[3]{e}^-} \frac{6\ln x}{3\ln x - 1} = -\infty$



2.6 Continuity

2.6.1

a. $a(t)$ is a continuous function during the time period from when she jumps from the plane and when she touches down on the ground, because her position is changing continuously with time.

b. $n(t)$ is not a continuous function of time. The function "jumps" at the times when a quarter must be added.

c. $T(t)$ is a continuous function, because temperature varies continuously with time.

d. $p(t)$ is not continuous – it jumps by whole numbers when a player scores a point.

2.6.2 In order for f to be continuous at $x = a$, the following conditions must hold:

- f must be defined at a (i.e. a must be in the domain of f),
- $\lim_{x\to a} f(x)$ must exist, and

- $\lim_{x \to a} f(x)$ must equal $f(a)$.

2.6.3 A function f is continuous on an interval I if it is continuous at all points in the interior of I, and it must be continuous from the right at the left endpoint (if the left endpoint is included in I) and it must be continuous from the left at the right endpoint (if the right endpoint is included in I.)

2.6.4 The words "hole" and "break" are not mathematically precise, so a strict mathematical definition can not be based on them.

2.6.5 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $f(1)$ is not defined (so the first condition is violated). At $x = 2$, $f(2)$ is defined and $\lim_{x \to 2} f(x)$ exists, but $\lim_{x \to 2} f(x) \neq f(2)$ (so condition 3 is violated). At $x = 3$, $\lim_{x \to 3} f(x)$ does not exist (so condition 2 is violated).

2.6.6 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x) \neq f(1)$ (so condition 3 is violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.7 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist, and $f(1)$ is not defined (so conditions 1 and 2 are violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.8 f is discontinuous at $x = 2$, at $x = 3$, and at $x = 4$. At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined and $\lim_{x \to 3} f(x)$ does not exist (so conditions 1 and 2 are violated). At $x = 4$, $\lim_{x \to 4} f(x) \neq f(4)$ (so condition 3 is violated).

2.6.9

a. A function f is continuous from the left at $x = a$ if a is in the domain of f, and $\lim_{x \to a^-} f(x) = f(a)$.

b. A function f is continuous from the right at $x = a$ if a is in the domain of f, and $\lim_{x \to a^+} f(x) = f(a)$.

2.6.10 If f is right-continuous at $x = 3$, then $f(3) = \lim_{x \to 3^+} f(x) = 6$, so $f(3) = 6$.

2.6.11 f is continuous on $(0, 1)$, on $(1, 2)$, on $(2, 3]$, and on $(3, 4)$. It is continuous from the left at 3.

2.6.12 f is continuous on $(0, 1)$, on $(1, 2]$, on $(2, 3)$, and on $(3, 4)$. It is continuous from the left at 2.

2.6.13 f is continuous on $[0, 1)$, on $(1, 2)$, on $[2, 3)$, and on $(3, 5)$. It is continuous from the right at 2.

2.6.14 f is continuous on $(0, 2]$, on $(2, 3)$, on $(3, 4)$, and on $(4, 5)$. It is continuous from the left at 2.

2.6.15 The domain of $f(x) = \dfrac{e^x}{x}$ is $(-\infty, 0) \cup (0, \infty)$, and f is continuous everywhere on this domain.

2.6.16 The function is continuous on $(0, 15]$, on $(15, 30]$, on $(30, 45]$, and on $(45, 60]$.

2.6.17 The number -5 is not in the domain of f, because the denominator is equal to 0 when $x = -5$. Thus, the function is not continuous at -5.

2.6.18 The function is defined at 5, in fact $f(5) = \frac{50+15+1}{25+25} = \frac{66}{50} = \frac{33}{25}$. Also, $\lim_{x \to 5} f(x) = \lim_{x \to 5} \dfrac{2x^2 + 3x + 1}{x^2 + 5x} = \dfrac{33}{25} = f(5)$. The function is continuous at $a = 5$.

2.6.19 f is discontinuous at 1, because 1 is not in the domain of f; $f(1)$ is not defined.

2.6.20 g is discontinuous at 3 because 3 is not in the domain of g; $g(3)$ is not defined.

2.6.21 f is discontinuous at 1, because $\lim_{x\to 1} f(x) \neq f(1)$. In fact, $f(1) = 3$, but $\lim_{x\to 1} f(x) = 2$.

2.6.22 f is continuous at 3, because $\lim_{x\to 3} f(x) = f(3)$. In fact, $f(3) = 2$ and $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x-1) = 2$.

2.6.23 f is discontinuous at 4, because 4 is not in the domain of f; $f(4)$ is not defined.

2.6.24 f is discontinuous at -1 because $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x(x+1)}{x+1} = \lim_{x\to -1} x = -1 \neq f(-1) = 2$.

2.6.25 Because p is a polynomial, it is continuous on all of $\mathbb{R} = (-\infty, \infty)$.

2.6.26 Because g is a rational function, it is continuous on its domain, which is all of $\mathbb{R} = (-\infty, \infty)$. (Because $x^2 + x + 1$ has no real roots.)

2.6.27 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

2.6.28 Because s is a rational function, it is continuous on its domain. Its domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2.6.29 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.30 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.31 Because $f(x) = \left(x^8 - 3x^6 - 1\right)^{40}$ is a polynomial, it is continuous everywhere, including at 0. Thus $\lim_{x\to 0} f(x) = f(0) = (-1)^{40} = 1$.

2.6.32 Because $f(x) = \left(\frac{3}{2x^5 - 4x^2 - 50}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 2$. So $\lim_{x\to 2} f(x) = f(2) = \frac{81}{16}$.

2.6.33 Because $x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x-4)(x+2)$, we have (as long as $x \neq 4$)

$$\sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \sqrt{x(x+2)}.$$

Thus, $\lim_{x\to 4} \sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \lim_{x\to 4} \sqrt{x(x+2)} = \sqrt{24}$, using Theorem 2.12 and the fact that the square root is a continuous function.

2.6.34 Note that $t - 4 = (\sqrt{t} - 2)(\sqrt{t} + 2)$, so for $t \neq 4$, we have

$$\frac{t-4}{\sqrt{t}-2} = \sqrt{t} + 2.$$

Thus, $\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2} = \lim_{t\to 4}(\sqrt{t} + 2) = 4$. Then using Theorem 2.12 and the fact that the tangent function is continuous at 4, we have $\lim_{t\to 4} \tan\left(\frac{t-4}{\sqrt{t}-2}\right) = \tan\left(\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2}\right) = \tan 4$.

2.6.35 Because $f(x) = \left(\frac{x+5}{x+2}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 1$. Thus $\lim_{x\to 1} f(x) = f(1) = 16$.

2.6.36 $\lim_{x\to\infty} \left(\dfrac{2x+1}{x}\right)^3 = \lim_{x\to\infty} (2 + (1/x))^3 = 2^3 = 8$.

2.6.37 Note that

$$\lim_{x\to 5} \frac{6(\sqrt{x^2-16}-3)}{5(x-5)} = \lim_{x\to 5} \frac{6(\sqrt{x^2-16}-3)}{5(x-5)} \cdot \frac{(\sqrt{x^2-16}+3)}{(\sqrt{x^2-16}+3)} = \lim_{x\to 5} \frac{6(x^2-25)}{5(x-5)(\sqrt{x^2-16}+3)}$$
$$= \lim_{x\to 5} \frac{6(x+5)}{5(\sqrt{x^2-16}+3)} = \frac{60}{30} = 2.$$

2.6.38 First note that

$$\lim_{x\to 0} \frac{x}{\sqrt{16x+1}-1} = \lim_{x\to 0} \frac{x}{(\sqrt{16x+1}-1)} \cdot \frac{(\sqrt{16x+1}+1)}{(\sqrt{16x+1}+1)} = \lim_{x\to 0} \frac{x(\sqrt{16x+1}+1)}{16x} = \frac{2}{16} = \frac{1}{8}.$$

Then because $f(x) = x^{1/3}$ is continuous at $1/8$, we have $\lim_{x\to 0} \left(\dfrac{x}{\sqrt{16x+1}-1}\right)^{1/3} = \left(\dfrac{1}{8}\right)^{1/3} = \dfrac{1}{2}$, by Theorem 2.12.

2.6.39

a. f is defined at 1. We have $f(1) = 1^2 + (3)(1) = 4$. To see whether or not $\lim_{x\to 1} f(x)$ exists, we investigate the two one-sided limits. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2$, and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 + 3x) = 4$, so $\lim_{x\to 1} f(x)$ does not exist. Thus f is discontinuous at $x = 1$.

b. f is continuous from the right, because $\lim_{x\to 1^+} f(x) = 4 = f(1)$.

c. f is continuous on $(-\infty, 1)$ and on $[1, \infty)$.

2.6.40

a. f is defined at 0, in fact $f(0) = 1$. However, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^3 + 4x + 1) = 1$, while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x^3 = 0$. So $\lim_{x\to 0} f(x)$ does not exist.

b. f is continuous from the left at 0, because $\lim_{x\to 0^-} f(x) = f(0) = 1$.

c. f is continuous on $(-\infty, 0]$ and on $(0, \infty)$.

2.6.41 f is defined and is continuous on $(-\infty, 5]$. It is continuous from the left at 5.

2.6.42 f is defined and is continuous on $[-5, 5]$. It is continuous from the right at -5 and is continuous from the left at 5.

2.6.43 f is continuous on $(-\infty, -\sqrt{8}]$ and on $[\sqrt{8}, \infty)$. It is continuous from the left at $-\sqrt{8}$ and from the right at $\sqrt{8}$.

2.6.44 $g(x) = \sqrt{x^2 - 3x + 2} = \sqrt{(x-1)(x-2)}$ is defined and is continuous on $(-\infty, 1]$ and on $[2, \infty)$. It is continuous from the left at 1 and from the right at 2.

2.6.45 Because f is the composition of two functions which are continuous on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$.

2.6.46 f is continuous on $(-\infty, -1]$ and on $[1, \infty)$. It is continuous from the left at -1 and from the right at 1.

2.6.47 Because f is the composition of two functions which are continuous on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$.

2.6.48 f is continuous on $[1, \infty)$. It is continuous from the right at 1.

2.6.49 $\lim_{x\to 2} \sqrt{\frac{4x+10}{2x-2}} = \sqrt{\frac{18}{2}} = 3.$

2.6.50 $\lim_{x\to -1} \left(x^2 - 4 + \sqrt[3]{x^2-9}\right) = (-1)^2 - 4 + \sqrt[3]{(-1)^2 - 9} = -3 + \sqrt[3]{-8} = -3 + -2 = -5.$

2.6.51 $\lim_{x\to\pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \lim_{x\to\pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1} = \lim_{x\to\pi} (\cos x + 2) = 1.$

2.6.52 $\lim_{x\to 3\pi/2} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \lim_{x\to 3\pi/2} \frac{(\sin x + 5)(\sin x + 1)}{(\sin x - 1)(\sin x + 1)} = \lim_{x\to 3\pi/2} \frac{\sin x + 5}{\sin x - 1} = \frac{4}{-2} = -2.$

2.6.53 $\lim_{x\to 3} \sqrt{x^2+7} = \sqrt{9+7} = 4.$

2.6.54 $\lim_{t\to 2} \frac{t^2+5}{1+\sqrt{t^2+5}} = \frac{9}{1+\sqrt{9}} = \frac{9}{4}.$

2.6.55 $\lim_{x\to\pi/2} \frac{\sin x - 1}{\sqrt{\sin x} - 1} = \lim_{x\to\pi/2} (\sqrt{\sin x} + 1) = 2.$

2.6.56 $\lim_{\theta\to 0} \frac{\frac{1}{2+\sin\theta} - \frac{1}{2}}{\sin\theta} \cdot \frac{(2)(2+\sin\theta)}{(2)(2+\sin\theta)} = \lim_{\theta\to 0} \frac{2-(2+\sin\theta)}{(\sin\theta)(2)(2+\sin\theta)} = \lim_{\theta\to 0} -\frac{1}{2(2+\sin\theta)} = -\frac{1}{4}.$

2.6.57 $\lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x} = \lim_{x\to 0} \frac{\cos x - 1}{1-\cos^2 x} = \lim_{x\to 0} \frac{\cos x - 1}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0} -\frac{1}{1+\cos x} = -\frac{1}{2}.$

2.6.58 $\lim_{x\to 0^+} \frac{1-\cos^2 x}{\sin x} = \lim_{x\to 0^+} \frac{\sin^2 x}{\sin x} = \lim_{x\to 0^+} \sin x = 0.$

2.6.59 $\lim_{x\to 0} \frac{e^{4x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^{2x}+1)(e^{2x}-1)}{e^x-1} = \lim_{x\to 0} \frac{(e^{2x}+1)(e^x-1)(e^x+1)}{e^x-1} =$
$\lim_{x\to 0} (e^{2x}+1)(e^x+1) = 2\cdot 2 = 4.$

2.6.60 $\lim_{x\to e^2} \frac{\ln^2 x - 5\ln x + 6}{\ln x - 2} = \lim_{x\to e^2} \frac{(\ln x - 2)(\ln x - 3)}{\ln x - 2} = \lim_{x\to e^2} (\ln x - 3) = -1.$

2.6.61 $f(x) = \csc x$ isn't defined at $x = k\pi$ where k is an integer, so it isn't continuous at those points. So it is continuous on intervals of the form $(k\pi, (k+1)\pi)$ where k is an integer. $\lim_{x\to\pi/4} \csc x = \sqrt{2}$. $\lim_{x\to 2\pi^-} \csc x = -\infty$.

2.6.62 f is defined on $[0, \infty)$, and it is continuous there, because it is the composition of continuous functions defined on that interval. $\lim_{x\to 4} f(x) = e^2$. $\lim_{x\to 0} f(x)$ does not exist—but $\lim_{x\to 0^+} f(x) = e^0 = 1$, because f is continuous from the right.

2.6.63 f isn't defined for any number of the form $\pi/2 + k\pi$ where k is an integer, so it isn't continuous there. It is continuous on intervals of the form $(\pi/2 + k\pi, \pi/2 + (k+1)\pi)$, where k is an integer.

$$\lim_{x\to\pi/2^-} f(x) = \infty. \quad \lim_{x\to 4\pi/3} f(x) = \frac{1-\sqrt{3}/2}{-1/2} = \sqrt{3} - 2.$$

2.6.64 The domain of f is $(0,1]$, and f is continuous on this interval because it is the quotient of two continuous functions and the function in the denominator isn't zero on that interval.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\ln x}{\sin^{-1}(x)} = \frac{\ln 1}{\sin^{-1}(1)} = \frac{0}{\pi/2} = 0.$$

2.6.65 This function is continuous on its domain, which is $(-\infty,0)\cup(0,\infty)$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty, \text{ while } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty.$$

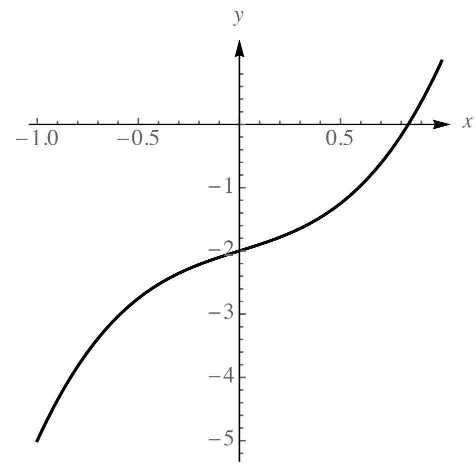
2.6.66 This function is continuous on its domain, which is $(-\infty,0)\cup(0,\infty)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = 2.$$

2.6.67

a. Note that $f(x) = 2x^3 + x - 2$ is continuous everywhere, so in particular it is continuous on $[-1,1]$. Note that $f(-1) = -5 < 0$ and $f(1) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between -1 and 1 where $f(c) = 0$.

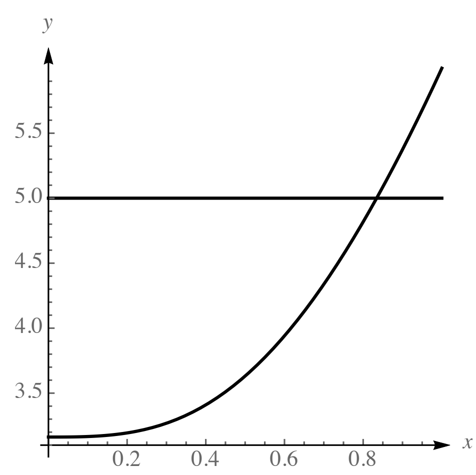
b. Using a graphing calculator and a computer algebra system, we see that the root of f is about 0.835.

c. 

2.6.68

a. Note that $f(x) = \sqrt{x^4 + 25x^3 + 10} - 5$ is continuous on its domain, so in particular it is continuous on $[0,1]$. Note that $f(0) = \sqrt{10} - 5 < 0$ and $f(1) = 6 - 5 = 1 > 0$. Because 0 is an intermediate value between $f(0)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between 0 and 1 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see the root of $f(x)$ is at about .834.

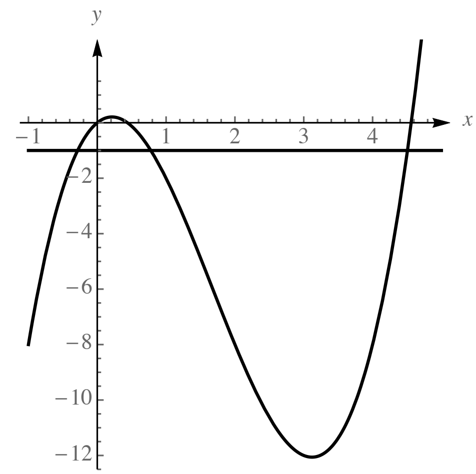
c. 

2.6.69

a. Note that $f(x) = x^3 - 5x^2 + 2x$ is continuous everywhere, so in particular it is continuous on $[-1, 5]$. Note that $f(-1) = -8 < -1$ and $f(5) = 10 > -1$. Because -1 is an intermediate value between $f(-1)$ and $f(5)$, the Intermediate Value Theorem guarantees a number c between -1 and 5 where $f(c) = -1$.

b. Using a graphing calculator and a computer algebra system, we see that there are actually three different values of c between -1 and 5 for which $f(c) = -1$. They are $c \approx -0.285$, $c \approx 0.778$, and $c \approx 4.507$.

c.

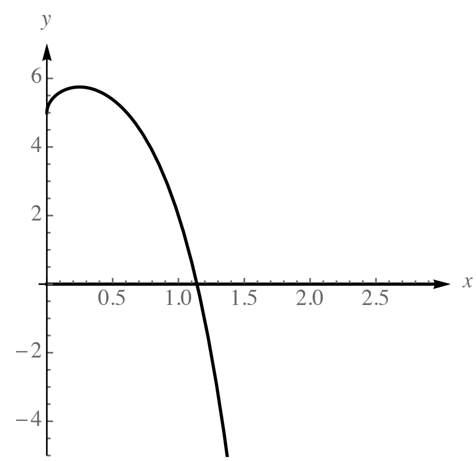


2.6.70

a. Note that $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$ is continuous on its domain, so in particular it is continuous on $[0, 3]$. Note that $f(0) = 5 > 0$ and $f(3) \approx -270.5 < 0$. Because 0 is an intermediate value between $f(0)$ and $f(3)$, the Intermediate Value Theorem guarantees a number c between 0 and 3 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.141.

c.

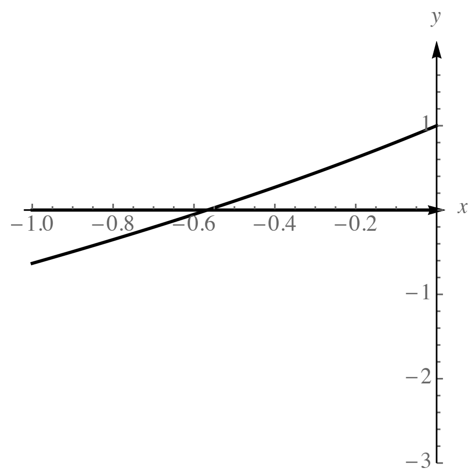


2.6.71

a. Note that $f(x) = e^x + x$ is continuous on its domain, so in particular it is continuous on $[-1, 0]$. Note that $f(-1) = \frac{1}{e} - 1 < 0$ and $f(0) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(0)$, the Intermediate Value Theorem guarantees a number c between -1 and 0 where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about -0.567.

c.

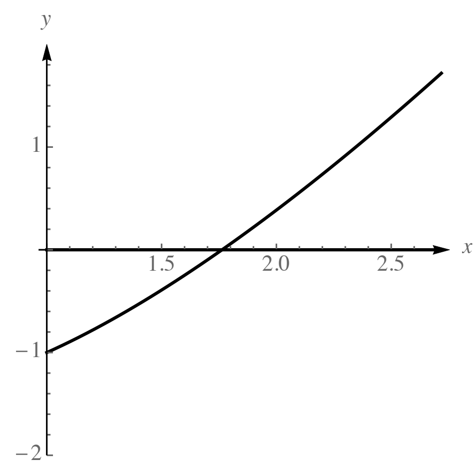


2.6.72

a. Note that $f(x) = x \ln x - 1$ is continuous on its domain, so in particular it is continuous on $[1, e]$. Note that $f(1) = \ln 1 - 1 = -1 < 0$ and $f(e) = e - 1 > 0$. Because 0 is an intermediate value between $f(1)$ and $f(e)$, the Intermediate Value Theorem guarantees a number c between 1 and e where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.76322.

c.



2.6.73

a. True. If f is right continuous at a, then $f(a)$ exists and the limit from the right at a exists and is equal to $f(a)$. Because it is left continuous, the limit from the left exists — so we now know that the limit as $x \to a$ of $f(x)$ exists, because the two one-sided limits are both equal to $f(a)$.

b. True. If $\lim_{x\to a} f(x) = f(a)$, then $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to a^-} f(x) = f(a)$.

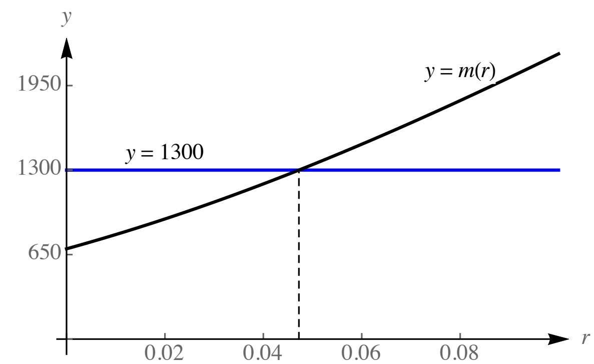
c. False. The statement would be true if f were continuous. However, if f isn't continuous, then the statement doesn't hold. For example, suppose that $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } 1 \le x \le 2, \end{cases}$ Note that $f(0) = 0$ and $f(2) = 1$, but there is no number c between 0 and 2 where $f(c) = 1/2$.

d. False. Consider $f(x) = x^2$ and $a = -1$ and $b = 1$. Then f is continuous on $[a, b]$, but $\frac{f(1)+f(-1)}{2} = 1$, and there is no c on (a, b) with $f(c) = 1$.

2.6.74

a. Because m is a continuous function of r on $[0.04, 0.05]$, and because $m(0.04) \approx 1193.54$ and $m(0.05) \approx 1342.05$, (and 1300 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0.04 and 0.05 where $m(r) = 1300$.

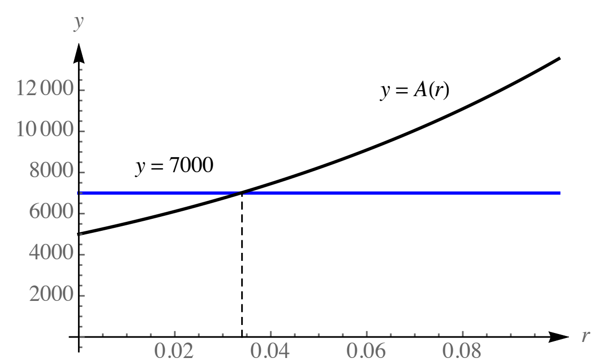
b. Using a computer algebra system, we see that the required interest rate is about 0.047.



2.6.75

a. Because A is a continuous function of r on $[0, 0.08]$, and because $A(0) = 5000$ and $A(0.08) \approx 11098.2$, (and 7000 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0 and 0.08 where $A(r) = 7000$.

b. Solving $5000(1 + (r/12))^{120} = 7000$ for r, we see that $(1+(r/12))^{120} = 7/5$, so $1+r/12 = \sqrt[120]{7/5}$, so $r = 12(\sqrt[120]{7/5} - 1) \approx 0.034$.



2.6.76

a. Note that $A(0.01) \approx 2615.55$ and $A(0.1) \approx 3984.36$. By the Intermediate Value Theorem, there must be a number r_0 between 0.01 and 0.1 so that $A(r_0) = 3500$.

b. The desired value is $r_0 \approx 0.0728$ or 7.28%.

2.6.77 Consider the function $f(x) = \cos x - 2x$ on the interval $[0, 1]$. Note that $f(0) = 1$ and $f(\pi/2) = -\pi < 0$. So by the Intermediate Value Theorem, there must be a root of f on the interval $[0, \pi/2]$. Using a computer algebra system, we find a root of approximately 0.45.

2.6.78 Let $f(x) = |x|$.

For values of a other than 0, it is clear that $\lim_{x\to a} |x| = |a|$ because f is defined to be either the polynomial x (for values greater than 0) or the polynomial $-x$ (for values less than 0.) For the value of $a = 0$, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0)$. Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = -0 = 0$. Thus $\lim_{x\to 0} f(x) = f(0)$, so f is continuous at 0.

2.6.79 Because $f(x) = x^3 + 3x - 18$ is a polynomial, it is continuous on $(-\infty, \infty)$, and because the absolute value function is continuous everywhere, $|f(x)|$ is continuous everywhere.

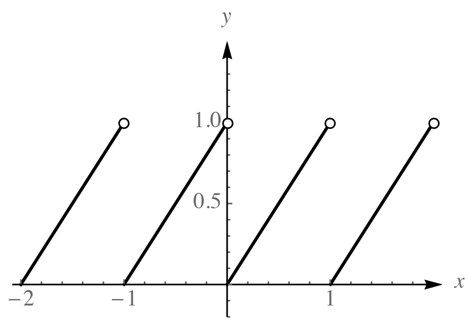
2.6.80 Let $f(x) = \dfrac{x+4}{x^2-4}$. Then f is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. So $g(x) = |f(x)|$ is also continuous on this set.

2.6.81 Let $f(x) = \dfrac{1}{\sqrt{x}-4}$. Then f is continuous on $[0, 16) \cup (16, \infty)$. So $h(x) = |f(x)|$ is continuous on this set as well.

2.6.82 Because $x^2 + 2x + 5$ is a polynomial, it is continuous everywhere, as is $|x^2 + 2x + 5|$. So $h(x) = |x^2 + 2x + 5| + \sqrt{x}$ is continuous on its domain, namely $[0, \infty)$.

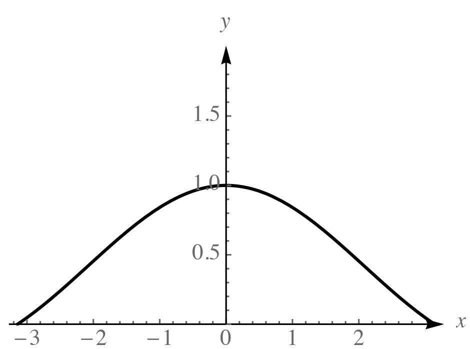
2.6.83

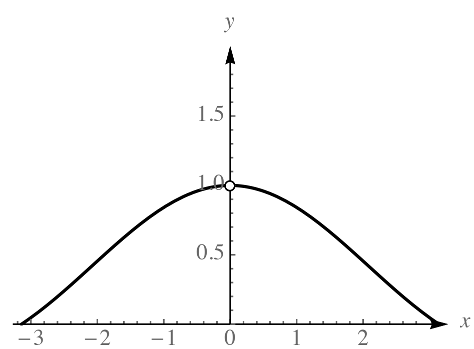
The graph shown isn't drawn correctly at the integers. At an integer a, the value of the function is 0, whereas the graph shown appears to take on all the values from 0 to 1.
Note that in the correct graph, $\lim_{x \to a^-} f(x) = 1$ and $\lim_{x \to a^+} f(x) = 0$ for every integer a.



2.6.84

The graph as drawn on most graphing calculators appears to be continuous at $x = 0$, but it isn't, of course (because the function isn't defined at $x = 0$). A better drawing would show the "hole" in the graph at $(0, 1)$.

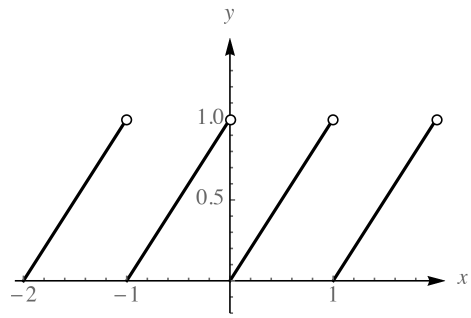
a. 

b. 

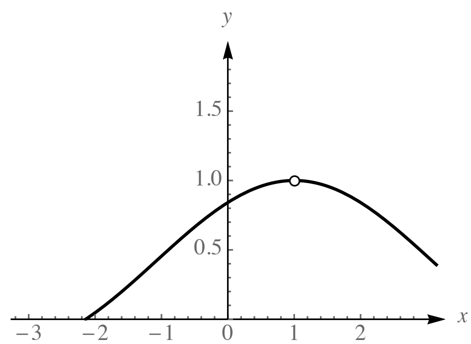
c. It appears that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

2.6.85 With slight modifications, we can use the examples from the previous two problems.

a. The function $y = x - \lfloor x \rfloor$ is defined at $x = 1$ but isn't continuous there.



b. The function $y = \frac{\sin(x-1)}{x-1}$ has a limit at $x = 1$, but isn't defined there, so isn't continuous there.



2.6.86 In order for this function to be continuous at $x = -1$, we require $\lim_{x\to-1} f(x) = f(-1) = a$. So the value of a must be equal to the value of $\lim_{x\to-1} \frac{x^2+3x+2}{x+1} = \lim_{x\to-1} \frac{(x+2)(x+1)}{x+1} = \lim_{x\to-1}(x+2) = 1$. Thus we must have $a = 1$.

2.6.87

a. In order for g to be continuous from the left at $x = 1$, we must have $\lim_{x\to1^-} g(x) = g(1) = a$. We have $\lim_{x\to1^-} g(x) = \lim_{x\to1^-}(x^2+x) = 2$. So we must have $a = 2$.

b. In order for g to be continuous from the right at $x = 1$, we must have $\lim_{x\to1^+} g(x) = g(1) = a$. We have $\lim_{x\to1^+} g(x) = \lim_{x\to1^+}(3x+5) = 8$. So we must have $a = 8$.

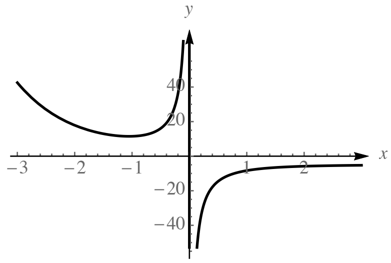
c. Because the limit from the left and the limit from the right at $x = 1$ don't agree, there is no value of a which will make the function continuous at $x = 1$.

2.6.88 $\lim_{x\to0^-} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to0^-} \frac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = \infty$.

$\lim_{x\to0^+} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to0^+} \frac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = -\infty$.

$\lim_{x\to-\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to-\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \frac{e^{-2x}}{e^{-2x}} = \lim_{x\to-\infty} \frac{2e^{-x}+5e^x}{1-e^x} = \infty$.

$\lim_{x\to\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \frac{e^{-3x}}{e^{-3x}} = \lim_{x\to\infty} \frac{2e^{-2x}+5}{e^{-x}-1} = -5$.

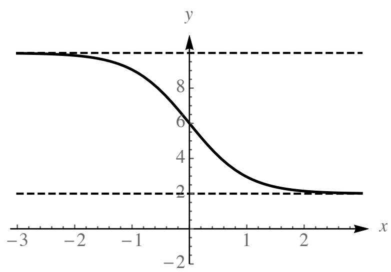


There is a vertical asymptote at $x = 0$, and the line $y = -5$ is a horizontal asymptote.

2.6.89 $\lim_{x\to 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \frac{12}{2} = 6.$

$\lim_{x\to -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to -\infty} \frac{2e^{2x} + 10}{e^{2x} + 1} = \frac{10}{1} = 10.$

$\lim_{x\to \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to \infty} \frac{2 + 10e^{-2x}}{1 + e^{-2x}} = \frac{2}{1} = 2.$



There are no vertical asymptotes. The lines $y = 2$ and $y = 10$ are horizontal asymptotes.

2.6.90 Let $f(x) = x^3 + 10x^2 - 100x + 50$. Note that $f(-20) < 0$, $f(-5) > 0$, $f(5) < 0$, and $f(10) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(-20, -5)$, at least one on $(-5, 5)$, and at least one on $(5, 10)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 \approx -16.32$, $x_2 \approx 0.53$ and $x_3 \approx 5.79$.

2.6.91 Let $f(x) = 70x^3 - 87x^2 + 32x - 3$. Note that $f(0) < 0$, $f(0.2) > 0$, $f(0.55) < 0$, and $f(1) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(0, 0.2)$, at least one on $(0.2, 0.55)$, and at least one on $(0.55, 1)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 = 1/7$, $x_2 = 1/2$ and $x_3 = 3/5$.

2.6.92

a. We have $f(0) = 0$, $f(2) = 3$, $g(0) = 3$ and $g(2) = 0$.

b. $h(t) = f(t) - g(t)$, $h(0) = -3$ and $h(2) = 3$.

c. By the Intermediate Value Theorem, because h is a continuous function and 0 is an intermediate value between -3 and 3, there must be a time c between 0 and 2 where $h(c) = 0$. At this point $f(c) = g(c)$, and at that time, the distance from the car is the same on both days, so the hiker is passing over the exact same point at that time.

2.6.93 We can argue essentially like the previous problem, or we can imagine an identical twin to the original monk, who takes an identical version of the original monk's journey up the winding path while the monk is taking the return journey down. Because they must pass somewhere on the path, that point is the one we are looking for.

2.6.94

a. Because $|-1| = 1$, $|g(x)| = 1$, for all x.

b. The function g isn't continuous at $x = 0$, because $\lim_{x\to 0^+} g(x) = 1 \neq -1 = \lim_{x\to 0^-} g(x)$.

c. This constant function is continuous everywhere, in particular at $x = 0$.

d. This example shows that in general, the continuity of $|g|$ does not imply the continuity of g.

2.6.95 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is a jump discontinuity.

2.6.96 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is an infinite discontinuity.

2.6.97 Note that $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x-5)}{x-2} = \lim_{x\to 2}(x - 5) = -3$. Because this limit exists, the discontinuity is removable.

2.6.98 Note that $\lim_{x\to 1} \dfrac{x^2 - 1}{1 - x} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{1-x} = \lim_{x\to 1}[-(x + 1)] = -2$. Because this limit exists, the discontinuity is removable.

2.6.99 Note that $h(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x-1)} = \dfrac{x(x-2)^2}{x(x-1)}$. Thus $\lim_{x\to 0} h(x) = -4$, and the discontinuity at $x = 0$ is removable. However, $\lim_{x\to 1} h(x)$ does not exist, and the discontinuity at $x = 1$ is not removable (it is infinite.)

2.6.100 This is a jump discontinuity, because $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$.

2.6.101

a. Note that $-1 \le \sin(1/x) \le 1$ for all $x \neq 0$, so $-x \le x\sin(1/x) \le x$ (for $x > 0$. For $x < 0$ we would have $x \le x\sin(1/x) \le -x$.) Because both $x \to 0$ and $-x \to 0$ as $x \to 0$, the Squeeze Theorem tells us that $\lim_{x\to 0} x\sin(1/x) = 0$ as well. Because this limit exists, the discontinuity is removable.

b. Note that as $x \to 0^+$, $1/x \to \infty$, and thus $\lim_{x\to 0^+} \sin(1/x)$ does not exist. So the discontinuity is not removable.

2.6.102 Because g is continuous at a, as $x \to a$, $g(x) \to g(a)$. Because f is continuous at $g(a)$, as $z \to g(a)$, $f(z) \to f(g(a))$. Let $z = g(x)$, and suppose $x \to a$. Then $g(x) = z \to g(a)$, so $f(z) = f(g(x)) \to f(g(a))$, as desired.

2.6.103

a. Consider $g(x) = x + 1$ and $f(x) = \frac{|x-1|}{x-1}$. Note that both g and f are continuous at $x = 0$. However $f(g(x)) = f(x + 1) = \frac{|x|}{x}$ is not continuous at 0.

b. The previous theorem says that the composition of f and g is continuous at a if g is continuous at a and f is continuous at $g(a)$. It does not say that if g and f are both continuous at a that the composition is continuous at a.

2.6.104 The Intermediate Value Theorem requires that our function be continuous on the given interval. In this example, the function f is not continuous on $[-2, 2]$ because it isn't continuous at 0.

2.6.105

a. Using the hint, we have

$$\sin x = \sin(a + (x - a)) = \sin a \cos(x - a) + \sin(x - a)\cos a.$$

Note that as $x \to a$, we have that $\cos(x - a) \to 1$ and $\sin(x - a) \to 0$.

So,

$$\lim_{x\to a} \sin x = \lim_{x\to a} \sin(a+(x-a)) = \lim_{x\to a} (\sin a \cos(x - a) + \sin(x - a)\cos a) = (\sin a)\cdot 1 + 0 \cdot \cos a = \sin a.$$

b. Using the hint, we have

$$\cos x = \cos(a + (x - a)) = \cos a \cos(x - a) - \sin a \sin(x - a).$$

So,

$$\begin{aligned}\lim_{x\to a} \cos x &= \lim_{x\to a} \cos(a + (x - a)) = \lim_{x\to a} ((\cos a)\cos(x - a) - (\sin a)\sin(x - a)) \\ &= (\cos a)\cdot 1 - (\sin a)\cdot 0 = \cos a.\end{aligned}$$

2.7 Precise Definitions of Limits

2.7.1 Note that all the numbers in the interval $(1, 3)$ are within 1 unit of the number 2. So $|x - 2| < 1$ is true for all numbers in that interval. In fact, $\{x : 0 < |x - 2| < 1\}$ is exactly the set $(1, 3)$ with $x \neq 2$.

2.7.2 Note that all the numbers in the interval $(2, 6)$ are within 2 units of the number 4. So $|f(x) - 4| < \varepsilon$ for $\varepsilon = 2$ (or any number greater than 2).

2.7.3

a. This is symmetric about $x = 5$, because $\dfrac{1+9}{2} = 5$.

b. This is symmetric about $x = 5$, because $\dfrac{4+6}{2} = 5$.

c. This is not symmetric about $x = 5$, because $\dfrac{3+8}{2} \neq 5$.

d. This is symmetric about $x = 5$, because $\dfrac{4.5+5.5}{2} = 5$.

2.7.4 The set $\{x : |x - a| < \delta\}$ is the interval $(a - \delta, a + \delta)$ and $\{x : 0 < |x - a| < \delta\}$ is the set of all points in the interval $(a - \delta, a + \delta)$ excluding the point a.

2.7.5 $\lim_{x\to a} f(x) = L$ if for any arbitrarily small positive number ε, there exists a number δ, so that $f(x)$ is within ε units of L for any number x within δ units of a (but not including a itself).

2.7.6 The set of all x for which $|f(x) - L| < \varepsilon$ is the set of numbers so that the value of the function f at those numbers is within ε units of L.

2.7.7 We are given that $|f(x) - 5| < 0.1$ for values of x in the interval $(0, 5)$, so we need to ensure that the set of x values we are allowing fall in this interval.

Note that the number 0 is two units away from the number 2 and the number 5 is three units away from the number 2. In order to be sure that we are talking about numbers in the interval $(0, 5)$ when we write $|x - 2| < \delta$, we would need to have $\delta = 2$ (or a number less than 2). In fact, the set of numbers for which $|x - 2| < 2$ is the interval $(0, 4)$ which is a subset of $(0, 5)$.

If we were to allow δ to be any number greater than 2, then the set of all x so that $|x - 2| < \delta$ would include numbers less than 0, and those numbers aren't on the interval $(0, 5)$.

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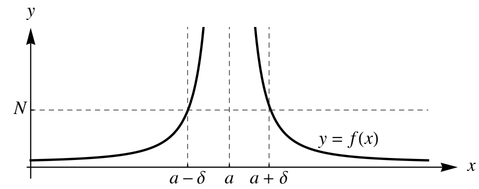
2.7.8

$\lim_{x \to a} f(x) = \infty$, if for any $N > 0$, there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.



2.7.9

a. In order for f to be within 2 units of 5, it appears that we need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1 unit of 5, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 0.5$.

2.7.10

a. In order for f to be within 1 unit of 4, it appears that we would need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1/2 unit of 4, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 1/2$.

2.7.11

a. In order for f to be within 3 units of 6, it appears that we would need x to be within 2 units of 3. So $\delta = 2$.

b. In order for f to be within 1 unit of 6, it appears that we would need x to be within 1/2 unit of 3. So $\delta = 1/2$.

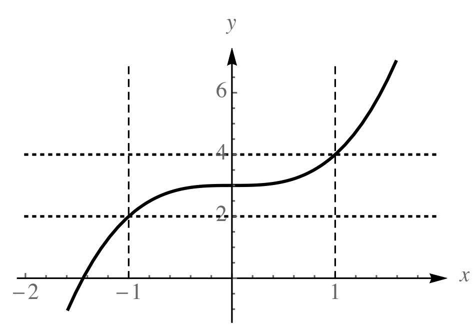
2.7.12

a. In order for f to be within 1 unit of 5, it appears that we would need x to be within 3 units of 4. So $\delta = 3$.

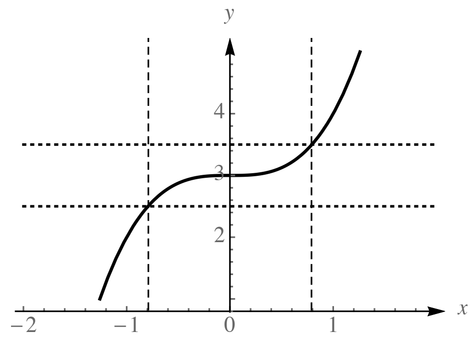
b. In order for f to be within 1/2 unit of 5, it appears that we would need x to be within 2 units of 4. So $\delta = 2$.

2.7.13

a. If $\varepsilon = 1$, we need $|x^3 + 3 - 3| < 1$. So we need $|x| < \sqrt[3]{1} = 1$ in order for this to happen. Thus $\delta = 1$ will suffice.

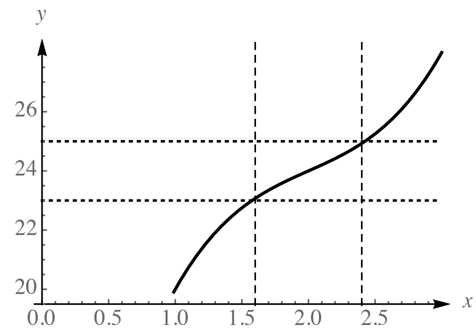


b. If $\varepsilon = 0.5$, we need $|x^3 + 3 - 3| < 0.5$. So we need $|x| < \sqrt[3]{0.5}$ in order for this to happen. Thus $\delta = \sqrt[3]{0.5} \approx 0.79$ will suffice.

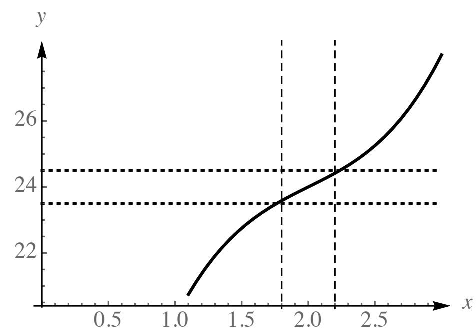


2.7.14

a. By looking at the graph, it appears that for $\varepsilon = 1$, we would need δ to be about 0.4 or less.



b. By looking at the graph, it appears that for $\varepsilon = 0.5$, we would need δ to be about 0.2 or less.



2.7.15

a. For $\varepsilon = 1$, the required value of δ would also be 1. A larger value of δ would work to the right of 2, but this is the largest one that would work to the left of 2.

b. For $\varepsilon = 1/2$, the required value of δ would also be $1/2$.

c. It appears that for a given value of ε, it would be wise to take $\delta = \min(\varepsilon, 2)$. This assures that the desired inequality is met on both sides of 2.

2.7.16

a. For $\varepsilon = 2$, the required value of δ would be 1 (or smaller). This is the largest value of δ that works on either side.

b. For $\varepsilon = 1$, the required value of δ would be $1/2$ (or smaller). This is the largest value of δ that works on the right of 4.

c. It appears that for a given value of ε, the corresponding value of $\delta = \min(5/2, \varepsilon/2)$.

2.7.17

a. For $\varepsilon = 2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of $\delta = \varepsilon/2$ or smaller appears to suffice.

2.7.18

a. For $\varepsilon = 1/2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1/4$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of 2ε or smaller appears to suffice.

2.7.19 For any $\varepsilon > 0$, let $\delta = \varepsilon/8$. Then if $0 < |x-1| < \delta$, we would have $|x-1| < \varepsilon/8$. Then $|8x-8| < \varepsilon$, so $|(8x+5)-13| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 1}(8x+5) = 13$.

2.7.20 For any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \varepsilon/2$. Then $|2x-6| < \varepsilon$, so $|-2x+6| < \varepsilon$, so $|(-2x+8)-2| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 3}(-2x+8) = 2$.

2.7.21 First note that if $x \neq 4$, $f(x) = \frac{x^2-16}{x-4} = x+4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-4| < \delta$. Then $x \neq 4$, so the function $f(x)$ can be described by $x+4$. Also, because $|x-4| < \delta$, we have $|x-4| < \varepsilon$. Thus $|(x+4)-8| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 4}\dfrac{x^2-16}{x-4} = 8$.

2.7.22 First note that if $x \neq 3$, $f(x) = \frac{x^2-7x+12}{x-3} = \frac{(x-4)(x-3)}{x-3} = x-4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-3| < \delta$. Then $x \neq 3$, so the function $f(x)$ can be described by $x-4$. Also, because $|x-3| < \delta$, we have $|x-3| < \varepsilon$. Thus $|(x-4)-(-1)| < \varepsilon$. This last inequality has the form $|f(x)-L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim_{x\to 3} f(x) = -1$.

2.7.23 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \varepsilon$. It follows that $||x|-0| = |x| = |x-0| < \delta = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \varepsilon$.

2.7.24 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \frac{\varepsilon}{5}$. It follows that $||5x|-0| = 5|x-0| < 5\delta = 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||5x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \frac{\varepsilon}{5}$.

2.7.25 Let $\varepsilon > 0$ be given and assume that $0 < |x-7| < \delta$ where $\delta = \varepsilon/3$. If $x < 7$, $|f(x)-9| = |3x-12-9| = 3|x-7| < 3\delta = 3(\varepsilon/3) = \varepsilon$; if $x > 7$, then $|f(x)-9| = |x+2-9| = |x-7| < \delta = \varepsilon/3 < \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-9| < \varepsilon$ whenever $0 < |x-7| < \delta$, provided $0 < \delta \le \varepsilon/3$.

2.7.26 Let $\varepsilon > 0$ be given and assume that $0 < |x-5| < \delta$ where $\delta = \varepsilon/4$. If $x < 5$, $|f(x)-4| = |2x-6-4| = 2|x-5| < 2\delta = 2(\varepsilon/4) = \varepsilon/2 < \varepsilon$; if $x > 5$, then $|f(x)-4| = |-4x+24-4| = |-4(x-5)| = 4|x-5| < 4\delta = 4(\varepsilon/4) = \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-4| < \varepsilon$ whenever $0 < |x-5| < \delta$, provided $0 < \delta \le \varepsilon/4$.

2.7.27 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-0| < \delta$, we would have $|x| < \sqrt{\varepsilon}$. But then $|x^2| < \varepsilon$, which has the form $|f(x)-L| < \varepsilon$. Thus, $\lim_{x\to 0} f(x) = 0$.

2.7.28 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \sqrt{\varepsilon}$. But then $|(x-3)^2| < \varepsilon$, which has the form $|f(x) - L| < \varepsilon$. Thus, $\lim_{x\to 3} f(x) = 0$.

2.7.29 Let $\varepsilon > 0$ be given and assume that $0 < |x-2| < \delta$ where $\delta = \min\{1, \varepsilon/8\}$. By factoring $x^2+3x-10$, we find that $|x^2+3x-10| = |x-2||x+5|$. Because $|x-2| < \delta$ and $\delta \le 1$, we have $|x-2| < 1$, which implies that $-1 < x-2 < 1$, or $1 < x < 3$. It follows that $|x+5| = x+5 \le 8$. We also know that $|x-2| < \varepsilon/8$ because $0 < |x-2| < \delta$ and $\delta \le \varepsilon/8$. Therefore $|x^2+3x-10| = |x-2||x+5| < (\varepsilon/8)\cdot 8 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2+3x-10| < \varepsilon$ whenever $0 < |x-2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/8\}$.

2.7.30 Let $\varepsilon > 0$ be given and assume that $0 < |x-4| < \delta$ where $\delta = \min\{1, \varepsilon/14\}$. Observe that $|2x^2-4x+1-17| = |2x^2-4x-16| = 2|x-4||x+2|$. Because $|x-4| < \delta$ and $\delta \le 1$, we have $|x-4| < 1$ which implies that $-1 < x-4 < 1$ or $3 < x < 5$. It follows that $|x+2| = x+2 \le 7$. We also know that $|x-4| < \varepsilon/14$ because $0 < |x-4| < \delta$ and $\delta \le \varepsilon/14$. Therefore $|2x^2-4x+1| = 2|x-4||x+2| < 2(\varepsilon/14)\cdot 7 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|2x^2-4x+1-17| < \varepsilon$ provided $0 < \delta \le \min\{1, \varepsilon/14\}$.

2.7.31 Let $\varepsilon > 0$ be given and assume that $0 < |x-(-3)| < \delta$ where $\delta = \varepsilon/2$. Using the inequality $||a|-|b|| \le |a-b|$ with $a = 2x$ and $b = -6$, it follows that $||2x|-6| = ||2x|-|-6|| \le |2x-(-6)| = 2|x-(-3)| < 2\delta = 2(\varepsilon/2) = \varepsilon$ and therefore $||2x|-6| < \varepsilon$. We have shown that for any $\varepsilon > 0$, $||2x|-6| < \varepsilon$ whenever $0 < |x-(-3)| < \delta$, provided $0 < \delta \le \varepsilon/2$.

2.7.32 Let $\varepsilon > 0$ be given and assume that $0 < |x-25| < \delta$ where $\delta = \min\{25, 5\varepsilon\}$. Because $|x-25| < \delta$ and $\delta \le 25$, we have $|x-25| < 25$, which implies that $-25 < x-25 < 25$ or $0 < x < 50$. Because $x > 0$, we have $\sqrt{x}+5 > 5$ and it follows that $\frac{1}{\sqrt{x}+5} < \frac{1}{5}$. Therefore

$$|\sqrt{x}-5| = \frac{|x-25|}{\sqrt{x}+5} \le \frac{|x-25|}{5} < \frac{\delta}{5} \le \frac{5\varepsilon}{5} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|\sqrt{x}-5| < \varepsilon$, provided $0 < \delta \le \min\{25, 5\varepsilon\}$.

2.7.33 Assume $|x-3| < 1$, as indicated in the hint. Then $2 < x < 4$, so $\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$, and thus $\left|\frac{1}{x}\right| < \frac{1}{2}$.

Also note that the expression $\left|\frac{1}{x} - \frac{1}{3}\right|$ can be written as $\left|\frac{x-3}{3x}\right|$.

Now let $\varepsilon > 0$ be given. Let $\delta = \min(6\varepsilon, 1)$. Now assume that $0 < |x-3| < \delta$. Then

$$|f(x) - L| = \left|\frac{x-3}{3x}\right| < \left|\frac{x-3}{6}\right| < \frac{6\varepsilon}{6} = \varepsilon.$$

Thus we have established that $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$ whenever $0 < |x-3| < \delta$.

2.7.34 Note that for $x \ne 4$, the expression $\frac{x-4}{\sqrt{x}-2} = \frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \sqrt{x}+2$. Also note that if $|x-4| < 1$, then x is between 3 and 5, so $\sqrt{x} > 0$. Then it follows that $\sqrt{x}+2 > 2$, and therefore $\frac{1}{\sqrt{x}+2} < \frac{1}{2}$. We will use this fact below.

Let $\varepsilon > 0$ be given. Let $\delta = \min(2\varepsilon, 1)$. Suppose that $0 < |x-4| < \delta$, so $|x-4| < 2\varepsilon$. We have

$$\begin{aligned} |f(x) - L| = |\sqrt{x}+2-4| = |\sqrt{x}-2| &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{|x-4|}{2} < \frac{2\varepsilon}{2} = \varepsilon. \end{aligned}$$

2.7.35 Assume $|x-(1/10)| < (1/20)$, as indicated in the hint. Then $1/20 < x < 3/20$, so $\frac{20}{3} < \frac{1}{x} < \frac{20}{1}$, and thus $\left|\frac{1}{x}\right| < 20$.

Also note that the expression $\left|\frac{1}{x} - 10\right|$ can be written as $\left|\frac{10x-1}{x}\right|$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(\varepsilon/200, 1/20)$. Now assume that $0 < |x-(1/10)| < \delta$. Then

$$\begin{aligned} |f(x) - L| = \left|\frac{10x-1}{x}\right| &< |(10x-1)\cdot 20| \\ &\le |x-(1/10)|\cdot 200 < \frac{\varepsilon}{200}\cdot 200 = \varepsilon. \end{aligned}$$

Thus we have established that $\left|\frac{1}{x} - 10\right| < \varepsilon$ whenever $0 < |x-(1/10)| < \delta$.

2.7.36 Multiplying both sides of the inequality $|\sin \frac{1}{x}| \le 1$ by $|x|$, we have $|x \sin \frac{1}{x}| \le |x|$. Let $\varepsilon > 0$ be given and assume that $0 < |x - 0| < \delta$ where $\delta = \varepsilon$. We have $|x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| \le |x| \le |x - 0| < \delta = \varepsilon$. Therefore it as been shown that for any $\varepsilon > 0$, $|x \sin \frac{1}{x} - 0| < \varepsilon$ whenever $0 < |x - 0| < \delta$, provided $0 < \delta \le \varepsilon$.

2.7.37 Let $\varepsilon > 0$ be given and assume that $0 < |x - 0| < \delta$ where $\delta = \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$. Because $|x - 0| < \delta$, we have $|x| < 1$ and $|x| < \sqrt{\varepsilon}/\sqrt{2}$, which implies that $x^2 < 1$ and $x^2 < \varepsilon/2$. It follows that $|x^2 + x^4 - 0| = x^2 + x^4 = x^2(1+x^2) \le \frac{\varepsilon}{2} \cdot 2 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2 + x^4 - 0| < \varepsilon$ whenever $0 < |x - 0| < \delta$, provided $0 < \delta \le \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$.

2.7.38 Let $f(x) = b$. Let $\varepsilon > 0$ be given and assume that $0 < |x - a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x) - b| = |b - b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b - b| < \varepsilon$ whenever $0 < |x - a| < \delta$, provided δ equals any positive number.

2.7.39 Let $m = 0$, then the proof is as follows: Let $\varepsilon > 0$ be given and assume that $0 < |x - a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x) - b| = |b - b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b - b| < \varepsilon$ whenever $0 < |x - a| < \delta$, provided δ equals any positive number.

Now assume that $m \ne 0$. Let $\varepsilon > 0$ be given and assume that $0 < |x - a| < \delta$ where $\delta = \varepsilon/|m|$. Then

$$|(mx + b) - (ma + b)| = |m||x - a| < |m|\delta = |m|(\varepsilon/|m|) = \varepsilon.$$

Therefore it has been shown that for any $\varepsilon > 0$, $|(mx + b) - (ma + b)| < \varepsilon$ whenever $0 < |x - a| < \delta$, provided $\delta = \varepsilon/|m|$.

2.7.40 Let $\varepsilon > 0$ be given and assume that $0 < |x - 3| < \delta$ where $\delta = \min\{1, \varepsilon/37\}$. By factoring $x^3 - 27$, we find that $|x^3 - 27| = |x - 3||x^2 + 3x + 9|$. Because $|x - 3| < \delta$ and $\delta \le 1$, we have $|x - 3| \le 1$, which implies that $-1 < x - 3 < 1$ or $2 < x < 4$. It follows that $|x^2 + 3x + 9| = x^2 + 3x + 9 \le 4^2 + 3(4) + 9 = 37$. We also know that $|x - 3| < \varepsilon/37$ because $0 < |x - 3| < \delta$ and $\delta \le \varepsilon/37$. Therefore $|x^3 - 27| = |x - 3||x^2 + 3x + 9| \le (\varepsilon/37) \cdot 37 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^3 - 27| < \varepsilon$ whenever $0 < |x - 3| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/37\}$.

2.7.41 Let $\varepsilon > 0$ be given and assume that $0 < |x - 1| < \delta$ where $\delta = \min\left\{\frac{1}{2}, \frac{8\varepsilon}{65}\right\}$. Observe that $|x^4 - 1| = |(x^2 - 1)(x^2 + 1)| = |x - 1||x + 1||x^2 + 1|$. Because $|x - 1| < \delta$ and $\delta \le \frac{1}{2}$, we have $|x - 1| < \frac{1}{2}$, which implies that $-\frac{1}{2} < x - 1 < \frac{1}{2}$, or $\frac{1}{2} < x < \frac{3}{2}$. It follows that $|x + 1| = x + 1 \le \frac{5}{2}$. Also $x^2 < \frac{9}{4}$, so $|x^2 + 1| = x^2 + 1 \le \frac{13}{4}$. We also know that $|x - 1| < \frac{8\varepsilon}{65}$ because $|x - 1| < \delta$ and $\delta \le \frac{8\varepsilon}{65}$. Therefore

$$|x^4 - 1| = |x - 1||x + 1||x^2 + 1| \le \frac{8\varepsilon}{65} \cdot \frac{5}{2} \cdot \frac{13}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|x^4 - 1| < \varepsilon$ whenever $0 < |x - 1| < \delta$, provided $0 < \delta = \min\left\{\frac{1}{2}, \frac{8\varepsilon}{65}\right\}$.

2.7.42 Note that if $|x - 5| < 1$, then $4 < x < 6$, so that $9 < x + 5 < 11$, so $|x + 5| < 11$. Note also that $16 < x^2 < 36$, so $\frac{1}{x^2} < \frac{1}{16}$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(1, \frac{400}{11}\varepsilon)$. Assume that $0 < |x - 5| < \delta$. Then

$$\begin{aligned} |f(x) - L| &= \left|\frac{1}{x^2} - \frac{1}{25}\right| = \frac{|x + 5||x - 5|}{25x^2} \\ &< \frac{11|x - 5|}{25x^2} < \frac{11}{25 \cdot 16}|x - 5| < \frac{11}{400}\frac{400\varepsilon}{11} = \varepsilon. \end{aligned}$$

2.7.43 Let $\varepsilon > 0$ be given.

Because $\lim_{x \to a} f(x) = L$, we know that there exists a $\delta_1 > 0$ so that $|f(x) - L| < \varepsilon/2$ when $0 < |x - a| < \delta_1$. Also, because $\lim_{x \to a} g(x) = M$, there exists a $\delta_2 > 0$ so that $|g(x) - M| < \varepsilon/2$ when $0 < |x - a| < \delta_2$.

Now let $\delta = \min(\delta_1, \delta_2)$.

Then if $0 < |x - a| < \delta$, we would have $|f(x) - g(x) - (L - M)| = |(f(x) - L) + (M - g(x))| \le |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. Note that the key inequality in this sentence follows from the triangle inequality.

2.7.44 First note that the theorem is trivially true if $c = 0$. So assume $c \neq 0$.

Let $\varepsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ so that if $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon/|c|$. But then $|c||f(x) - L| = |cf(x) - cL| < \varepsilon$, as desired. Thus, $\lim_{x\to a} cf(x) = cL$.

2.7.45 Let $N > 0$ be given. Let $\delta = 1/\sqrt{N}$. Then if $0 < |x - 4| < \delta$, we have $|x - 4| < 1/\sqrt{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x-4|} > \sqrt{N}$, and squaring both sides of this inequality yields $\dfrac{1}{(x-4)^2} > N$. Thus $\lim_{x\to 4} f(x) = \infty$.

2.7.46 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Then if $0 < |x-(-1)| < \delta$, we have $|x+1| < 1/\sqrt[4]{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x+1|} > \sqrt[4]{N}$, and raising both sides to the 4th power yields $\dfrac{1}{(x+1)^4} > N$. Thus $\lim_{x\to -1} f(x) = \infty$.

2.7.47 Let $N > 1$ be given. Let $\delta = 1/\sqrt{N-1}$. Suppose that $0 < |x - 0| < \delta$. Then $|x| < 1/\sqrt{N-1}$, and taking the reciprocal of both sides, we see that $1/|x| > \sqrt{N-1}$. Then squaring both sides yields $dsfrac1x^2 > N - 1$, so $\dfrac{1}{x^2} + 1 > N$. Thus $\lim_{x\to 0} f(x) = \infty$.

2.7.48 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N+1}$. Then if $0 < |x - 0| < \delta$, we would have $|x| < 1/\sqrt[4]{N+1}$. Taking the reciprocal of both sides yields $\dfrac{1}{|x|} > \sqrt[4]{N+1}$, and then raising both sides to the 4th power gives $\dfrac{1}{x^4} > N+1$, so $\dfrac{1}{x^4} - 1 > N$. Now because $-1 \leq \sin x \leq 1$, we can surmise that $\frac{1}{x^4} - \sin x > N$ as well, because $\dfrac{1}{x^4} - \sin x \geq \dfrac{1}{x^4} - 1$. Hence $\lim_{x\to 0}\left(\dfrac{1}{x^4} - \sin x\right) = \infty$.

2.7.49

a. False. In fact, if the statement is true for a specific value of δ_1, then it would be true for any value of $\delta < \delta_1$. This is because if $0 < |x - a| < \delta$, it would automatically follow that $0 < |x - a| < \delta_1$.

b. False. This statement is not equivalent to the definition – note that it says "for an arbitrary δ there exists an ε" rather than "for an arbitrary ε there exists a δ."

c. True. This is the definition of $\lim_{x\to a} f(x) = L$.

d. True. Both inequalities describe the set of x's which are within δ units of a.

2.7.50

a. We want it to be true that $|f(x)-2| < 0.25$. So we need $|x^2-2x+3-2| = |x^2-2x+1| = (x-1)^2 < 0.25$. Therefore we need $|x - 1| < \sqrt{0.25} = 0.5$. Thus we should let $\delta = 0.5$.

b. We want it to be true that $|f(x) - 2| < \varepsilon$. So we need $|x^2 - 2x + 3 - 2| = |x^2 - 2x + 1| = (x - 1)^2 < \varepsilon$. Therefore we need $|x - 1| < \sqrt{\varepsilon}$. Thus we should let $\delta = \sqrt{\varepsilon}$.

2.7.51 Because we are approaching a from the right, we are only considering values of x which are close to, but a little larger than a. The numbers x to the right of a which are within δ units of a satisfy $0 < x-a < \delta$.

2.7.52 Because we are approaching a from the left, we are only considering values of x which are close to, but a little smaller than a. The numbers x to the left of a which are within δ units of a satisfy $0 < a-x < \delta$.

2.7.53

a. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/2$. Suppose that $0 < x < \delta$. Then $0 < x < \varepsilon/2$ and

$$\begin{aligned}|f(x) - L| &= |2x - 4 - (-4)| = |2x| = 2|x| \\ &= 2x < \varepsilon.\end{aligned}$$

b. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/3$. Suppose that $0 < 0 - x < \delta$. Then $-\delta < x < 0$ and $-\varepsilon/3 < x < 0$, so $\varepsilon > -3x$. We have

$$\begin{aligned}|f(x) - L| &= |3x - 4 - (-4)| = |3x| = 3|x| \\ &= -3x < \varepsilon.\end{aligned}$$

c. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/3$. Because $\varepsilon/3 < \varepsilon/2$, we can argue that $|f(x) - L| < \varepsilon$ whenever $0 < |x| < \delta$ exactly as in the previous two parts of this problem.

2.7.54

a. This statement holds for $\delta = 2$ (or any number less than 2).

b. This statement holds for $\delta = 2$ (or any number less than 2).

c. This statement holds for $\delta = 1$ (or any number less than 1).

d. This statement holds for $\delta = .5$ (or any number less than 0.5).

2.7.55 Let $\varepsilon > 0$ be given, and let $\delta = \varepsilon^2$. Suppose that $0 < x < \delta$, which means that $x < \varepsilon^2$, so that $\sqrt{x} < \varepsilon$. Then we have

$$|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon.$$

as desired.

2.7.56

a. Suppose that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. Let $\varepsilon > 0$ be given. There exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta_1$, and there exists a number δ_2 so that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. It immediately follows that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, as desired.

b. Suppose $\lim_{x\to a} f(x) = L$, and let $\varepsilon > 0$ be given. We know that a δ exists so that $|f(x)-L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In particular, it must be the case that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$ and also that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$. Thus $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

2.7.57

a. We say that $\lim_{x\to a^+} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a < x < a + \delta.$$

b. We say that $\lim_{x\to a^-} f(x) = -\infty$ if for each negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad a - \delta < x < a.$$

c. We say that $\lim_{x\to a^-} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a - \delta < x < a.$$

2.7.58 Let $N < 0$ be given. Let $\delta = -1/N$, and suppose that $1 < x < 1 + \delta$. Then $1 < x < \frac{N-1}{N}$, so $\frac{1-N}{N} < -x < -1$, and therefore $1 + \frac{1-N}{N} < 1 - x < 0$, which can be written as $\frac{1}{N} < 1 - x < 0$. Taking reciprocals yields the inequality $N > \frac{1}{1-x}$, as desired.

2.7.59 Let $N > 0$ be given. Let $\delta = 1/N$, and suppose that $1 - \delta < x < 1$. Then $\frac{N-1}{N} < x < 1$, so $\frac{1-N}{N} > -x > -1$, and therefore $1 + \frac{1-N}{N} > 1 - x > 0$, which can be written as $\frac{1}{N} > 1 - x > 0$. Taking reciprocals yields the inequality $N < \frac{1}{1-x}$, as desired.

2.7.60 Let $M < 0$ be given. Let $\delta = \sqrt{-2/M}$. Suppose that $0 < |x-1| < \delta$. Then $(x-1)^2 < -2/M$, so $\frac{1}{(x-1)^2} > \frac{M}{-2}$, and $\frac{-2}{(x-1)^2} < M$, as desired.

2.7.61 Let $M < 0$ be given. Let $\delta = \sqrt[4]{-10/M}$. Suppose that $0 < |x+2| < \delta$. Then $(x+2)^4 < -10/M$, so $\frac{1}{(x+2)^4} > \frac{M}{-10}$, and $-\frac{10}{(x+2)^4} < M$, as desired.

2.7.62 Let $N > 0$ be given and let $N_1 = \max\{1, N - c\}$. Because $\lim_{x\to a} f(x) = \infty$ there exists $\delta > 0$ such that $f(x) > N_1$ whenever $0 < |x-a| < \delta$. It follows that $f(x) + c > N_1 + c \geq N - c + c = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + c > N$ whenever $0 < |x-a| < \delta$.

2.7.63 Let $N > 0$ be given. Because $\lim_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $f(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_1$. Similarly, because $\lim_{x\to a} g(x) = \infty$, there exists $\delta_2 > 0$ such that $g(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and assume that $0 < |x-a| < \delta$. Because $\delta = \min\{\delta_1, \delta_2\}$, $\delta \leq \delta_1$ and $\delta \leq \delta_2$. It follows that $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ and therefore $f(x) + g(x) > \frac{N}{2} + \frac{N}{2} = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + g(x) > N$ whenever $0 < |x-a| < \delta$.

2.7.64 Let $\varepsilon > 0$ be given. Let $N = \frac{10}{\varepsilon}$. Suppose that $x > N$. Then $x > \frac{10}{\varepsilon}$ so $0 < \frac{10}{x} < \varepsilon$. Thus, $|\frac{10}{x} - 0| < \varepsilon$, as desired.

2.7.65 Let $\varepsilon > 0$ be given. Let $N = 1/\varepsilon$. Suppose that $x > N$. Then $\frac{1}{x} < \varepsilon$, and so

$$|f(x) - L| = |2 + \frac{1}{x} - 2| < \varepsilon.$$

2.7.66 Let $M > 0$ be given. Let $N = 100M$. Suppose that $x > N$. Then $x > 100M$, so $\frac{x}{100} > M$, as desired.

2.7.67 Let $M > 0$ be given. Let $N = M - 1$. Suppose that $x > N$. Then $x > M - 1$, so $x + 1 > M$, and thus $\frac{x^2 + x}{x} > M$, as desired.

2.7.68 Let $\varepsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_1$. And because $\lim_{x\to a} h(x) = L$, there exists a number δ_2 so that $|h(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, and suppose that $0 < |x-a| < \delta$. Because $f(x) \leq g(x) \leq h(x)$ for x near a, we also have that $f(x) - L \leq g(x) - L \leq h(x) - L$. Now whenever x is within δ units of a (but $x \neq a$), we also note that $-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$. Therefore $|g(x) - L| < \varepsilon$, as desired.

2.7.69 Let $\varepsilon > 0$ be given. Let $N = \lfloor (1/\varepsilon) \rfloor + 1$. By assumption, there exists an integer $M > 0$ so that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Let $\delta = 1/M$.

Now assume $0 < |x - a| < \delta$. Then $|x - a| < 1/M$, and thus $|f(x) - L| < 1/N$. But then

$$|f(x) - L| < \frac{1}{\lfloor (1/\varepsilon) \rfloor + 1} < \varepsilon,$$

as desired.

2.7.70 Suppose that $\varepsilon = 1$. Then no matter what δ is, there are numbers in the set $0 < |x - 2| < \delta$ so that $|f(x) - 2| > \varepsilon$. For example, when x is only slightly greater than 2, the value of $|f(x) - 2|$ will be 2 or more.

2.7.71 Let $f(x) = \dfrac{|x|}{x}$ and suppose $\lim_{x \to 0} f(x)$ does exist and is equal to L. Let $\varepsilon = 1/2$. There must be a value of δ so that when $0 < |x| < \delta$, $|f(x) - L| < 1/2$. Now consider the numbers $\delta/3$ and $-\delta/3$, both of which are within δ of 0. We have $f(\delta/3) = 1$ and $f(-\delta/3) = -1$. However, it is impossible for both $|1 - L| < 1/2$ and $|-1 - L| < 1/2$, because the former implies that $1/2 < L < 3/2$ and the latter implies that $-3/2 < L < -1/2$. Thus $\lim_{x \to 0} f(x)$ does not exist.

2.7.72 Suppose that $\lim_{x \to a} f(x)$ exists and is equal to L. Let $\varepsilon = 1/2$. By the definition of limit, there must be a number δ so that $|f(x) - L| < \frac{1}{2}$ whenever $0 < |x - a| < \delta$. Now in every set of the form $(a, a + \delta)$ there are both rational and irrational numbers, so there will be value of f equal to both 0 and 1. Thus we have $|0 - L| < 1/2$, which means that L lies in the interval $(-1/2, 1/2)$, and we have $|1 - L| < 1/2$, which means that L lies in the interval $(1/2, 3/2)$. Because these both can't be true, we have a contradiction.

2.7.73 Because f is continuous at a, we know that $\lim_{x \to a} f(x)$ exists and is equal to $f(a) > 0$. Let $\varepsilon = f(a)/3$. Then there is a number $\delta > 0$ so that $|f(x) - f(a)| < f(a)/3$ whenever $|x - a| < \delta$. Then whenever x lies in the interval $(a - \delta, a + \delta)$ we have $-f(a)/3 \le f(x) - f(a) \le f(a)/3$, so $2f(a)/3 \le f(x) \le 4f(a)/3$, so f is positive in this interval.

2.7.74 Using the triangle inequality, we have $|a| = |(a-b)+b| \le |a-b|+|b|$. This implies that $|a| \le |a-b|+|b|$ or $|a| - |b| \le |a - b|$. A similar argument shows that $|b| - |a| \le |a - b|$. Because the expression $||a| - |b||$ is equal to either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \le |a - b|$.

Chapter Two Review

1

a. False. Because $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1} = \lim_{x \to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$, f doesn't have a vertical asymptote at $x = 1$.

b. False. In general, these methods are too imprecise to produce accurate results.

c. False. For example, the function $f(x) = \begin{cases} 2x & \text{if } x < 0; \\ 1 & \text{if } x = 0; \\ 4x & \text{if } x > 0 \end{cases}$ has a limit of 0 as $x \to 0$, but $f(0) = 1$.

d. True. When we say that a limit exists, we are saying that there is a real number L that the function is approaching. If the limit of the function is ∞, it is still the case that there is no real number that the function is approaching. (There is no real number called "infinity.")

e. False. It could be the case that $\lim_{x \to a^-} f(x) = 1$ and $\lim_{x \to a^+} f(x) = 2$.

f. False.

g. False. For example, the function $f(x) = \begin{cases} 2 & \text{if } 0 < x < 1; \\ 3 & \text{if } 1 \le x < 2, \end{cases}$ is continuous on $(0,1)$, and on $[1,2)$, but isn't continuous on $(0,2)$.

h. True. $\lim_{x\to a} f(x) = f(a)$ if and only if f is continuous at a.

2 $s(1) = 48$ and $s(1.5) = 60$, so the average velocity over the time period is

$$\frac{s(1.5)-s(1)}{1.5-1} = \frac{60-48}{0.5} = 24\,\text{ft/s}.$$

3 For various values of b, we calculate $v_{\text{avg}} = \dfrac{s(b)-s(1.5)}{b-1.5}$.

b	1.6	1.51	1.501	1.5001	1.50001
v_{avg}	10.4	11.84	11.984	11.9984	11.9998

We estimate that the instantaneous velocity is 12.

4

a. $f(-1) = 1$

b. $\lim_{x\to -1^-} f(x) = 3.$

c. $\lim_{x\to -1^+} f(x) = 1.$

d. $\lim_{x\to -1} f(x)$ does not exist.

e. $f(1) = 5.$

f. $\lim_{x\to 1} f(x) = 5.$

g. $\lim_{x\to 2} f(x) = 4.$

h. $\lim_{x\to 3^-} f(x) = 3.$

i. $\lim_{x\to 3^+} f(x) = 5.$

j. $\lim_{x\to 3} f(x)$ does not exist.

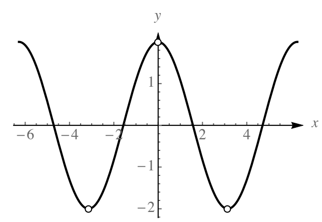
5 This function is discontinuous at $x=-1$, at $x=1$, and at $x=3$. At $x=-1$ it is discontinuous because $\lim_{x\to -1} f(x)$ does not exist. At $x=1$, it is discontinuous because $\lim_{x\to 1} f(x) \ne f(1)$. At $x=3$, it is discontinuous because $f(3)$ does not exist, and because $\lim_{x\to 3} f(x)$ does not exist.

6

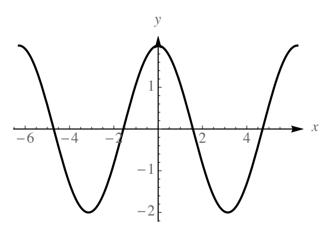
a. The graph drawn by most graphing calculators and computer algebra systems doesn't show the discontinuities where $\sin\theta = 0$.

b. It appears to be equal to 2

c. Using a trigonometric identity, $\lim_{\theta\to 0} \dfrac{\sin 2\theta}{\sin\theta} = \lim_{\theta\to 0} \dfrac{2\sin\theta\cos\theta}{\sin\theta}$. This can then be seen to be $\lim_{\theta\to 0} 2\cos\theta = 2.$



True graph, showing discontinuities where $\sin\theta = 0$.



Graph shown without discontinuities.

7

a.

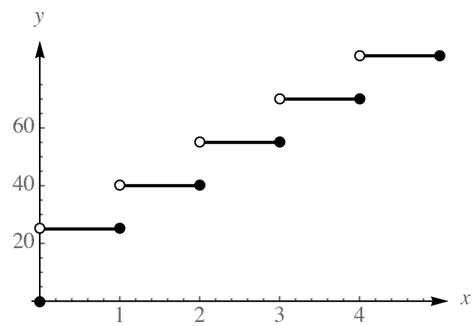
x	$0.9\pi/4$	$0.99\pi/4$	$0.999\pi/4$	$0.9999\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

x	$1.1\pi/4$	$1.01\pi/4$	$1.001\pi/4$	$1.0001\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

The limit appears to be approximately 1.4142.

b. $\displaystyle\lim_{x\to\pi/4} \frac{\cos 2x}{\cos x - \sin x} = \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \lim_{x\to\pi/4} (\cos x + \sin x) = \sqrt{2}.$

8



a.

b. $\displaystyle\lim_{t\to 2.9} f(t) = 55.$

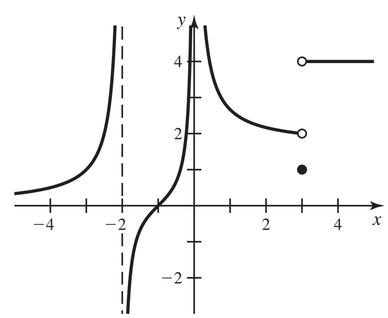
c. $\displaystyle\lim_{t\to 3^-} f(t) = 55$ and $\displaystyle\lim_{t\to 3^+} f(t) = 70.$

d. The cost of the rental jumps by \$15 exactly at $t = 3$. A rental lasting slightly less than 3 days cost \$55 and rentals lasting slightly more than 3 days cost \$70.

e. The function f is continuous everywhere except at the integers. The cost of the rental jumps by \$15 at each integer.

9

There are infinitely many different correct functions which you could draw. One of them is:



10 $\displaystyle\lim_{x\to 1000} 18\pi^2 = 18\pi^2.$

11 $\displaystyle\lim_{x\to 1} \sqrt{5x+6} = \sqrt{11}.$

12

$$\lim_{h\to 0} \frac{\sqrt{5x+5h}-\sqrt{5x}}{h} \cdot \frac{\sqrt{5x+5h}+\sqrt{5x}}{\sqrt{5x+5h}+\sqrt{5x}} = \lim_{h\to 0} \frac{(5x+5h)-5x}{h(\sqrt{5x+5h}+\sqrt{5x})} = \lim_{h\to 0} \frac{5}{\sqrt{5x+5h}+\sqrt{5x}} = \frac{5}{2\sqrt{5x}}.$$

13

$$\lim_{h\to 0} \frac{(h+6)^2+(h+6)-42}{h} = \lim_{h\to 0} \frac{h^2+12h+36+h+6-42}{h} = \lim_{h\to 0} \frac{h^2+13h}{h} = \lim_{h\to 0}(h+13) = 13.$$

14 Factoring the numerator as the difference of squares, we have

$$\begin{aligned}\lim_{x\to a}\frac{(3x+1)^2-(3a+1)^2}{x-a} &= \lim_{x\to a}\frac{((3x+1)-(3a+1))((3x+1)+(3a+1))}{x-a}\\ &= \lim_{x\to a}\frac{(3x-3a)(3x+3a+2)}{x-a}\\ &= 3\lim_{x\to a}(3x+3a+2)\\ &= 3(3a+3a+2) = 18a+6.\end{aligned}$$

15 $\lim_{x\to 1}\frac{x^3-7x^2+12x}{4-x} = \frac{1-7+12}{4-1} = \frac{6}{3} = 2.$

16 $\lim_{x\to 4}\frac{x^3-7x^2+12x}{4-x} = \lim_{x\to 4}\frac{x(x-3)(x-4)}{4-x} = \lim_{x\to 4} x(3-x) = -4.$

17 $\lim_{x\to 1}\frac{1-x^2}{x^2-8x+7} = \lim_{x\to 1}\frac{(1-x)(1+x)}{(x-7)(x-1)} = \lim_{x\to 1}\frac{-(x+1)}{x-7} = \frac{1}{3}.$

18 $\lim_{x\to 3}\frac{\sqrt{3x+16}-5}{x-3}\cdot\frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5} = \lim_{x\to 3}\frac{3(x-3)}{(x-3)(\sqrt{3x+16}+5)} = \lim_{x\to 3}\frac{3}{\sqrt{3x+16}+5} = \frac{3}{10}.$

19

$$\begin{aligned}\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) &= \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2(x-3)\sqrt{x+1}}\cdot\frac{(2+\sqrt{x+1})}{(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{4-(x+1)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{-(x-3)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3} -\frac{1}{2\sqrt{x+1}(2+\sqrt{x+1})} = -\frac{1}{16}.\end{aligned}$$

20 $\lim_{t\to 1/3}\frac{t-\frac{1}{3}}{(3t-1)^2} = \lim_{t\to 1/3}\frac{3t-1}{3(3t-1)^2} = \lim_{t\to 1/3}\frac{1}{3(3t-1)}$, which does not exist.

21 $\lim_{x\to 3}\frac{x^4-81}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{x-3} = \lim_{x\to 3}(x+3)(x^2+9) = 108.$

22 Note that $\frac{p^5-1}{p-1} = p^4+p^3+p^2+p+1$. (Use long division.)
$\lim_{p\to 1}\frac{p^5-1}{p-1} = \lim_{p\to 1}(p^4+p^3+p^2+p+1) = 5.$

23 $\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{x-81} = \lim_{x\to 81}\frac{\sqrt[4]{x}-3}{(\sqrt{x}+9)(\sqrt[4]{x}+3)(\sqrt[4]{x}-3)} = \lim_{x\to 81}\frac{1}{(\sqrt{x}+9)(\sqrt[4]{x}+3)} = \frac{1}{108}.$

24 $\lim_{\theta\to \pi/2}\frac{\sin^2\theta-5\sin\theta+4}{\sin^2\theta-1} = \lim_{\theta\to \pi/2}\frac{(\sin\theta-4)(\sin\theta-1)}{(\sin\theta-1)(\sin\theta+1)} = \lim_{\theta\to \pi/2}\frac{\sin\theta-4}{\sin\theta+1} = \frac{1-4}{1+1} = -\frac{3}{2}.$

25 $\lim_{x\to \pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\pi/2} = \frac{0}{\pi} = 0.$

26 The domain of $f(x) = \sqrt{\frac{x-1}{x-3}}$ is $(-\infty, 1]$ and $(3,\infty)$, so $\lim_{x\to 1^+} f(x)$ doesn't exist.
However, we have $\lim_{x\to 1^-} f(x) = 0.$

27 $\lim_{x\to 5} \frac{x-7}{x(x-5)^2} = -\infty.$

28 $\lim_{x\to -5^+} \frac{x-5}{x+5} = -\infty.$

29 $\lim_{x\to 3^-} \frac{x-4}{x^2-3x} = \lim_{x\to 3^-} \frac{x-4}{x(x-3)} = \infty.$

30 $\lim_{x\to 0^+} \frac{u-1}{\sin u} = -\infty.$

31 $\lim_{x\to 1^+} \frac{4x^3-4x^2}{|x-1|} = \lim_{x\to 1^+} \frac{4x^2(x-1)}{x-1} = \lim_{x\to 1^+} 4x^2 = 4.$

32 The expression $2x-4 = 2(x-2)$ is negative for $x < 2$, so $|2x-4| = -2(x-2)$. Therefore, $\lim_{x\to 2^-} \frac{|2x-4|}{x^2-5x+6} = \lim_{x\to 2^-} \frac{-2(x-2)}{(x-2)(x-3)} = \lim_{x\to 2^-} \frac{-2}{x-3} = 2.$

33 $\lim_{x\to 0^-} \frac{2}{\tan x} = -\infty.$

34 First note that for all x, $\sqrt{x^4} = x^2$. Then we have

$$\lim_{x\to\infty} \frac{(4x^2+3x+1)}{\sqrt{8x^4+2}} \cdot \frac{1/x^2}{1/\sqrt{x^4}} = \lim_{x\to\infty} \frac{4+(3/x)+(1/x^2)}{\sqrt{8+(2/x^4)}} = \frac{4}{\sqrt{8}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

35 $\lim_{x\to\infty} \frac{2x-3}{4x+10} = \lim_{x\to\infty} \frac{2-(3/x)}{4+(10/x)} = \frac{2}{4} = \frac{1}{2}.$

36 $\lim_{x\to\infty} \frac{x^4-1}{x^5+2} = \lim_{x\to\infty} \frac{(1/x)-(1/x^5)}{1+(2/x^5)} = \frac{0-0}{1+0} = 0.$

37 Note that for $x < 0$, $x = -\sqrt{x^2}$. Then we have

$$\lim_{x\to -\infty} \frac{(3x+1)}{\sqrt{ax^2+2}} \cdot \frac{1/x}{-1/\sqrt{x^2}} = -\lim_{x\to -\infty} \frac{3+(1/x)}{\sqrt{a+(2/x^2)}} = -\frac{3}{\sqrt{a}}.$$

38 We have

$$\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2-b} = \lim_{x\to\infty} \frac{(\sqrt{x^2+ax} - \sqrt{x^2-b})}{1} \cdot \frac{(\sqrt{x^2+ax} + \sqrt{x^2-b})}{(\sqrt{x^2+ax} + \sqrt{x^2-b})}$$
$$= \lim_{x\to\infty} \frac{ax+b}{\sqrt{x^2+ax} + \sqrt{x^2-b}}.$$

Now noting that $x = \sqrt{x^2}$ for $x > 0$ we have

$$\lim_{x\to\infty} \frac{(ax+b)}{(\sqrt{x^2+ax} + \sqrt{x^2-b})} \cdot \frac{1/x}{1/\sqrt{x^2}} = \lim_{x\to\infty} \frac{a+b/x}{\sqrt{1+a/x} + \sqrt{1-b/x}} = \frac{a}{2}.$$

39 We multiply the numerator and denominator by the conjugate of the denominator (i.e., the expression $\sqrt{x^2-ax} + \sqrt{x^2-x}$). This gives

$$\lim_{x\to\infty} \frac{\sqrt{x^2-ax} + \sqrt{x^2-x}}{(x^2-ax)-(x^2-x)} = \lim_{x\to\infty} \frac{\sqrt{x^2-ax} + \sqrt{x^2-x}}{-ax+x}.$$

We now multiply by $\frac{1/\sqrt{x^2}}{1/x}$ to obtain

$$\lim_{x\to\infty} \frac{\sqrt{1-(a/x)} + \sqrt{1-(1/x)}}{-a+1} = \frac{2}{1-a}.$$

40 $\lim_{z\to\infty}\left(e^{-2z}+\frac{2}{z}\right)=0+0=0.$

41 $\lim_{x\to\infty}(3\tan^{-1}x+2)=\frac{3\pi}{2}+2.$

42 $\lim_{x\to-\infty}(-3x^3+5)=\infty.$

43 For $x<1$, $|x-1|+x=-(x-1)+x=1$. Therefore $\lim_{x\to-\infty}(|x-1|+x)=\lim_{x\to-\infty}1=1.$
For $x>1$, $|x-1|+x=x-1+x=2x-1$. Therefore $\lim_{x\to\infty}(|x-1|+x)=\lim_{x\to\infty}2x-1=\infty.$

44 For $x<2$, $|x-2|+x=-(x-2)+x=2$. Therefore $\lim_{x\to-\infty}\frac{|x-2|+x}{x}=\lim_{x\to-\infty}\frac{2}{x}=0.$
For $x>2$, $|x-2|+x=x-2+x=2x-2$. Therefore $\lim_{x\to\infty}\frac{|x-2|+x}{x}=\lim_{x\to\infty}\frac{2x-2}{x}=2.$

45 $\lim_{w\to\infty}\frac{\ln w^2}{\ln w^3+1}=\lim_{w\to\infty}\frac{2\ln w}{(3\ln w+1)}\cdot\frac{1/\ln w}{1/\ln w}=\lim_{w\to\infty}\frac{2}{3+(1/\ln w)}=\frac{2}{3}.$

46 $\lim_{r\to-\infty}\frac{1}{2+e^r}=\frac{1}{2+0}=\frac{1}{2}.$
$\lim_{r\to\infty}\frac{1}{2+e^r}=0.$

47 $\lim_{r\to\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{5r}}{1/e^{5r}}=\lim_{r\to\infty}\frac{(2/e^r)+3}{(7/e^r)-9}=\frac{3}{-9}=-\frac{1}{3}.$
$\lim_{r\to-\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{4r}}{1/e^{4r}}=\lim_{r\to-\infty}\frac{2+3e^r}{7-9e^r}=\frac{2}{7}.$

48 Because $-1\le\sin x\le1$, $-e^x\le e^x\sin x\le e^x$. Because $\lim_{x\to-\infty}-e^x=\lim_{x\to-\infty}e^x=0$, we can conclude that $\lim_{x\to-\infty}e^x\sin x=0$ by the Squeeze Theorem.

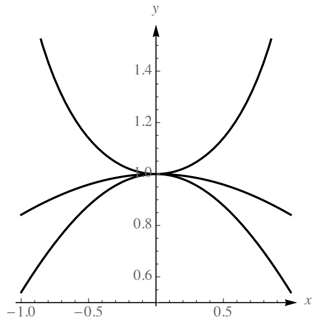
49 We know that $0\le\cos^4x\le1$. Dividing each part of this inequality by x^2+x+1 and then adding 5, we have $5\le5+\frac{\cos^4x}{x^2+x+1}\le5+\frac{1}{x^2+x+1}$. Note that $\lim_{x\to\infty}5=5$ and $\lim_{x\to\infty}\left(5+\frac{1}{x^2+x+1}\right)=5$, so by the Squeeze Theorem we can conclude that $\lim_{x\to\infty}\left(5+\frac{\cos^4x}{x^2+x+1}\right)=5.$

50 Recall that $-1\le\cos t\le1$, and that $e^{3t}>0$ for all t. Thus $-\frac{1}{e^{3t}}\le\frac{\cos t}{e^{3t}}\le\frac{1}{e^{3t}}$. Because $\lim_{t\to\infty}\frac{1}{e^{3t}}=\lim_{t\to\infty}-\frac{1}{e^{3t}}=0$, we can conclude $\lim_{t\to\infty}\frac{\cos t}{e^{3t}}=0$ by the Squeeze Theorem.

51 $\lim_{x\to1^-}\frac{x}{\ln x}=-\infty.$

52 Note that $\lim_{x\to0}(\sin^2x+1)=1$. Thus if $1\le g(x)\le\sin^2x+1$, the Squeeze Theorem assures us that $\lim_{x\to0}g(x)=1$ as well.

53

a. 

b. Because $\lim_{x\to 0} \cos x = \lim_{x\to 0} \frac{1}{\cos x} = 1$, the Squeeze Theorem assures us that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ as well.

54

First note that $f(x) = \frac{x^2-5x+6}{x^2-2x} = \frac{(x-3)(x-2)}{x(x-2)}$.

a. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x-3)(x-2)}{x(x-2)} = \infty$.

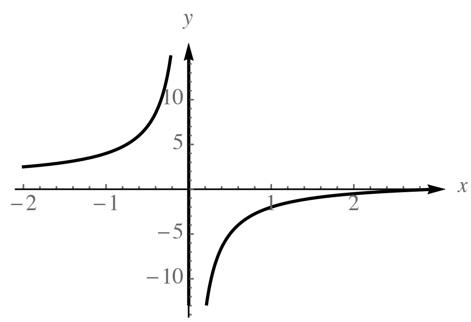
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x-3)(x-2)}{x(x-2)} = -\infty$.

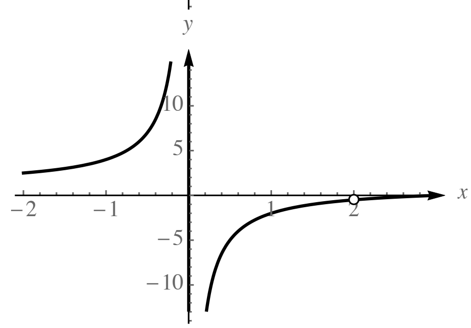
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-3}{x} = -\frac{1}{2}$.

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-3}{x} = -\frac{1}{2}$.

b. By the above calculations and the definition of vertical asymptote, f has a vertical asymptote at $x = 0$.

c. Note that the actual graph has a "hole" at the point $(2, -1/2)$, because $x = 2$ isn't in the domain, but $\lim_{x\to 2} f(x) = -1/2$.





55 $\lim_{x\to\infty} \frac{4x^3+1}{1-x^3} = \lim_{x\to\infty} \frac{4+(1/x^3)}{(1/x^3)-1} = \frac{4+0}{0-1} = -4$. A similar result holds as $x \to -\infty$. Thus, $y = -4$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

56 Note that $\sqrt{x^{12}} = x^6$ for all x. We have $\lim_{x\to\pm\infty} \frac{(x^6+1)}{\sqrt{16x^{14}+1}} \cdot \frac{1/x^6}{1/\sqrt{x^{12}}} = \lim_{x\to\pm\infty} \frac{1+(1/x^6)}{\sqrt{16x^2+(1/x^{12})}} = 0$.

57 $\lim_{x\to\infty} (1-e^{-2x}) = 1$, while $\lim_{x\to-\infty} (1-e^{-2x}) = -\infty$.
$y = 1$ is a horizontal asymptote as $x \to \infty$.

58 $\lim_{x\to\infty} \frac{1}{\ln x^2} = 0$, and $\lim_{x\to-\infty} \frac{1}{\ln x^2} = 0$, so $y = 0$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

59 $\lim_{x\to\infty} \frac{(6e^x+20)}{(3e^x+4)} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{6+(20/e^x)}{3+(4/e^x)} = \frac{6}{3} = 2$.
$\lim_{x\to-\infty} \frac{6e^x+20}{3e^x+4} = \frac{0+20}{0+4} = 5$.

60 First note that $\sqrt{\frac{1}{x^2}} = \left|\frac{1}{x}\right| = \begin{cases} \frac{1}{x} & \text{if } x > 0; \\ -\frac{1}{x} & \text{if } x < 0. \end{cases}$.

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{\sqrt{9+\frac{1}{x}}} = \frac{1}{3}.$$

On the other hand, $\lim_{x\to-\infty} \frac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \frac{1+(1/x)}{-\sqrt{9+\frac{1}{x}}} = -\frac{1}{3}$.

So $y = \frac{1}{3}$ is a horizontal asymptote as $x \to \infty$, and $y = -\frac{1}{3}$ is a horizontal asymptote as $x \to -\infty$.

61

a. $\lim_{x\to\infty} \frac{(3x^2+5x+7)}{(x+1)} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{3x+5+(7/x)}{1+(1/x)} = \infty$.

$\lim_{x\to-\infty} \frac{(3x^2+5x+7)}{(x+1)} \cdot \frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{3x+5+(7/x)}{1+(1/x)} = -\infty$.

b. By long division, we see that $\frac{3x^2+5x+7}{x+1} = 3x+2+\frac{5}{x+1}$, so the line $y = 3x+2$ is a slant asymptote.

62

a. $\lim_{x\to\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}$.

$\lim_{x\to-\infty} \frac{9x^2+4}{(2x-1)^2} = \lim_{x\to-\infty} \frac{9x^2+4}{4x^2-4x+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{9+4/x^2}{4-4/x+1/x^2} = \frac{9}{4}$.

b. Because there is a horizontal asymptote, there is not a slant asymptote.

63

a. $\lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = -\infty$.

$\lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to-\infty} \frac{1+x-2x^2-x^3}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1/x^2+1/x-2-x}{1+1/x^2} = \infty$.

b. By long division, we can write $f(x)$ as $f(x) = -x-2+\frac{2x+3}{x^2+1}$, so the line $y = -x-2$ is the slant asymptote.

64

a. $\lim_{x\to\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty$.

$\lim_{x\to-\infty} \frac{x(x+2)^3}{3x^2-4x} = \lim_{x\to-\infty} \frac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{x^2+6x+12+8/x}{3-4/x} = \infty$.

b. Because the degree of the numerator of this rational function is two more than the degree of the denominator, there is no slant asymptote.

65

a. $\lim_{x\to\infty} \frac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = \infty$.

$\lim_{x\to-\infty} \frac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = -\infty$.

b. By long division, we can write $\frac{4x^3+x^2+7}{x^2-x+1} = 4x+5+\frac{x+2}{x^2-x+1}$. Therefore $y = 4x+5$ is a slant asymptote.

66 Note that $f(x) = \frac{2x^2+6}{2x^2+3x-2} = \frac{2(x^2+3)}{(2x-1)(x+2)}$.

We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+6/x^2}{2+3/x-2/x^2} = 1$. A similar result holds as $x \to -\infty$.
$\lim_{x\to 1/2^-} f(x) = -\infty$. $\lim_{x\to 1/2^+} f(x) = \infty$.
$\lim_{x\to -2^-} f(x) = \infty$. $\lim_{x\to -2^+} f(x) = -\infty$.
Thus, $y = 1$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$. Also, $x = \frac{1}{2}$ and $x = -2$ are vertical asymptotes.

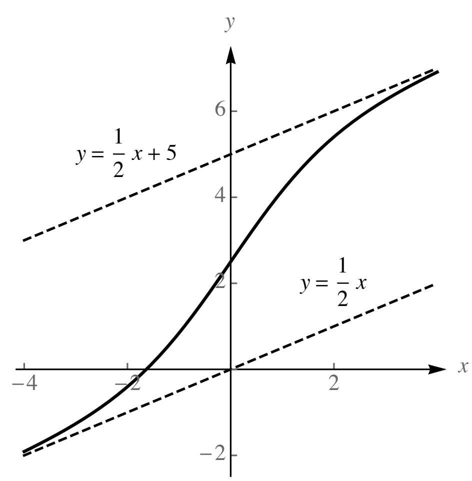
67 Recall that $\tan^{-1} x = 0$ only for $x = 0$. The only vertical asymptote is $x = 0$.
$\lim_{x\to\infty} \frac{1}{\tan^{-1} x} = \frac{1}{\pi/2} = \frac{2}{\pi}$.
$\lim_{x\to-\infty} \frac{1}{\tan^{-1} x} = \frac{1}{-\pi/2} = -\frac{2}{\pi}$. So $y = \frac{2}{\pi}$ is a horizontal asymptote as $x \to \infty$ and $y = -\frac{2}{\pi}$ is a horizontal asymptote as $x \to -\infty$.

68 By long division, we can write $\frac{2x^2-7}{x-2} = 2x+4+\frac{1}{x-2}$, so $y = 2x+4$ is a slant asymptote. Also, $\lim_{x\to 2^+} \frac{2x^2-7}{x-2} = \infty$, so $x = 2$ is a vertical asymptote.

69 Observe that

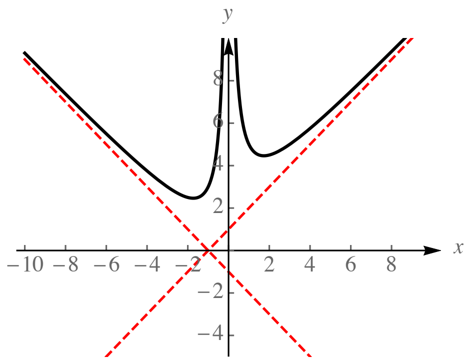
$$f(x) = \frac{x+xe^x+10e^x}{2(e^x+1)} = \frac{x(1+e^x)+10e^x}{2(e^x+1)} = \frac{1}{2}x + \frac{5e^x}{e^x+1}.$$

Because $\lim_{x\to\infty} \frac{5e^x}{e^x+1} = \lim_{x\to\infty} \frac{5}{1+(1/e^x)} = 5$, the graph of f and the line $y = \frac{1}{2}x+5$ approach each other as $x \to \infty$. Similarly, $\lim_{x\to-\infty} \frac{5e^x}{e^x+1} = \frac{0}{0+1} = 0$ and therefore the graph of f and the line $y = \frac{1}{2}x$ approach each other as $x \to -\infty$.



70 Observe that $\lim_{x\to0^+} \frac{x^2+x+3}{|x|} = \lim_{x\to0^+} \frac{x^2+x+3}{x} = \lim_{x\to0^+} x+1+(3/x) = \infty$ and $\lim_{x\to0^-} \frac{x^2+x+3}{|x|} = \lim_{x\to0^-} \frac{x^2+x+3}{-x} = -\lim_{x\to0^-} (x+1+(3/x)) = -\infty$. For $x > 0$, we have $f(x) = \frac{x^2+x+3}{x} = x+1+\frac{3}{x}$, so $y = x+1$ is a slant asymptote as $x \to \infty$. For $x < 0$, we have $f(x) = \frac{x^2+x+3}{-x} = -x-1-\frac{3}{x}$, so

$y = -x - 1$ is a slant asymptote as $x \to -\infty$. So the function has one vertical asymptote $x = 0$ and two slant asymptotes, $y = x + 1$ and $y = -x - 1$.



71 The function f is not continuous at 5 because $f(5)$ is not defined.

72 g is discontinuous at 4 because $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = 8 \neq g(4)$.

73 Observe that $h(5) = -2(5) + 14 = 4$. Because $\lim_{x\to 5^-} h(x) = \lim_{x\to 5^-}(-2x + 14) = 4$ and $\lim_{x\to 5^+} h(x) = \lim_{x\to 5^+} \sqrt{x^2 - 9} = \sqrt{25 - 9} = 4$, we have $\lim_{x\to 5} h(x) = 4$. Thus f is continuous at $x = 5$.

74 Observe that $g(2) = -2$ and $\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^3 - 5x^2 + 6x}{x-2} = \lim_{x\to 2} \frac{x(x-2)(x-3)}{x-2} = \lim_{x\to 2} x(x-3) = -2$. Therefore g is continuous at $x = 2$.

75 The domain of f is $(-\infty, -\sqrt{5}]$ and $[\sqrt{5}, \infty)$, and f is continuous on that domain. It is left continuous at $-\sqrt{5}$ and right continuous at $\sqrt{5}$.

76 The domain of g is $[2, \infty)$, and it is continuous on that domain. It is continuous from the right at $x = 2$.

77 The domain of h is $(-\infty, -5)$, $(-5, 0)$, $(0, 5)$, $(5, \infty)$, and like all rational functions, it is continuous on its domain.

78 g is the composition of two functions which are defined and continuous on $(-\infty, \infty)$, so g is continuous on that interval as well.

79 In order for g to be left continuous at 1, it is necessary that $\lim_{x\to 1^-} g(x) = g(1)$, which means that $a = 3$. In order for g to be right continuous at 1, it is necessary that $\lim_{x\to 1^+} g(x) = g(1)$, which means that $a + b = 3 + b = 3$, so $b = 0$.

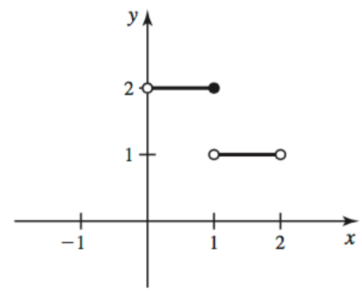
80

a. Because the domain of h is $(-\infty, -3]$ and $[3, \infty)$, there is no way that h can be left continuous at 3.

b. h is right continuous at 3, because $\lim_{x\to 3^+} h(x) = 0 = h(3)$.

81

One such possible graph is pictured to the right.



82

a. Consider the function $f(x) = x^5 + 7x + 5$. f is continuous everywhere, and $f(-1) = -3 < 0$ while $f(0) = 5 > 0$. Therefore, 0 is an intermediate value between $f(-1)$ and $f(0)$. By the Intermediate Value Theorem, there must a number c between 0 and 1 so that $f(c) = 0$.

b. Using a computer algebra system, one can find that $c \approx -0.691671$ is a root.

83

a. Rewrite The equation as $x - \cos x = 0$ and let $f(x) = x - \cos x$. Because x and $\cos x$ are continuous on the given interval, so is f. Because $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$, it follows from the Intermediate Value Theorem that the equation has a solution on $(0, \pi/2)$.

b. Using a computer algebra system, one can find that $c \approx 0.739085$ is a root.

84 Temperature changes gradually, so it is reasonable to assume that T is a continuous function and therefore f is also continuous. Because$f(0) = -33 < 0$ and $f(12) = 33 > 0$, it follows from the Intermediate Value Theorem that there is a value t_0 in $(0, 12)$ satisfying $f(t_0) = 0$. . Therefore, $T(t_0) - T(t_0 + 12) = 0$, or $T(t_0) = T(t_0 + 12)$.

85

a. Note that $m(0) = 0$ and $m(5) \approx 38.34$ and $m(15) \approx 21.2$. Thus, 30 is an intermediate value between both $m(0)$ and $m(5)$, and $m(5)$ and $m(15)$. Note also that m is a continuous function. By the IVT, there must be a number c_1 between 0 and 5 with $m(c_1) = 30$, and a number c_2 between 5 and 15 with $m(c_2) = 30$.

b. A little trial and error leads $c_1 \approx 2.4$ and $c_2 \approx 10.8$.

c. No. The graph of the function on a graphing calculator suggests that it peaks at about 38.5

86 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/5$. Now suppose that $0 < |x - 1| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= |(5x - 2) - 3| = |5x - 5| \\ &= 5|x - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

87 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Now suppose that $0 < |x - 5| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= \left|\frac{x^2 - 25}{x - 5} - 10\right| = \left|\frac{(x-5)(x+5)}{x - 5} - 10\right| = |x + 5 - 10| \\ &= |x - 5| < \varepsilon. \end{aligned}$$

88 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{4}$ and assume that $0 < |x - 3| < \delta$. For $x < 3$,

$$|f(x) - 5| = |3x - 4 - 5| = 3|x - 3| < 3\delta = 3 \cdot \frac{\varepsilon}{4} < \varepsilon.$$

For $x > 3$,

$$|f(x) - 5| = |-4x + 17 - 5| = 4|x - 3| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|f(x) - 5| < \varepsilon$ whenever $0 < |x - 2| < \delta$, provided $0 < \delta \le \frac{\varepsilon}{4}$.

89 Let $\varepsilon > 0$ be given. Let $\delta = \min\{1, \varepsilon/15\}$ and assume that $0 < |x - 2| < \delta$. Then $|3x^2 - 4 - 8| = 3|x^2 - 4| = 3|x - 2||x + 2|$. Because $0 < |x - 2| < \delta$ and $\delta \le 1$, $-1 < x - 2 < 1$ and so $1 < x < 3$. It follows that $x + 2 < 5$. Therefore $|3x^2 - 4 - 8| = 3|x - 2||x + 2| < 3 \cdot \frac{\varepsilon}{15} \cdot 5 = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|3x^2 - 4 - 8| < \varepsilon$ whenever $0 < |x - 2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/15\}$.

90 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon^2}{4}$ and assume that $0 < x - 2 < \delta$. Then $|\sqrt{4x - 8} - 0| = 2\sqrt{x - 2} < 2\sqrt{\delta} = 2\sqrt{\frac{\varepsilon^2}{4}} = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|\sqrt{4x - 8} - 0| < \varepsilon$ whenever $0 < x - 2 < \delta$, provided $0 < \delta \le \frac{\varepsilon^2}{4}$.

91 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Suppose that $0 < |x - 2| < \delta$. Then $|x - 2| < \dfrac{1}{\sqrt[4]{N}}$, so $\dfrac{1}{|x - 2|} > \sqrt[4]{N}$, and $\dfrac{1}{(x - 2)^4} > N$, as desired.

92

a. Assume $L > 0$. (If $L = 0$, the result follows immediately because that would imply that the function f is the constant function 0, and then $f(x)g(x)$ is also the constant function 0.) Assume that δ_1 is a number so that $|f(x)| \le L$ for $|x - a| < \delta_1$.

Let $\varepsilon > 0$ be given. Because $\lim_{x \to a} g(x) = 0$, we know that there exists a number $\delta_2 > 0$ so that $|g(x)| < \varepsilon/L$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$.

Then

$$|f(x)g(x) - 0| = |f(x)||g(x)| < L \cdot \frac{\varepsilon}{L} = \varepsilon,$$

whenever $0 < |x - a| < \delta$.

b. Let $f(x) = \dfrac{x^2}{x - 2}$. Then

$$\lim_{x \to 2} f(x)(x - 2) = \lim_{x \to 2} \frac{x^2(x - 2)}{x - 2} = \lim_{x \to 2} x^2 = 4 \ne 0.$$

This doesn't violate the previous result because the given function f is not bounded near $x = 2$.

c. Because $|H(x)| \le 1$ for all x, the result follows directly from part a) of this problem (using $L = 1$, $a = 0$, $f(x) = H(x)$, and $g(x) = x$).

Guided Project 2: Constant rate problems

Topics and skills: Algebra

Continuing with the theme of problem solving, we now give you an opportunity to apply Pólya's method (see Guided Project 1: *Problem-solving skills*) to a specific type of problem. Constant rate problems require only algebra (solving problems that involve variable rates is a major reason for studying calculus).
Exercises 1–16 below involve constant speeds. Constant speed problems use the fact that

$$\text{Distance traveled} = \text{speed} \times \text{time elapsed} \quad \text{or} \quad d = s \times t.$$

Exercises 17–25 deal with more general constant rates, such as work rates and flow rates; but the same ideas may be used. If a problem involves a quantity Q (such as gallons of water or number of bagels) and its rate of change is the constant r, the constant rate formula is

$$\text{Amount of } Q = \text{rate of change of } Q \times \text{time elapsed} \quad \text{or} \quad Q = r \times t.$$

Write out complete solutions to the following problems and discuss whether and how Pólya's method was useful.

Problem-Solving Exercises

<u>Constant speed problems</u>

1. A car went 4 miles up a hill at 40 mi/hr and 6 mi down the back of the hill at 60 mi/hr. What was the average speed of the round trip?

2. A one-mile-long train went through a one-mile-long tunnel at 15 mi/hr. How long did it take the entire train to pass through the tunnel?

3. If a lady walks to work and drives home, it takes one and a half hours. When she drives both ways, it takes half an hour. At the same speeds, how much time does a round trip while walking require?

4. At full speed, a motor boat can go upstream 10 mi (against the current) in 15 min and downstream 10 mi (with the current) in 9 min. At full speed, how much time is required for the boat to go 10 mi with no current?

5. Two trains travel toward each other on parallel tracks. Train A is 0.5 mi long and travels at 20 mi/hr. Train B is 1/3 mi long and travels at 30 mi/hr. How long does it take the trains to pass each other completely (from the instant the engines meet to the instant that the cabooses pass each other)?

6. At midnight a train left Denver bound for Omaha, a distance of 500 mi, at a speed of 80 mi/hr and another train left Omaha bound for Denver at a speed of 100 mi/hr. When the trains passed each other, what fraction of its trip had the Denver train completed?

7. At midnight a train left Denver bound for Chicago and another train left Chicago bound for Denver, both traveling at constant speeds on adjacent tracks. The first train took 12 hr to complete the trip and the second train took 16 hr to complete the trip. At what time did the trains pass each other?

8. A train left Denver bound for Omaha, a distance of 500 mi, at a speed of 80 mi/hr. Two hours later, another train left Omaha bound for Denver at a speed of 100 mi/hr. When the trains passed each other, how far had the Denver train traveled?

9. A train left Boston for New York, a distance of 220 mi, at 70 mi/hr. One hour later, a train left New York for Boston at 60 mi/hr. How far apart were the trains one hour before they met?

10. Two cyclists racing on parallel roads maintain constant speeds of 30 mi/hr and 25 mi/hr. The faster cyclist crosses the finish line one hour before the slower cyclist. How long was the race (in miles)?

11. Because Boat A travels 1.5 times faster than Boat B, Boat B was given a 1.5-hr head start in a race. How long did it take Boat A to catch Boat B?

12. A plane flew into a headwind and made the outbound trip in 84 min. It turned around and made the return trip with a tailwind in 9 min less than it would have taken with no wind. Assuming that the plane's ground speed and the wind speed are constant, what are the possible times for the return trip?

13. A woman usually takes the 5:30 train home from work, arriving at the station at 6:00 where her husband meets her to drive her home. One day she left work early and took the 5:00 train, arrived at the station at 5:30, and began walking home. Her husband, leaving home at the usual time, met his wife along the way and brought her home 10 min earlier than usual. How long did the woman walk?

14. Bob was traveling 80 km/hr behind a truck traveling 65 km/hr. How far behind the truck was Bob one minute before the crash?

15. At his usual rate Bernie can row 15 mi downstream in five hr less time than it takes him to row 15 mi upstream. If he doubles his usual rate, his time downstream is only one hour less than the time upstream. What is the rate of the current in miles per hour?

16. A cyclist began at the tail of a parade that is 4 km long and rode in the direction that the parade was moving. By the time the cyclist reached the head of the parade and returned to the tail, the parade had moved 6 km. Assuming that the cyclist and the parade moved at constant (but different) speeds, how far did the cyclist ride?

Other constant rate problems

17. Work rates Working together (but independently), Arlen, Ben, and Carla can complete a job in 1 hr. Working alone, Ben and Carla can complete the same job in 2.5 and 3.5 hr, respectively. How long would it take Arlen to complete the same job working alone?

18. Work rates Twenty people can make 4 hats in 2 hr. How long will it take 15 people to make 30 hats? How many people are needed to make 40 hats in 4 hr? How many hats can be made by 5 people in 12 hr?

19. Work rates Ann and Betty can do a job in 10 days; Ann and Carol can do the same job in 12 days; Betty and Carol can do the same job in 20 days. How long will it take Carol to do the job alone?

20. Machine rates Working alone, photocopy machine C requires 40 min to complete an 800-page job. Working together machines B and C require 25 min for the same 800-page job. With machines A, B, and C working together, the job takes 10 min. How long does it take machine A to complete the job working alone?

21. Filling a tank Pipes A and B can fill a tank in 2 hr and 3 hr, respectively. Pipe C can empty the same full tank in 5 hr. If all pipes are opened at the same time when the tank is empty, how long will it take to fill the tank?

22. Open and shut valves Each of valves A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hr. With only valves A and C open, it takes 1.5 hr to fill the tank, and with only valves B and C open, it takes 2 hr. How long does it take to fill the tank with valves A and B open?

23. Filling a tank Joe opened two input pipes to a tank, but forgot to close the drain. The tank was half full when he noticed his error and closed the drain. If it takes one input pipe 10 hr to fill the tank and the other input pipe 8 hr to fill the tank (with the drain closed), and if it takes the drain 6 hr to empty the tank when it is full (with no input pipes open), how long did it take Joe to fill the tank on this occasion?

24. Burning issue Two candles of equal length were lit at the same time. One candle took 6 hr to burn out and the other candle took 3 hr to burn out. After how much time was one candle exactly twice as long as the other candle?

25. Dueling candles Two candles of length L and $L + 1$ were lit at 6:00 and 4:30, respectively. At 8:30 they had the same length. The longer candle died at 10:30 and the shorter candle died at 10:00. Find L.

Solution to Guided Project 2: Constant rate problems

Constant speed problems

1. The car traveled 10 miles in 9 minutes (3/20 hr) for an average speed of $\frac{160}{3} \approx 53.3$ mi/hr.

2. When the front of the train enters the tunnel, the end of the train has two miles to travel before exiting the tunnel. Dividing this distance by the speed of 15 mi/hr gives that the train takes 2/15 hr or 8 minutes to pass through the tunnel.

3. Driving one way takes 15 minutes, so walking one way takes one and a quarter hours. The round trip walking takes two and one half hours.

4. Upstream, the boat travels 40 mi/hr. Downstream, it travels 200/3 mi/hr. These speeds are the boat speed minus the current speed and the boat speed plus the current speed, respectively. Combining these facts gives that the speed of the boat alone is 160/3 mi/hr. Therefore, traveling 10 mi with no current takes 3/16 hr or 11.25 min.

5. Consider the point at which the engines meet to be the origin. Then the position of the caboose of Train A after t hr is $-1/2+20t$ and the position of the caboose of Train B is $1/3-30t$. These positions are equal at time $t=1/60$ hr, meaning the trains completely pass each other after one minute.

6. Consider Denver to be the origin. After t hr, the train bound for Omaha is at position $100t$. At that same time, the train bound for Denver is at position $500-80t$. These positions are equal when $t=25/9$, at which point the train bound for Denver will have traveled $(25/9)\cdot 80$ mi, which is 4/9 of its trip.

7. Consider Denver to be the origin. After t hr, the first train is at position $(d/12)\cdot t$, where d is the distance between Denver and Chicago. At that same time, the second train is at position $d-(d/16)\cdot t$. The trains meet at $t=48/7 \approx 6.86$ hr, meaning the trains passed each other at approximately 6:51 a.m.

8. Consider Denver to be the origin. After t hours, the Omaha-bound train is at position $80t$. For $t \ge 2$, the Denver-bound train's position is $500-100(t-2)$. Therefore, the trains meet at time $t=35/9$ hr, meaning the Omaha-bound train had traveled $(35/9-2)\cdot 100=(17/9)\cdot 100 \approx 189$ mi.

9. Consider Boston to be the origin. After t hours, the New York train is at position $70t$. For $t \ge 1$, the Boston train is at position $220-60(t-1)$. Therefore, the trains meet at time $t=28/13$ hr. One hour prior, the distance between the trains is 130 mi.

10. Let x be the length of the race (in miles). The first cyclist completes the race in $x/30$ hr, while the second requires $x/25$ hr. We are given that $x/30+1=x/25$, which implies $x=150$ mi.

11. Let the speed of Boat B be s mi/hr. After t hr, Boat B has traveled st mi. Boat A has traveled $15s(t-15)$ mi t hr after Boat B started. Therefore, Boat A overtakes Boat B 4.5hr after Boat B began, so it took Boat A 3 hr to catch Boat B.

12. Let g be the plane's ground speed, w the wind speed, and d the distance the plane travels (one way). From the outbound trip, we have $84(g-w)=d$, so $w=g-d/84$. For the return trip, we have $(g+w)(d/g-9)=d$. Substituting for w and solving for g gives that $g=d/21$ or $g=d/72$. The time for the return trip is $d/g-9$, so it must be either 12 minutes or 63 minutes.

13. Let t_w be the time the woman spent walking and t_c the time spent in the car on this day. Let t be the usual amount of time it takes her husband to drive her home from the train station. Since she left 30 minutes early but only got home 10 minutes early, $t_w+t_c-20=t$. Since her husband left at the usual time but got home 10 minutes early, $2t_c=2t-10$. Combining these facts, we see that $t_w=25$ minutes.

14. At time t hr before the crash, the truck is $65t$ km away from the crash site and Bob is $80t$ km away from the crash site. Here $t=1/60$, and so the distance between these positions is 1/4 km.

15. Let r be Bernie's usual rowing rate and c the speed of the current. The relationship at his usual rowing rate gives that $15=(r+c)(15/(r-c)-5)$. At double his usual rate, we have $15=(2r+c)(15(2r-c)-1)$. Combining and solving for c gives $c=r/2$. Substituting this, we are able to solve for r and find $r=4$ miles per hour, so $c=2$ miles per hour.

16. Let p be the rate at which the parade moves and c the rate at which the cyclist moves. Fixing the origin at the point where the back of the parade began, before the cyclist turns around, the cyclist's position is ct and the front of the parade is at $pt+4$. The cyclist turns around at time $t_0 = 4/(c-p)$. For $t > t_0$, the cyclist's position is $ct_0 - ct$. The back of the parade is at $pt + pt_0$. Therefore, the cyclist reaches the back of the parade when $ct_0 - ct = pt + pt_0$. Solving for *t*, we find that the cyclist reaches the back of the parade t_1 hours after the turnaround where $t_1 = 4/(c+p)$. Since the parade moves 6 km in time $t_0 + t_1$, we have $p(t_0 + t_1) = 6$, and therefore $c = \left(\sqrt{13}+2\right)p/3$. The distance the cyclist rode is $c(t_0 + t_1) = 4 + 2\sqrt{13} \approx 11.21$ km.

Other Constant Rate Problems

17. Let a, b, and c be the work rates of Arlen, Ben, and Carla, respectively. Denote by j the amount of work required to complete the job. Then $a \approx 0.314j$. If it takes Arlen t time to do the job alone, then $at = j$, which allows us to find that $t \approx 3.18$ hr.

18. From the given information, one worker produces 1/10 hat per hour. Therefore, it will take 20 hr for 15 people to make 30 hats. To make 40 hats in 4 hr will require 100 people. Five people working for 12 hr will make 6 hats.

19. Let Q denote the total quantity of work done, and let a, b, and c denote the work rates of Ann, Betty, and Carol, respectively. Then $10(a+b) = Q$, $12(a+c) = Q$, and $20(b+c) = Q$. Solving, we find $c = Q/60$, and therefore it will take Carol 60 days to do the job alone.

20. Machine C works at 20 pages per min, which allows us to determine that machine B works at 12 pages per min. Together, this gives that machine A works at 48 pages per min. Therefore, machine A working alone will take $50/3 \approx 16.67$ min to complete the job.

21. Let Q be the quantity of water the tank holds. Then pipe A fills at a rate of $Q/2$, pipe B fills at a rate of $Q/3$, and pipe C drains at a rate of $Q/5$. Filling the tank with all three pipes open takes time $Q/(Q/2 + Q/3 - Q/5) = 30/19 \approx 1.58$.

22. Let Q denote the volume of the tank, and let a, b, and c denote the fill rates of valves A, B, and C, respectively. Then we have $a+b+c = Q$, $15(a+c) = Q$, and $2(b+c) = Q$. Therefore $a = Q/2$ and $b = Q/3$ and it takes 1.2 hr for the tank to fill with valves A and B open.

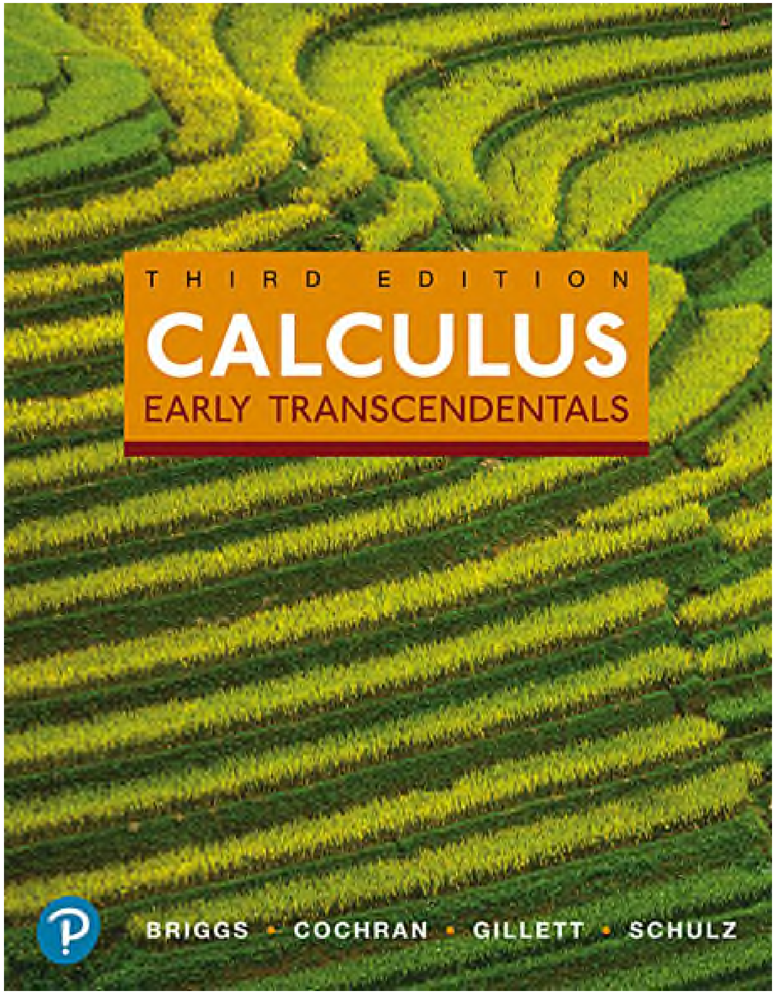
23. Let Q denote the volume of the tank. The first pipe fills at a rate of $Q/10$ and the second fills at a rate of $Q/8$. The drain empties the tank at a rate of $Q/6$. To get the tank half full with both input pipes and the drain open then takes $60/7 \approx 8.57$ hr. To fill half the tank with both input pipes open and the drain closed takes $20/9 \approx 2.22$ hr. Therefore, the total time in this case is approximately 10.79 hr.

24. Let L denote the length of the candle. The first candle burns at a rate of $L/6$, and the second burns at a rate of $L/3$. After t hr, the length of the first candle is $L-(L/6)t$ and the length of the second candle is $L-(L/3)t$. The first candle is twice the length of the second then after 2 hr.

25. The shorter candle burns at a rate of $L/4$, while the longer burns at a rate of $(L+1)/6$. Knowing that they had the same length at 8:30 gives the equation $L - 2.5L/4 = L + 1 - 4(L+1)/6$, and solving this gives $L = 8$.

Chapter 2

Limits



2.1

The Idea of Limits

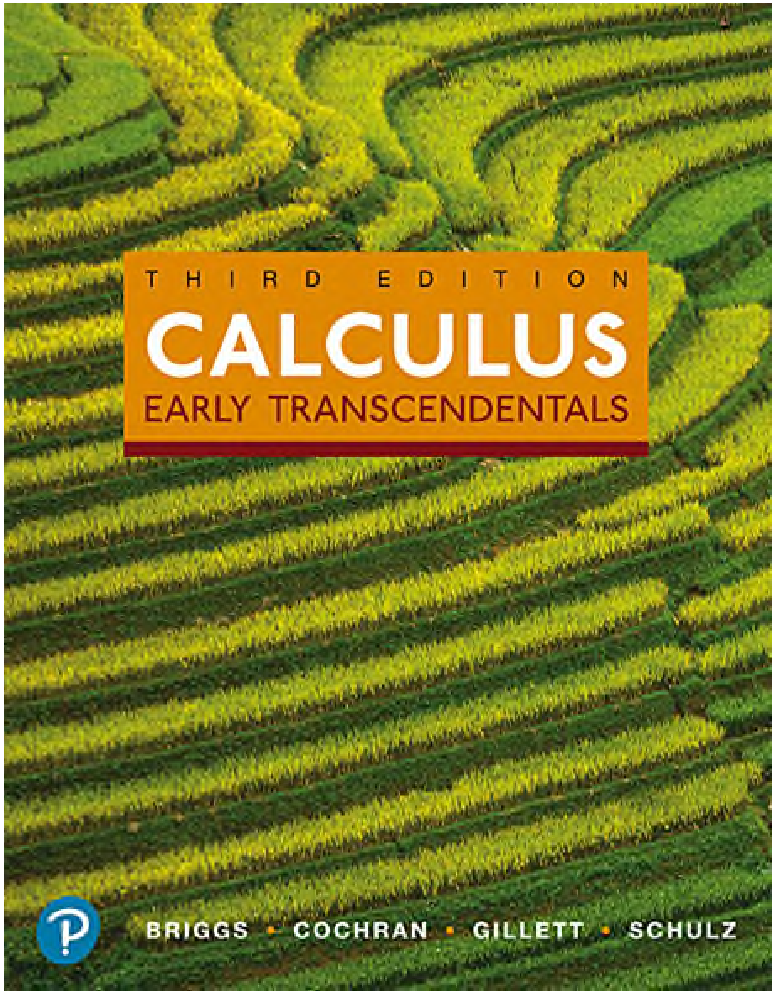
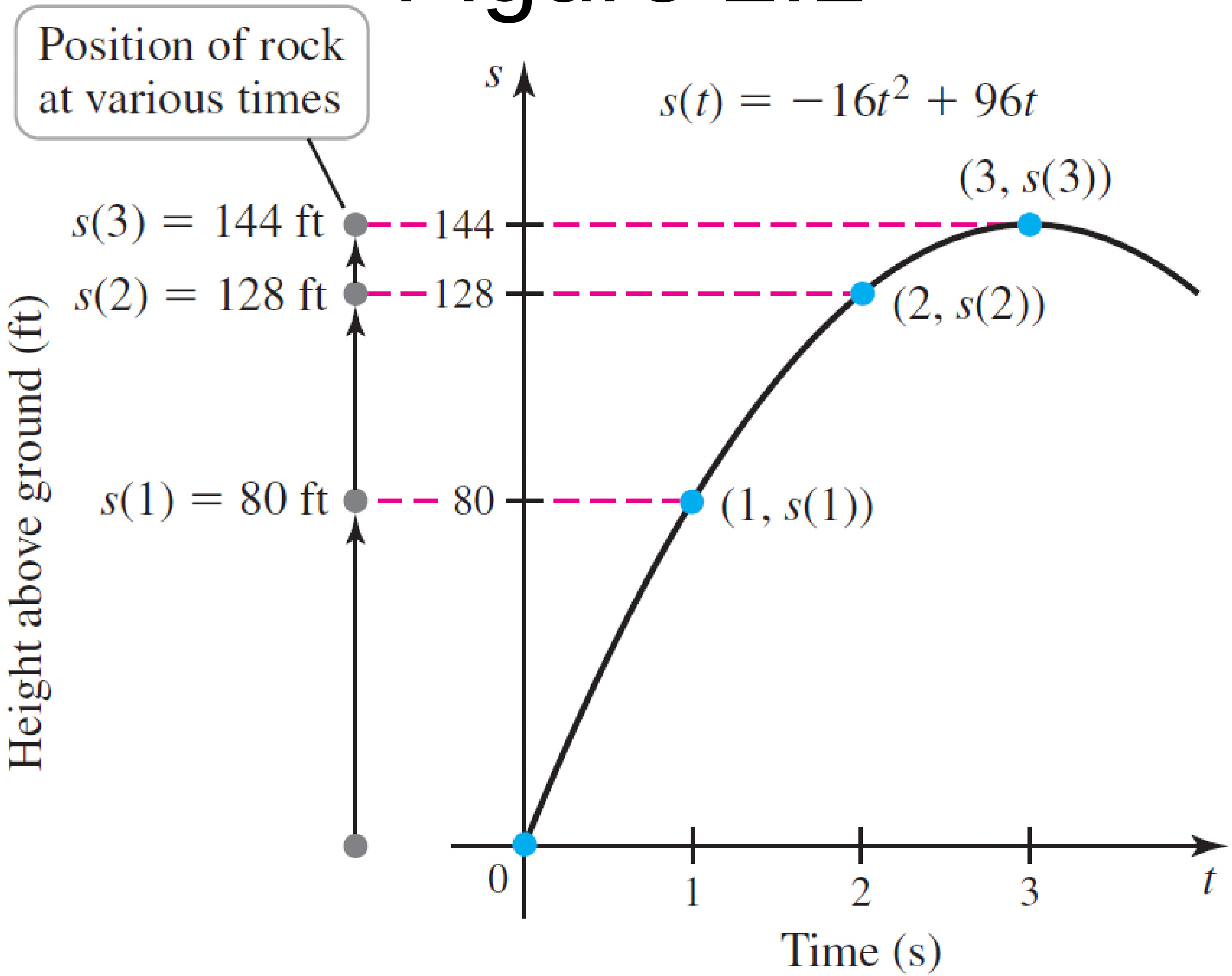


Figure 2.1


Position of rock at various times
s
$s(t) = -16t^2 + 96t$
(3, s(3))
s(3) = 144 ft
144
s(2) = 128 ft
128
(2, s(2))
s(1) = 80 ft
80
(1, s(1))
Height above ground (ft)
0
1
2
3
t
Time (s)

CLICK HERE TO ACCESS THE COMPLETE Solutions

Figure 2.2 (a)

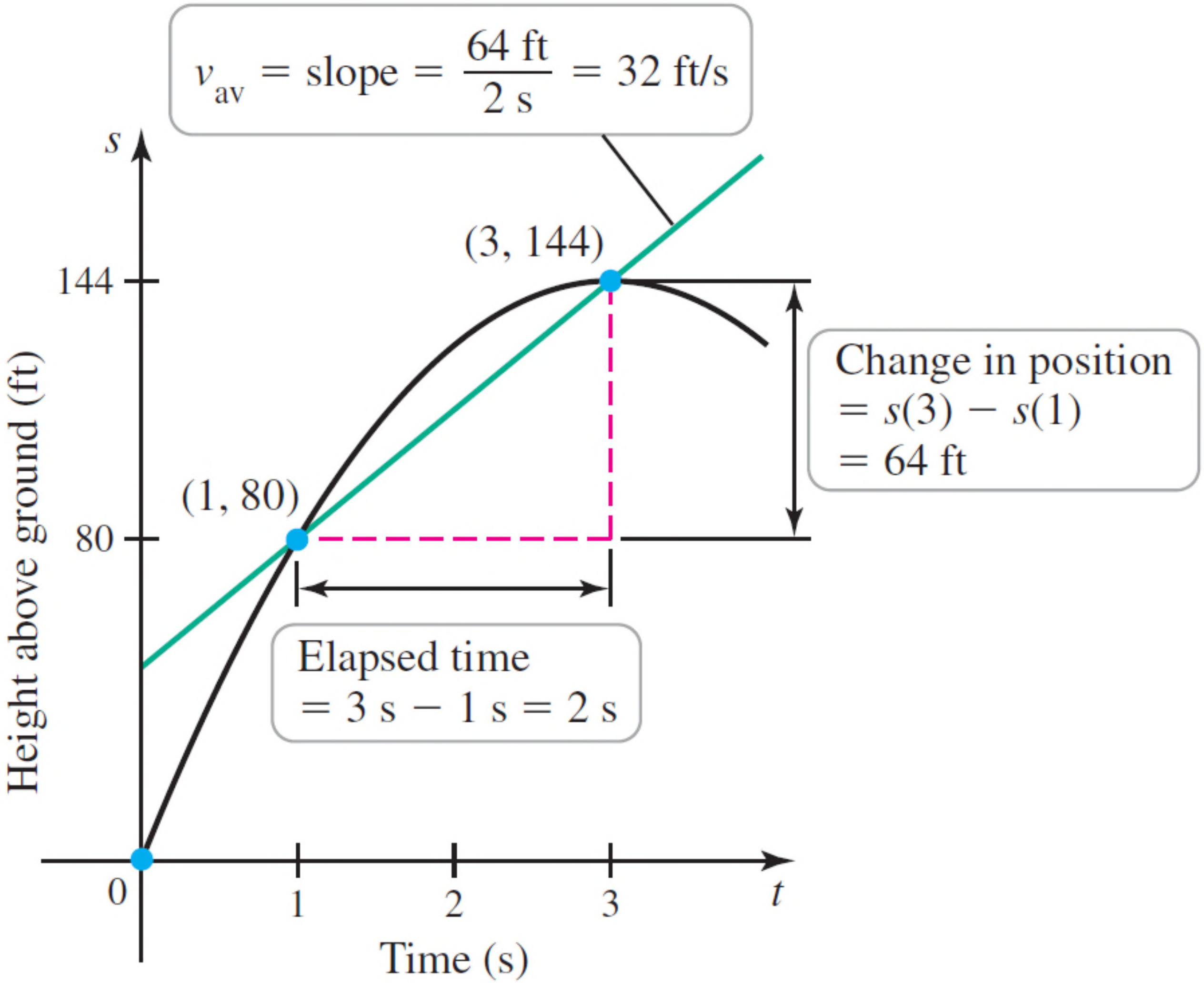
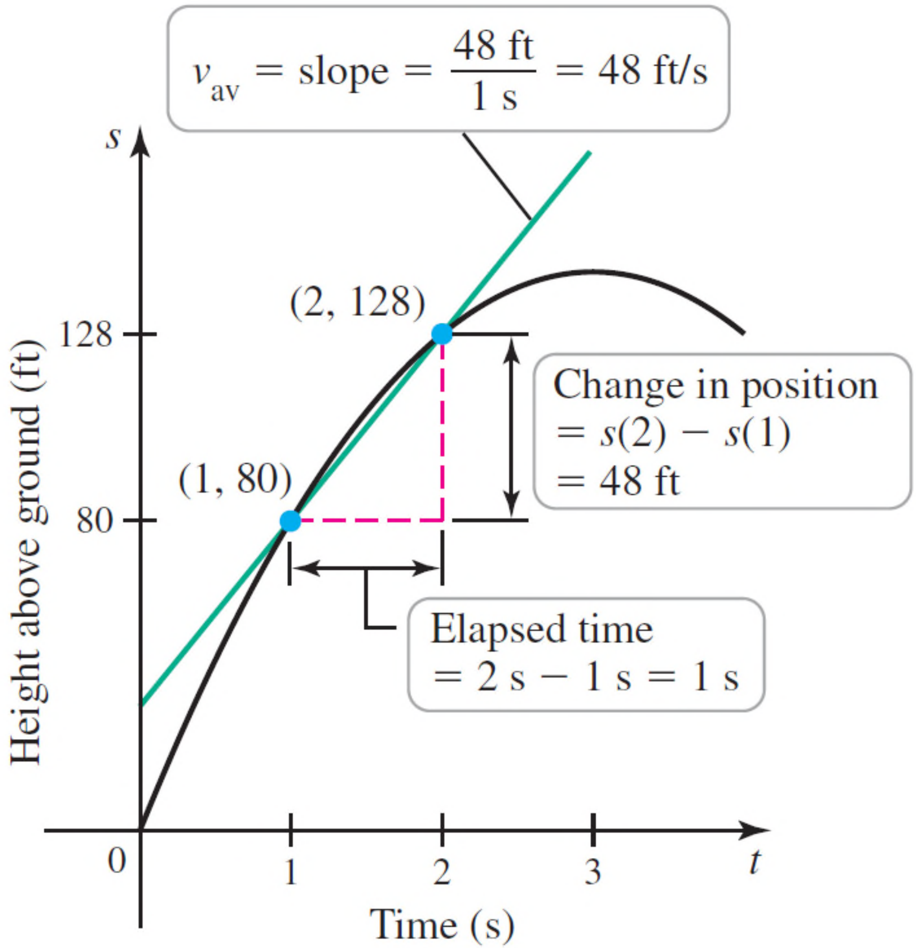
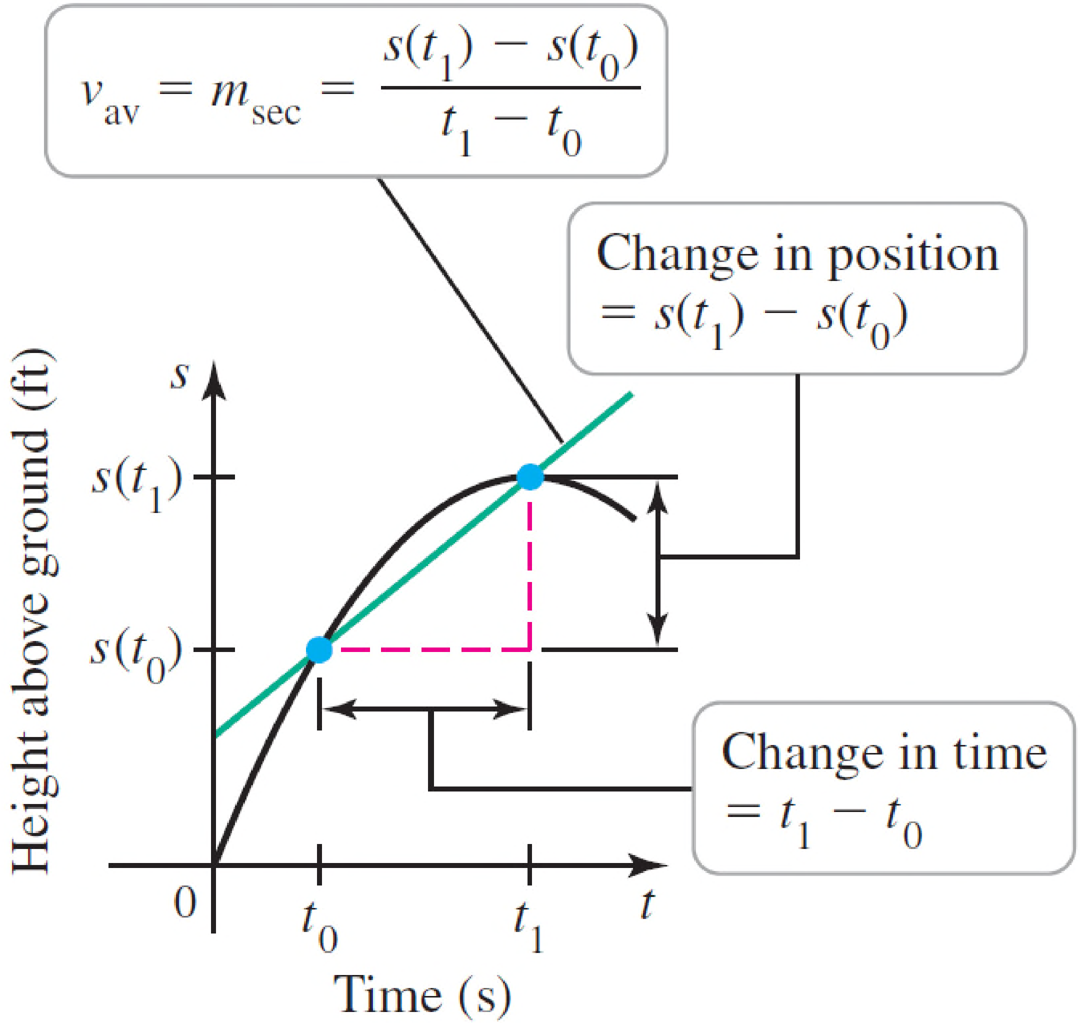


Figure 2.2 (b)



v_{av} = slope = $\frac{48 \text{ ft}}{1 \text{ s}}$ = 48 ft/s
s
Height above ground (ft)
128
80
(2, 128)
(1, 80)
Change in position
= s(2) − s(1)
= 48 ft
Elapsed time
= 2 s − 1 s = 1 s
0
1
2
3
t
Time (s)

Figure 2.3



$v_{av} = m_{sec} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$
Change in position
$= s(t_1) - s(t_0)$
Change in time
$= t_1 - t_0$
s
$s(t_1)$
$s(t_0)$
Height above ground (ft)
0
t_0
t_1
t
Time (s)

Table 2.1

Time interval	Average velocity
$[1, 2]$	48 ft/s
$[1, 1.5]$	56 ft/s
$[1, 1.1]$	62.4 ft/s
$[1, 1.01]$	63.84 ft/s
$[1, 1.001]$	63.984 ft/s
$[1, 1.0001]$	63.9984 ft/s

Figure 2.4

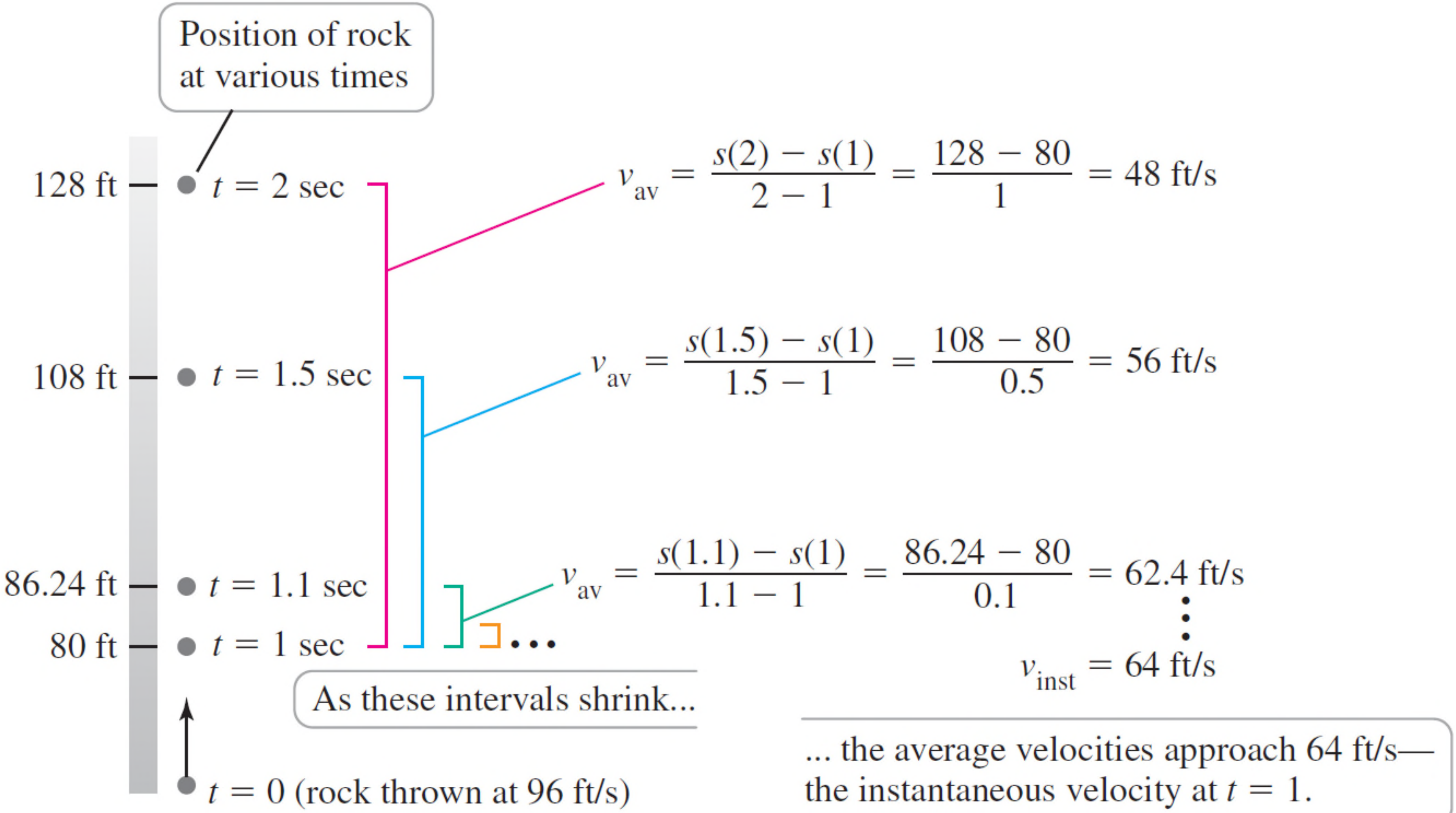

Position of rock
at various times
128 ft
108 ft
86.24 ft
80 ft
t = 2 sec
t = 1.5 sec
t = 1.1 sec
t = 1 sec
t = 0 (rock thrown at 96 ft/s)
$v_{av} = \frac{s(2) - s(1)}{2 - 1} = \frac{128 - 80}{1} = 48$ ft/s
$v_{av} = \frac{s(1.5) - s(1)}{1.5 - 1} = \frac{108 - 80}{0.5} = 56$ ft/s
$v_{av} = \frac{s(1.1) - s(1)}{1.1 - 1} = \frac{86.24 - 80}{0.1} = 62.4$ ft/s
$v_{inst} = 64$ ft/s
As these intervals shrink...
... the average velocities approach 64 ft/s—
the instantaneous velocity at t = 1.

Figure 2.5

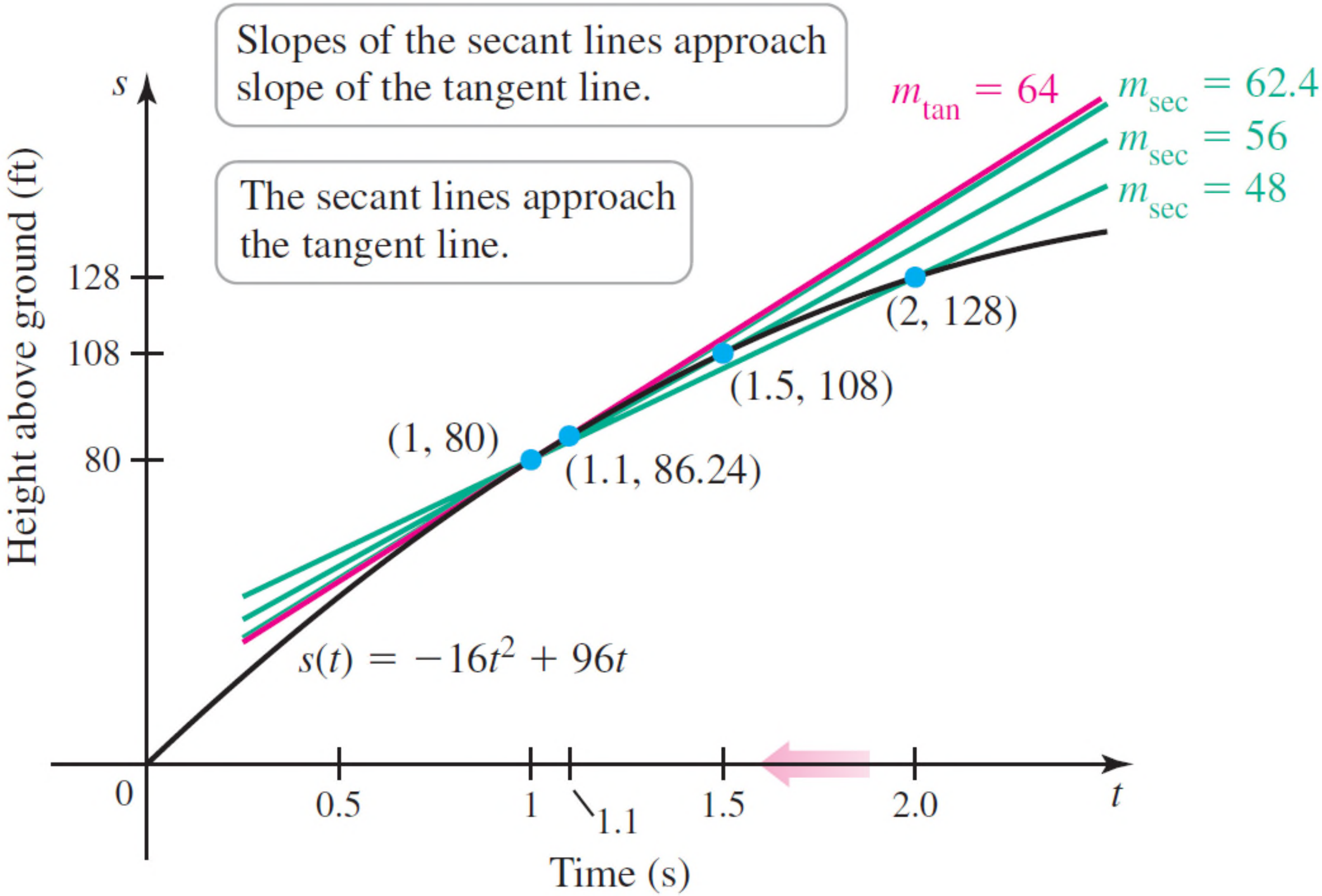

Slopes of the secant lines approach slope of the tangent line.
The secant lines approach the tangent line.
$m_{tan} = 64$
$m_{sec} = 62.4$
$m_{sec} = 56$
$m_{sec} = 48$
s
Height above ground (ft)
128
108
80
0
(2, 128)
(1.5, 108)
(1, 80)
(1.1, 86.24)
$s(t) = -16t^2 + 96t$
0.5
1
1.1
1.5
2.0
t
Time (s)

Figure 2.6 (1 of 3)

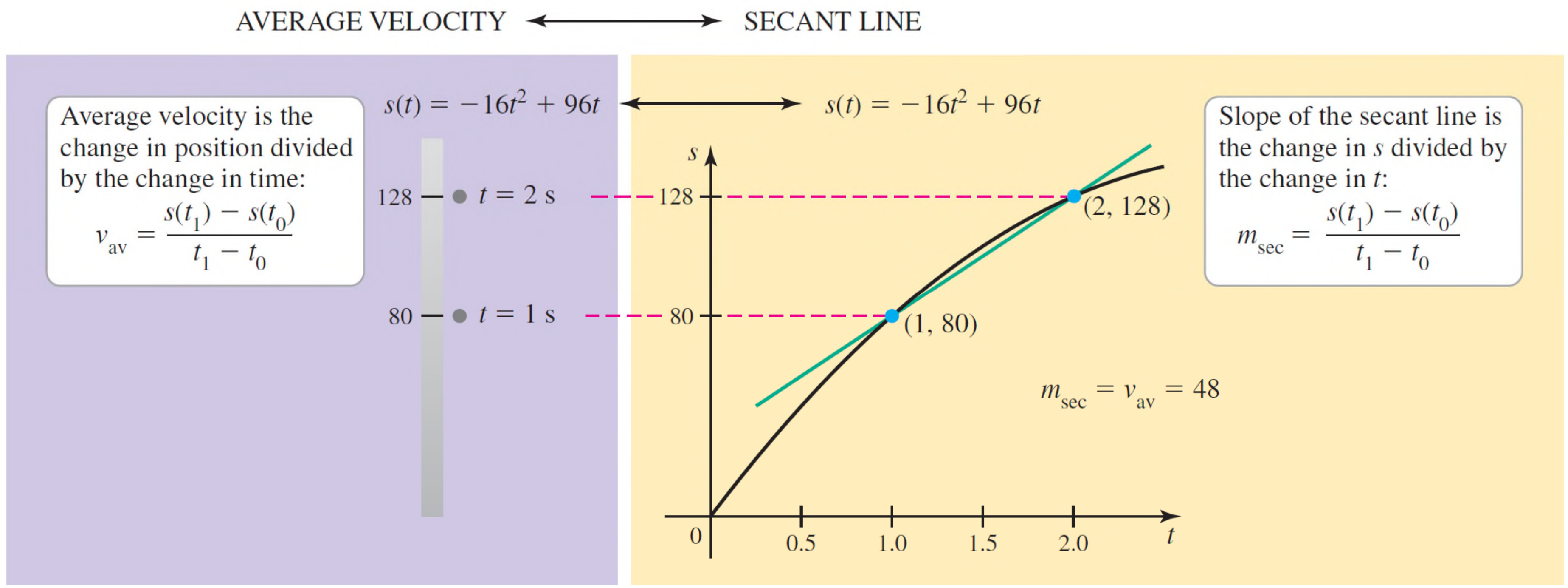


Figure 2.6 (2 of 3)

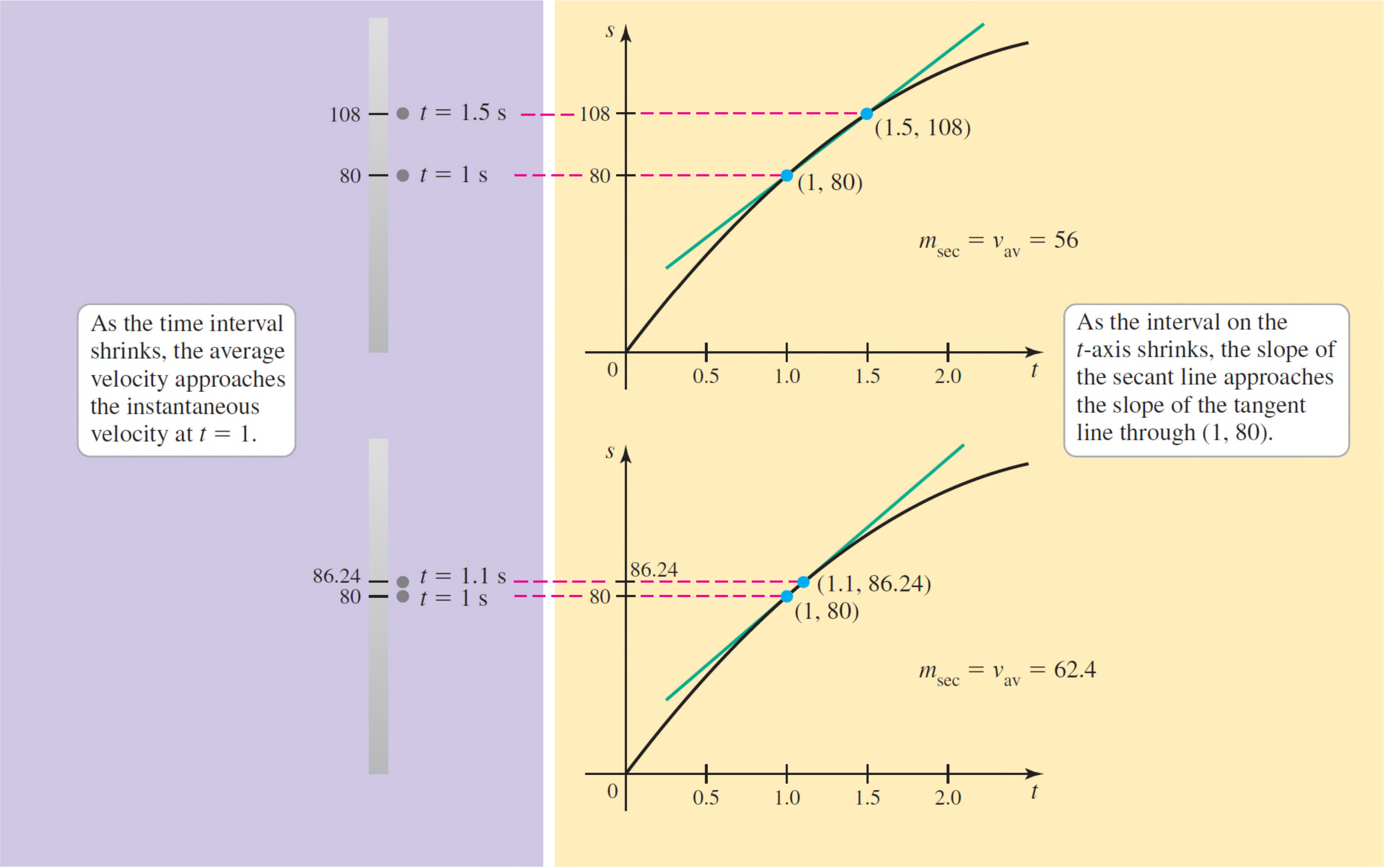


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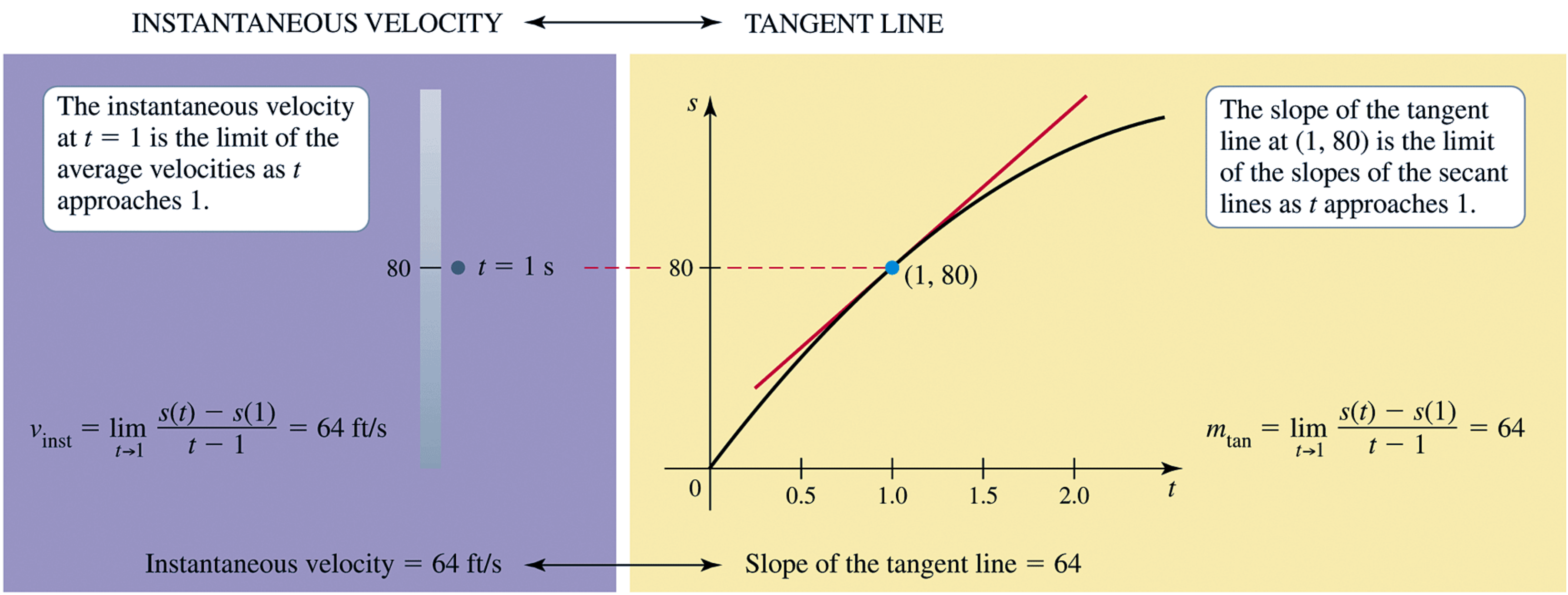
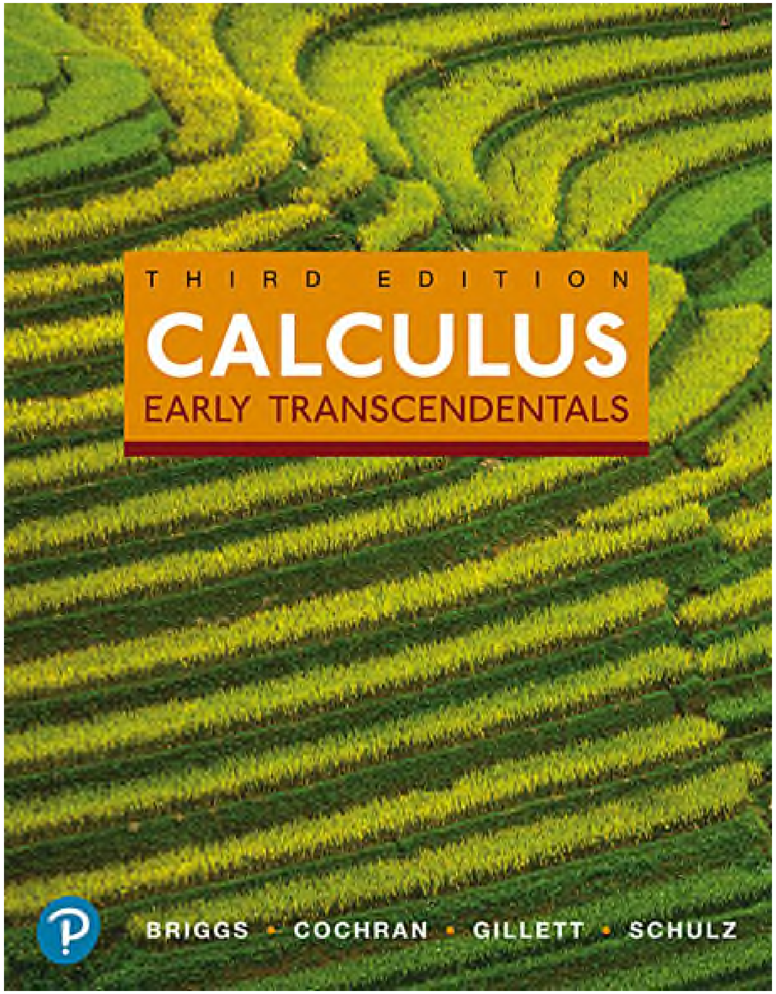
Figure 2.6 (3 of 3)



2.2

Definitions of Limits



DEFINITION Limit of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

Figure 2.7

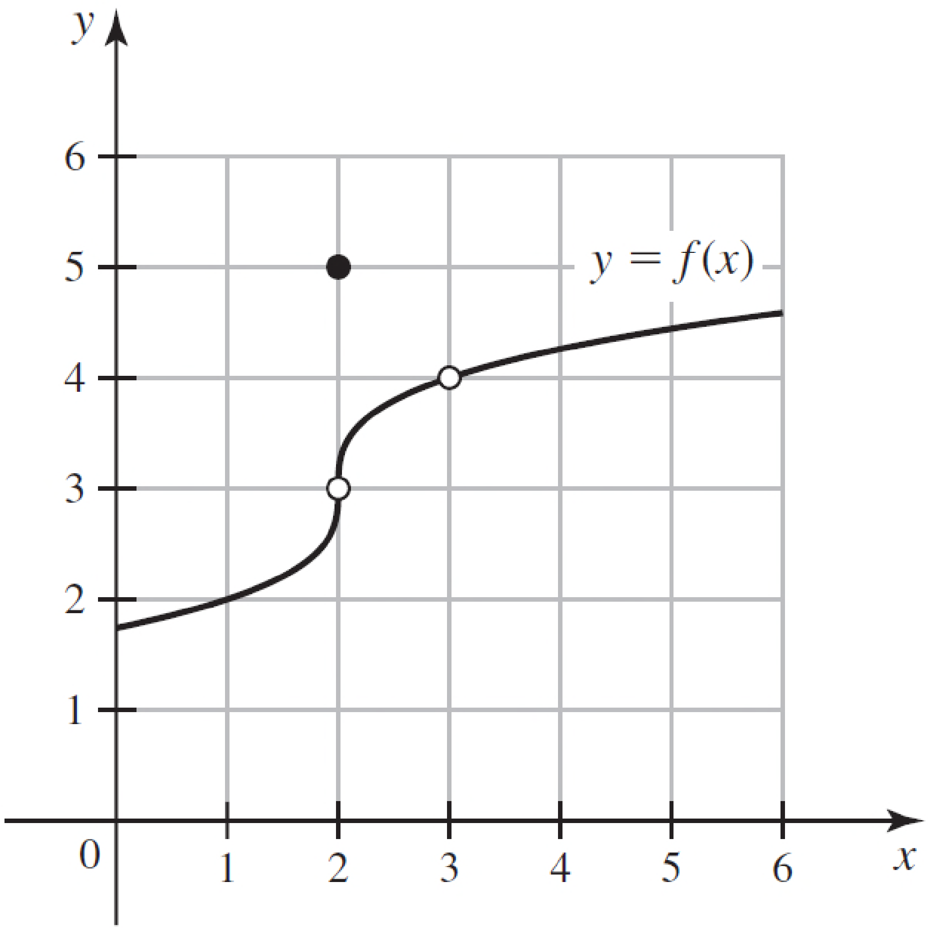


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Figure 2.8

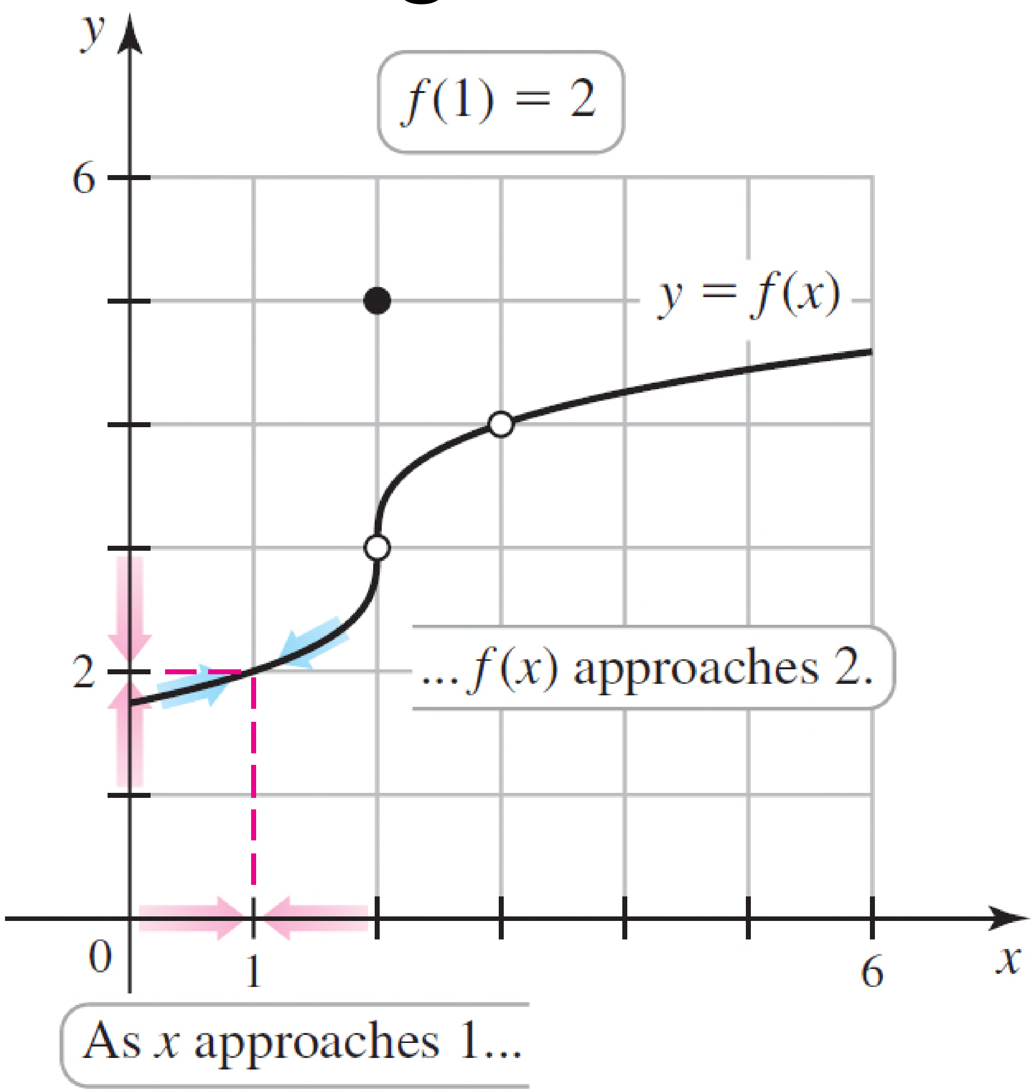


Figure 2.9

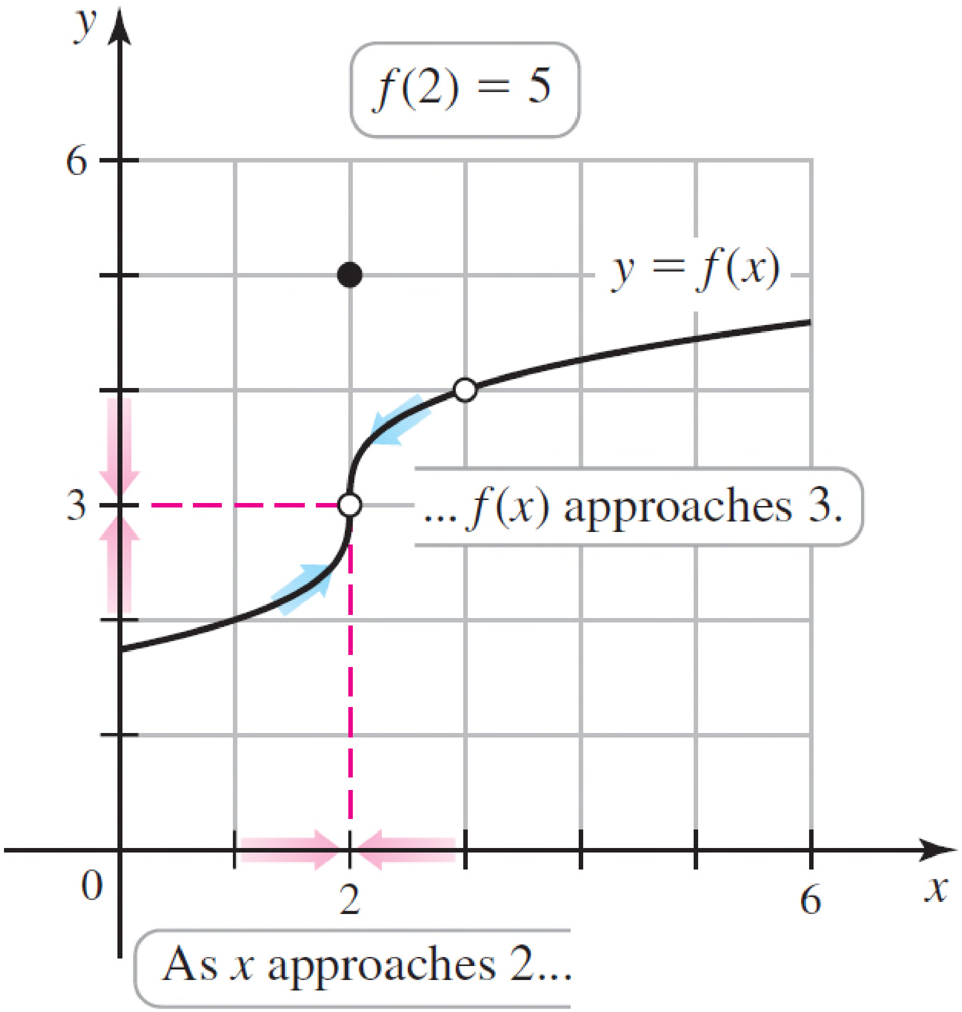
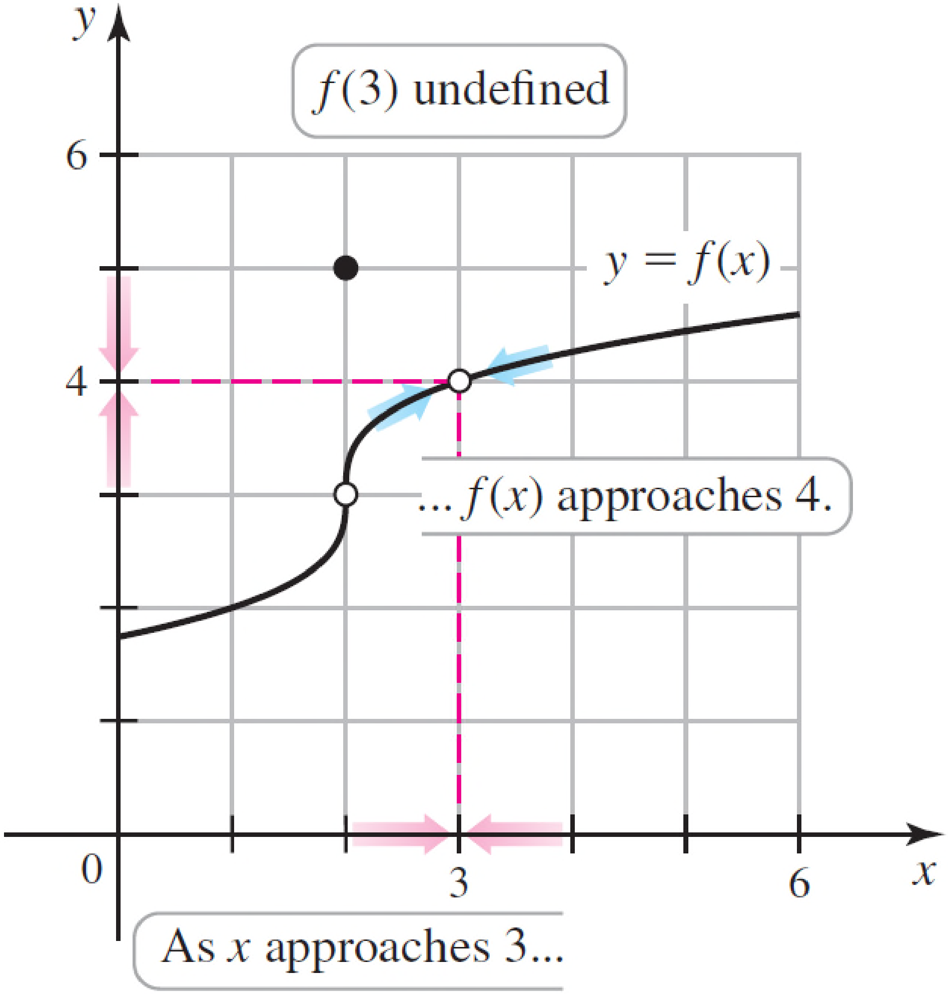


Figure 2.10



CLICK HERE TO ACCESS THE COMPLETE Solutions

Table 2.2

	$\longrightarrow$		1	$\longleftarrow$				
x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$	0.5131670	0.5012563	0.5001251	0.5000125	0.4999875	0.4998751	0.4987562	0.4880885

DEFINITION One-Sided Limits

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Figure 2.11 (a & b)

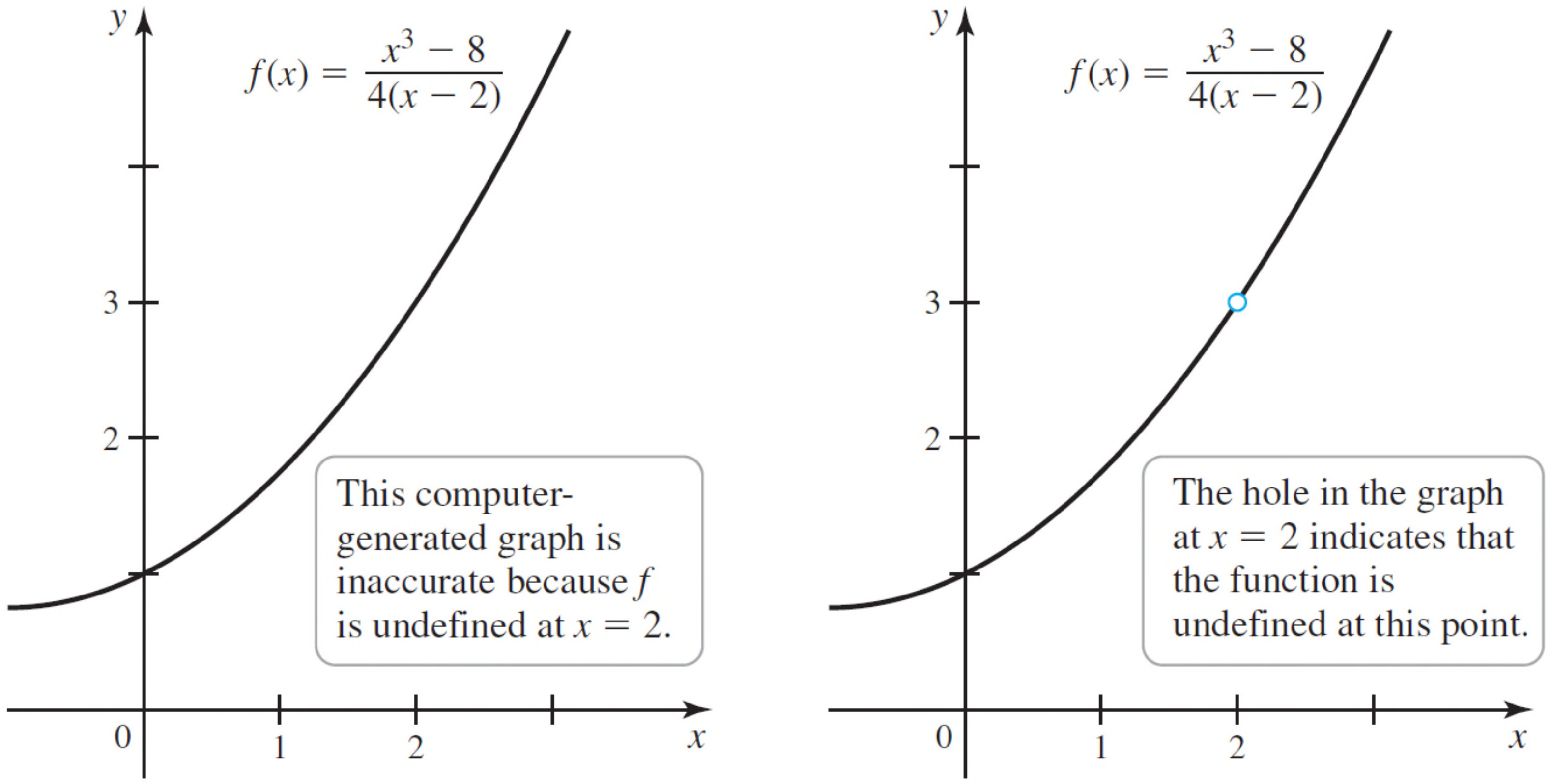
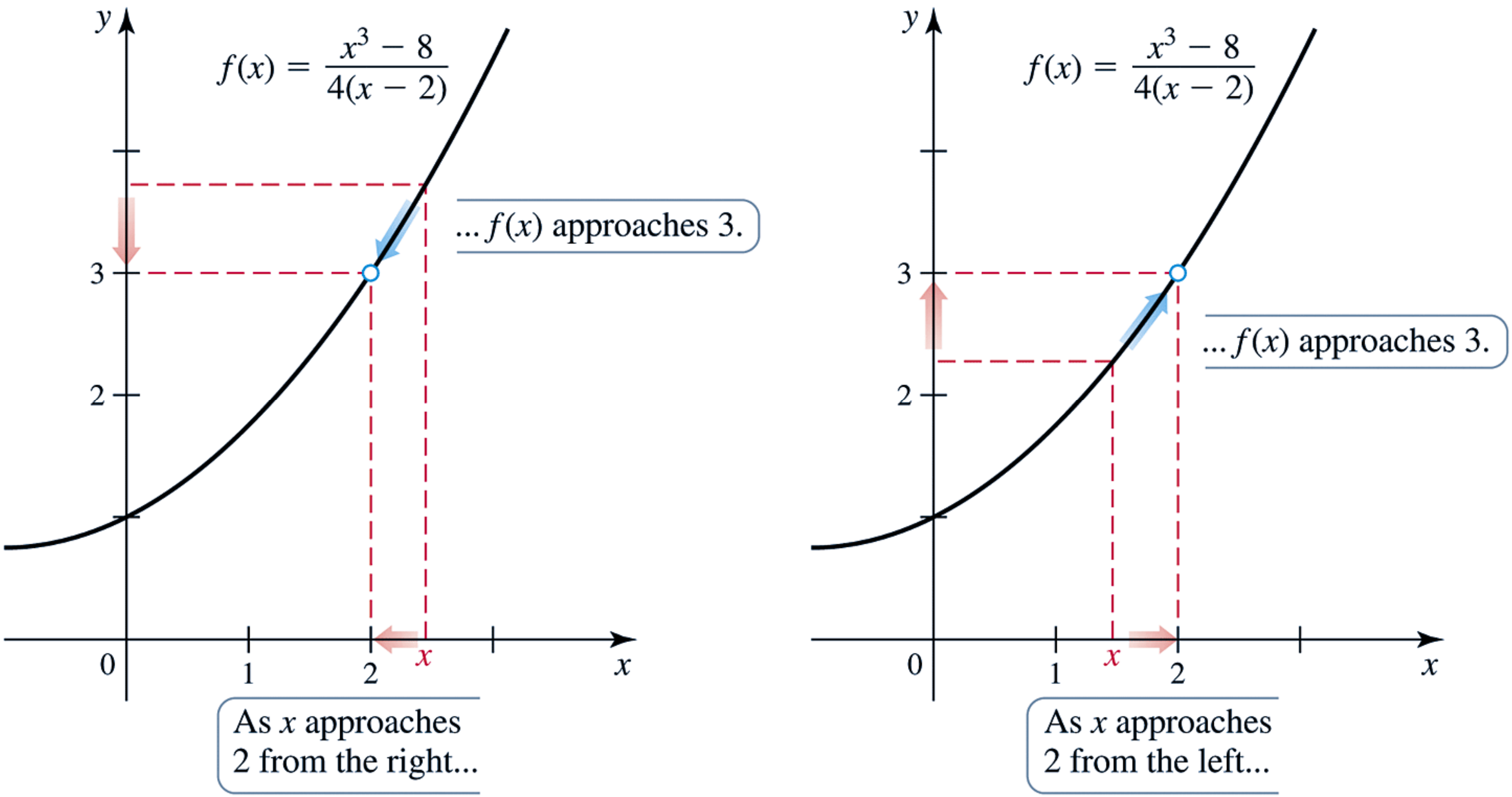


Figure 2.12 (a & b)



CLICK HERE TO ACCESS THE COMPLETE Solutions

Table 2.3

Table 2.3

	$\longrightarrow$				2			$\longleftarrow$
x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x) = \dfrac{x^3 - 8}{4(x - 2)}$	2.8525	2.985025	2.99850025	2.99985000	3.00015000	3.00150025	3.015025	3.1525

THEOREM 2.1 Relationship Between One-Sided and Two-Sided Limits

Assume f is defined for all x near a except possibly at a. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Figure 2.13

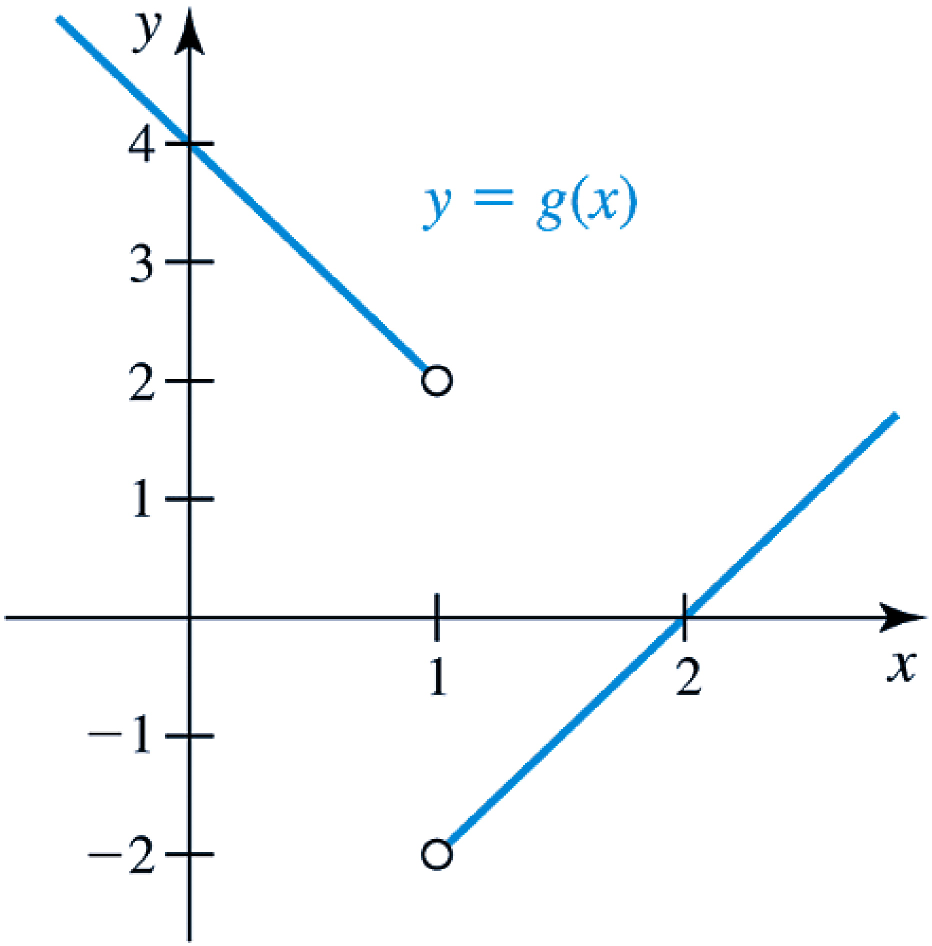
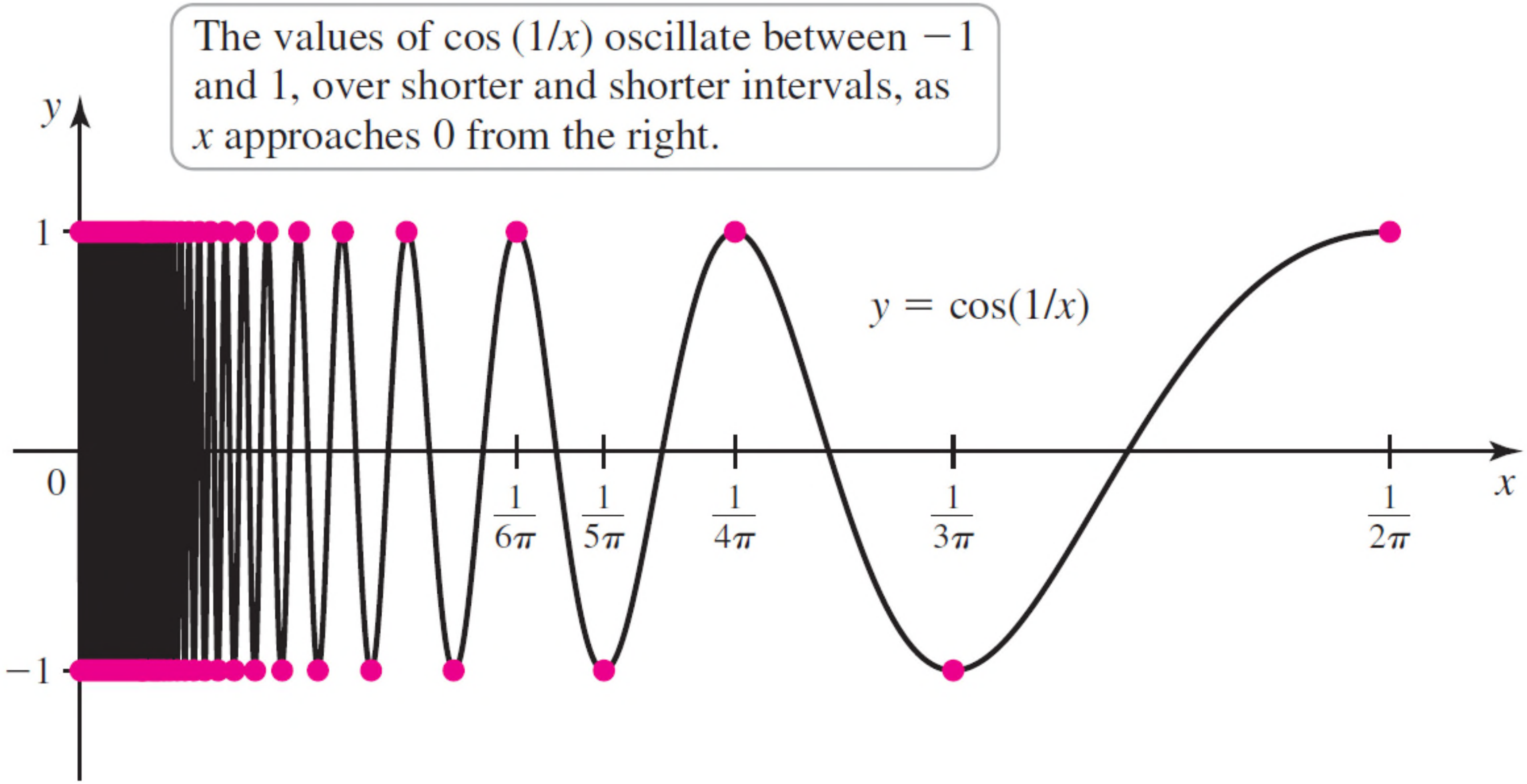


Table 2.4

x	$\cos(1/x)$
0.001	0.56238
0.0001	−0.95216
0.00001	−0.99936
0.000001	0.93675
0.0000001	−0.90727
0.00000001	−0.36338

We might *incorrectly* conclude that $\cos(1/x)$ approaches -1 as x approaches 0 from the right.

Figure 2.14


The values of cos (1/x) oscillate between −1 and 1, over shorter and shorter intervals, as x approaches 0 from the right.
y
1
−1
0
y = cos(1/x)
1/6π
1/5π
1/4π
1/3π
1/2π
x

2.3

Techniques for Computing Limits

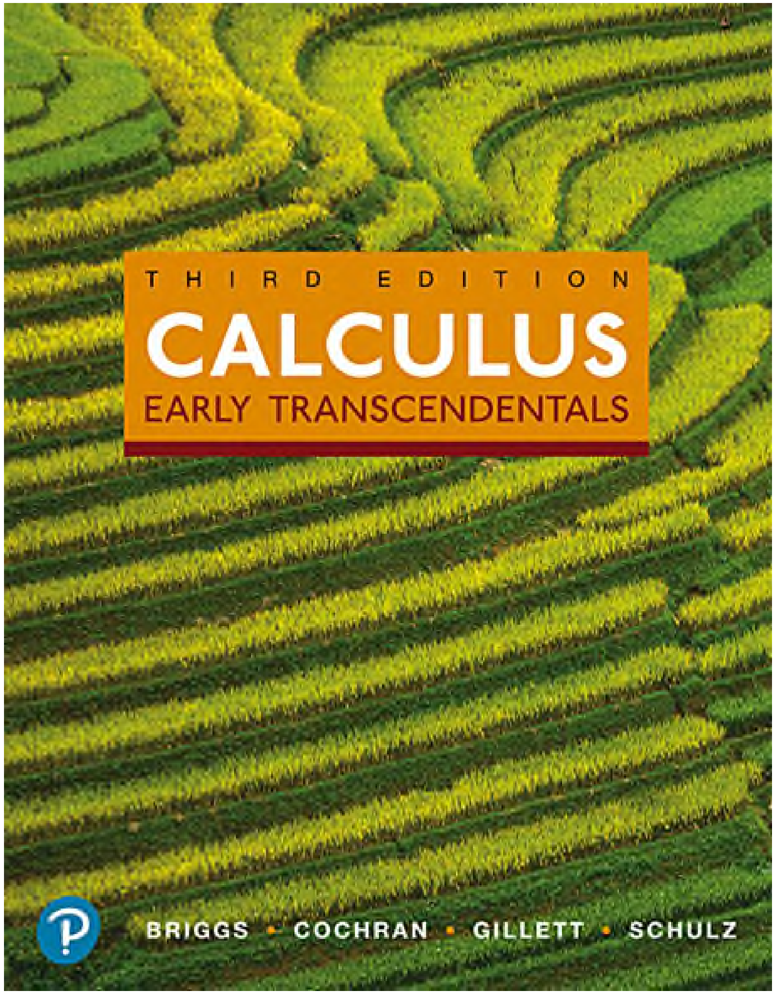
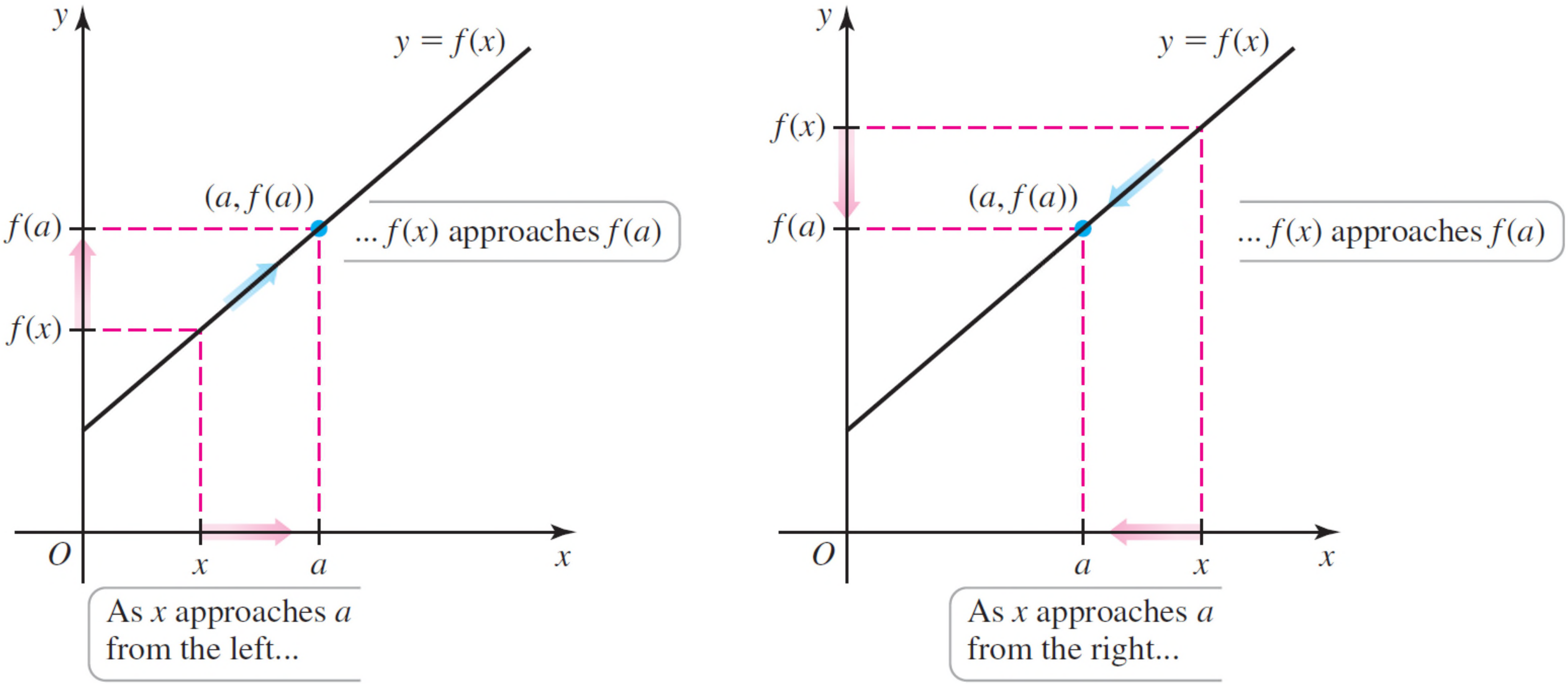


Figure 2.15



$\lim_{x \to a} f(x) = f(a)$ because $f(x) \to f(a)$ as $x \to a$ from both sides of a.

THEOREM 2.2 Limits of Linear Functions

Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

THEOREM 2.3 Limit Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. **Constant multiple** $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$
4. **Product** $\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
5. **Quotient** $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
6. **Power** $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$
7. **Root** $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n}$, provided $f(x) > 0$, for x near a, if n is even

THEOREM 2.4 Limits of Polynomial and Rational Functions

Assume p and q are polynomials and a is a constant.

a. Polynomial functions: $\lim_{x \to a} p(x) = p(a)$

b. Rational functions: $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$

THEOREM 2.3 (CONTINUED) Limit Laws for One-Sided Limits

Laws 1–6 hold with $\lim_{x \to a}$ replaced with $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$. Law 7 is modified as follows. Assume $n > 0$ is an integer.

7. Root

a. $\lim_{x \to a^+} (f(x))^{1/n} = \left(\lim_{x \to a^+} f(x)\right)^{1/n}$, provided $f(x) \geq 0$, for x near a with $x > a$, if n is even

b. $\lim_{x \to a^-} (f(x))^{1/n} = \left(\lim_{x \to a^-} f(x)\right)^{1/n}$, provided $f(x) \geq 0$, for x near a with $x < a$, if n is even

Figure 2.16

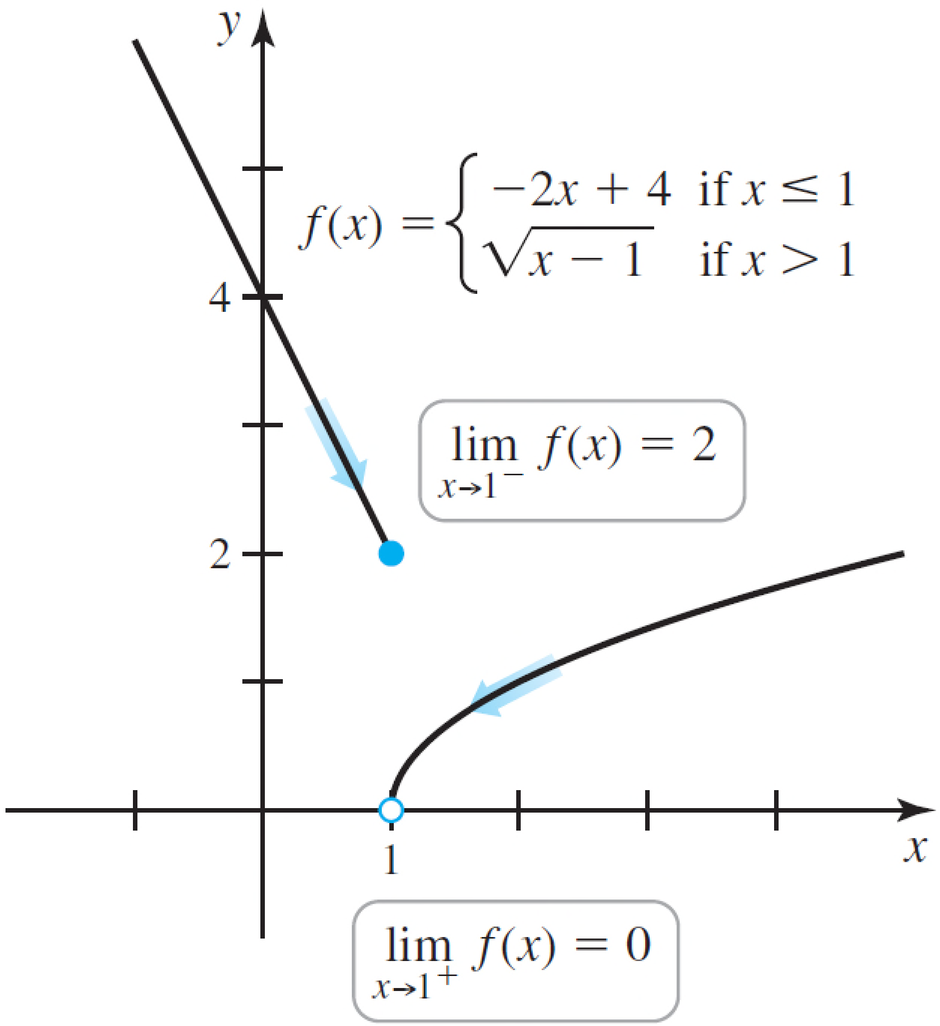

y
$f(x) = \begin{cases} -2x + 4 & \text{if } x \le 1 \\ \sqrt{x - 1} & \text{if } x > 1 \end{cases}$
4
$\lim_{x \to 1^-} f(x) = 2$
2
1
x
$\lim_{x \to 1^+} f(x) = 0$

Figure 2.17

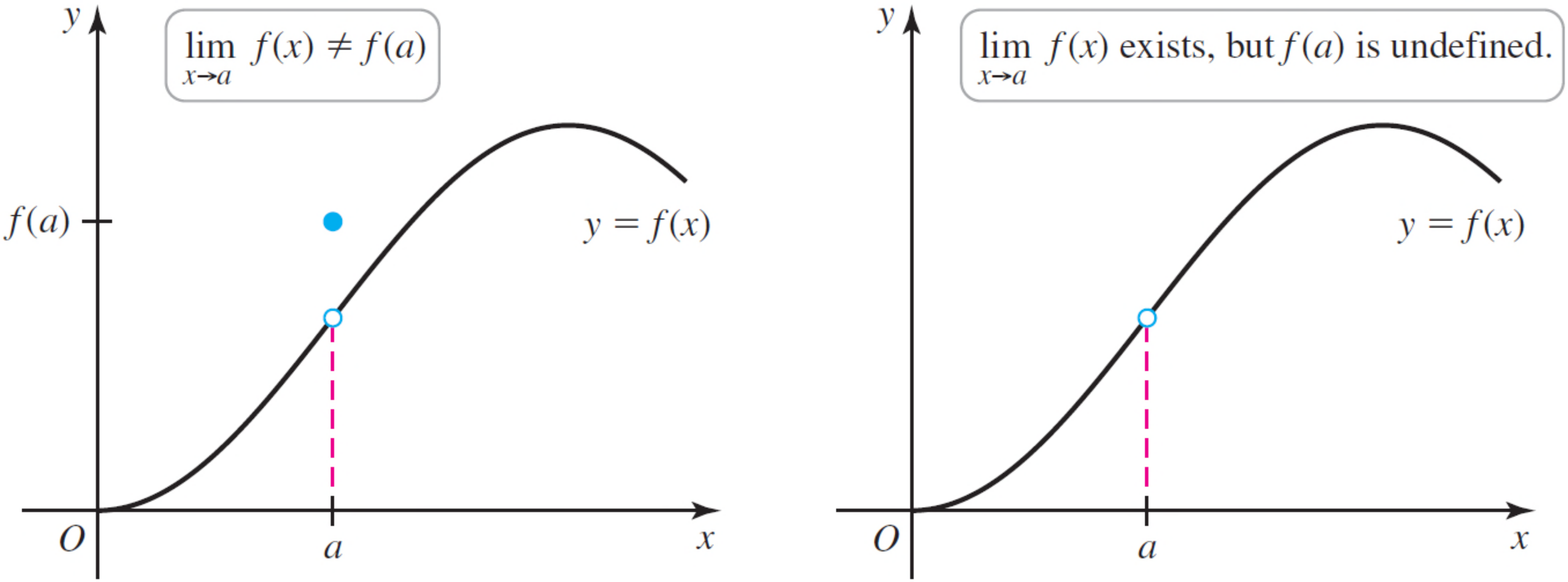
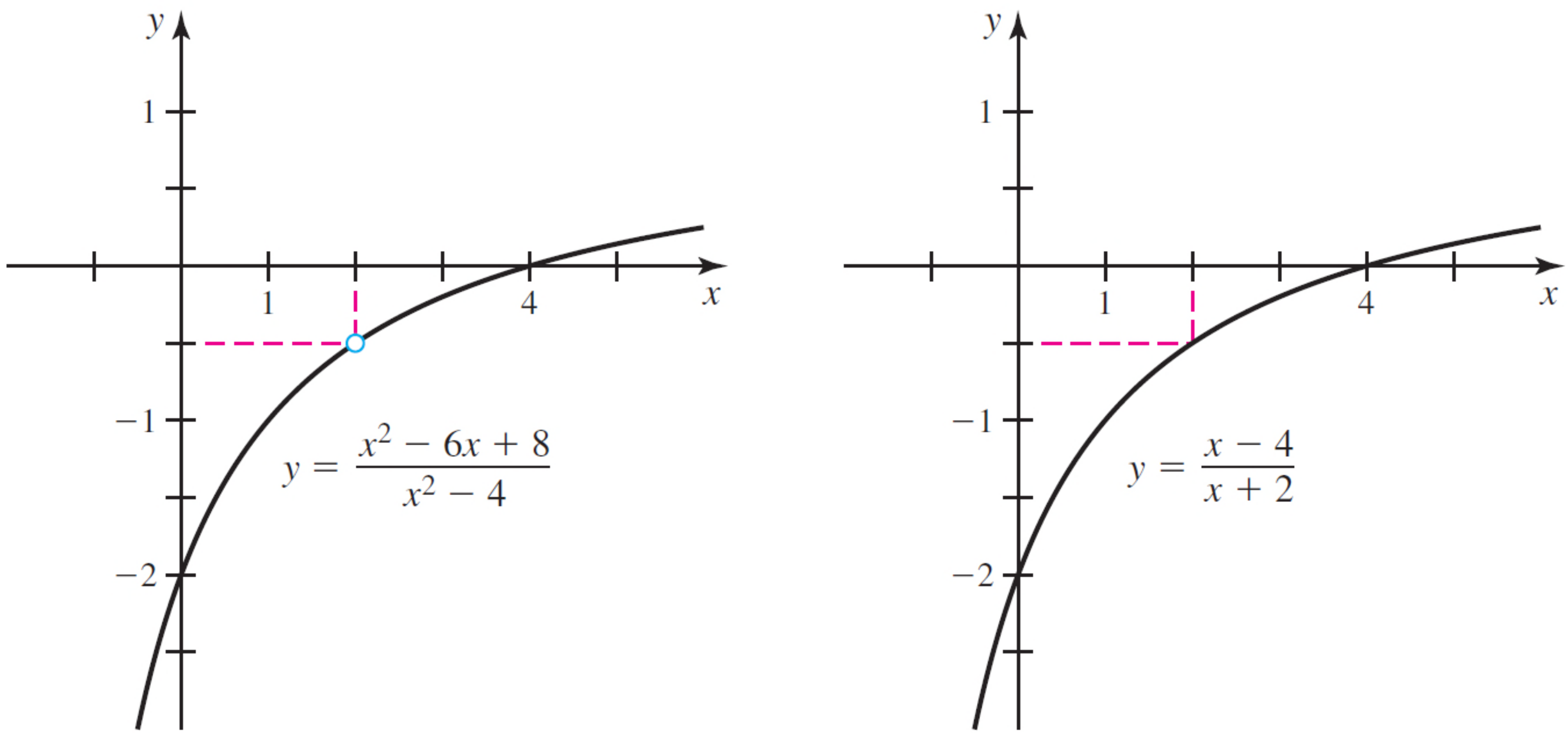

y
lim f(x) ≠ f(a)
x→a
f(a)
y = f(x)
O
a
x
y
lim f(x) exists, but f(a) is undefined.
x→a
y = f(x)
O
a
x

Figure 2.18



$$\lim_{x \to 2} \frac{x^2 - 6x + 8}{x^2 - 4} = \lim_{x \to 2} \frac{x - 4}{x + 2} = -\frac{1}{2}$$

Figure 2.19 (a)

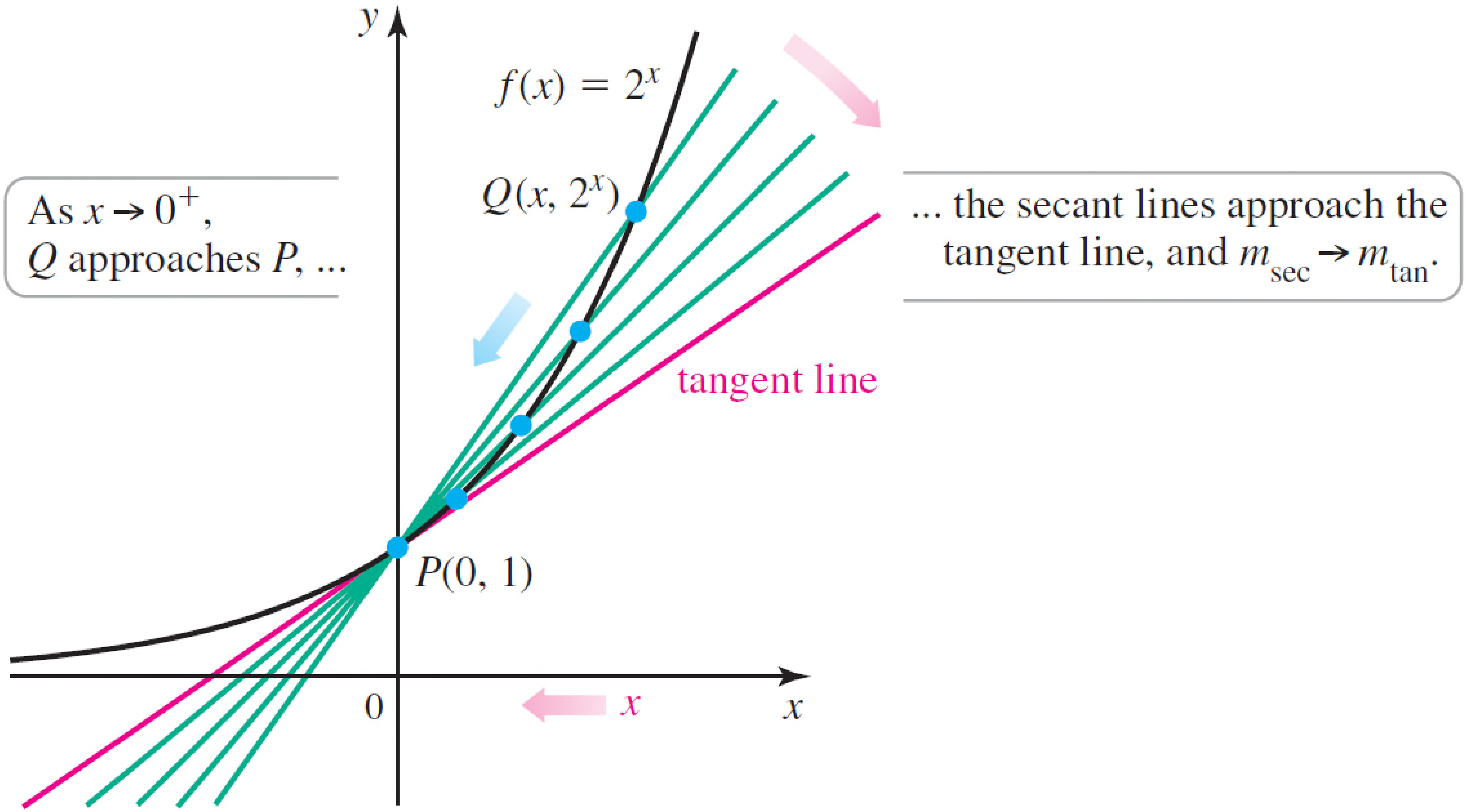
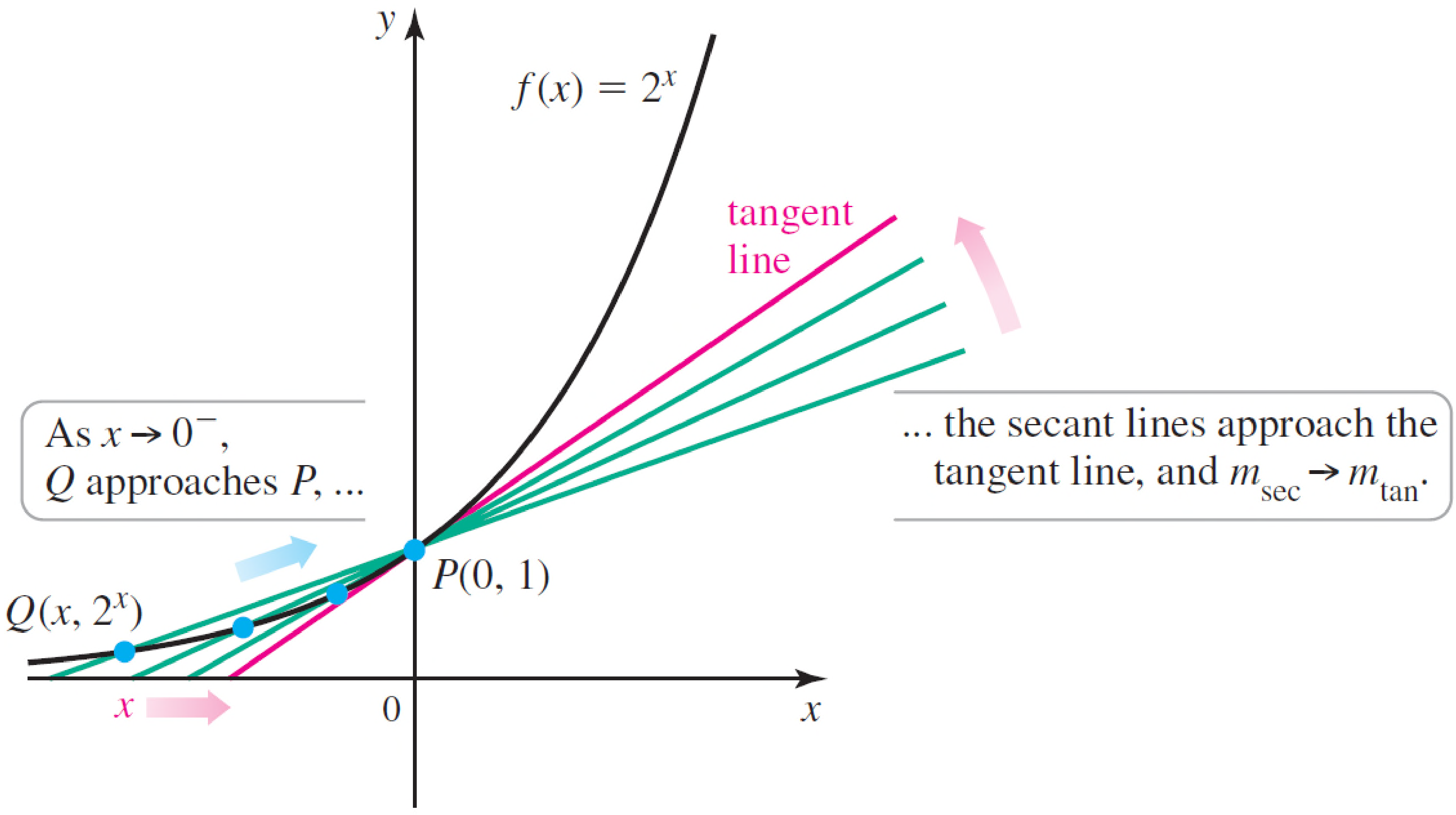

As $x \to 0^+$,
Q approaches P, ...
... the secant lines approach the tangent line, and $m_{sec} \to m_{tan}$.
$f(x) = 2^x$
$Q(x, 2^x)$
$P(0, 1)$
tangent line
y
x
0
x

Figure 2.19 (b)

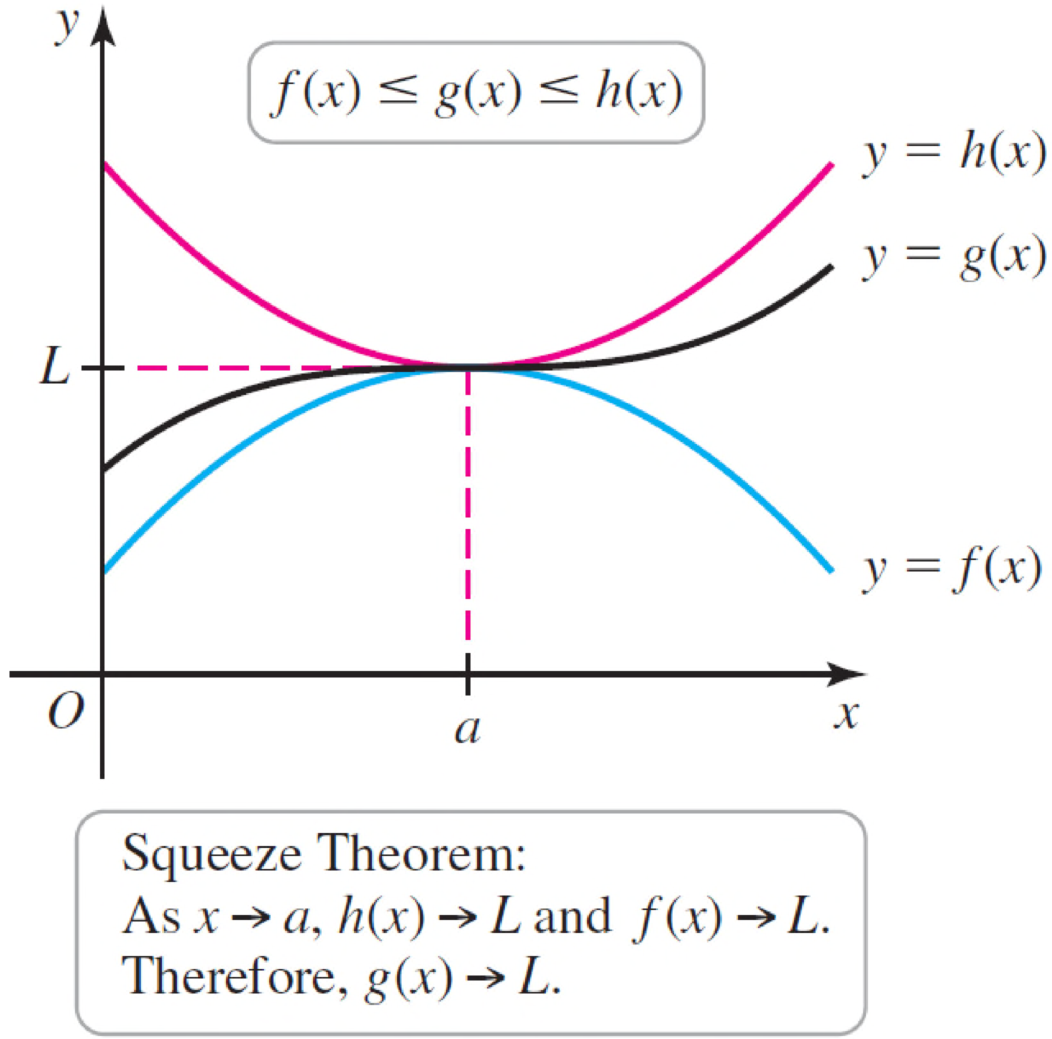


y
$f(x) = 2^x$
tangent line
As $x \to 0^-$,
Q approaches P, ...
... the secant lines approach the tangent line, and $m_{sec} \to m_{tan}$.
$P(0, 1)$
$Q(x, 2^x)$
x
0
x

Table 2.5

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$m_{sec} = \dfrac{2^x - 1}{x}$	1.000000	0.7177	0.6956	0.6934	0.6932	0.6931

Figure 2.20



y
f(x) ≤ g(x) ≤ h(x)
y = h(x)
y = g(x)
L
y = f(x)
O
a
x
Squeeze Theorem:
As x → a, h(x) → L and f(x) → L.
Therefore, g(x) → L.

THEOREM 2.5 The Squeeze Theorem

Assume the functions f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Figure 2.21

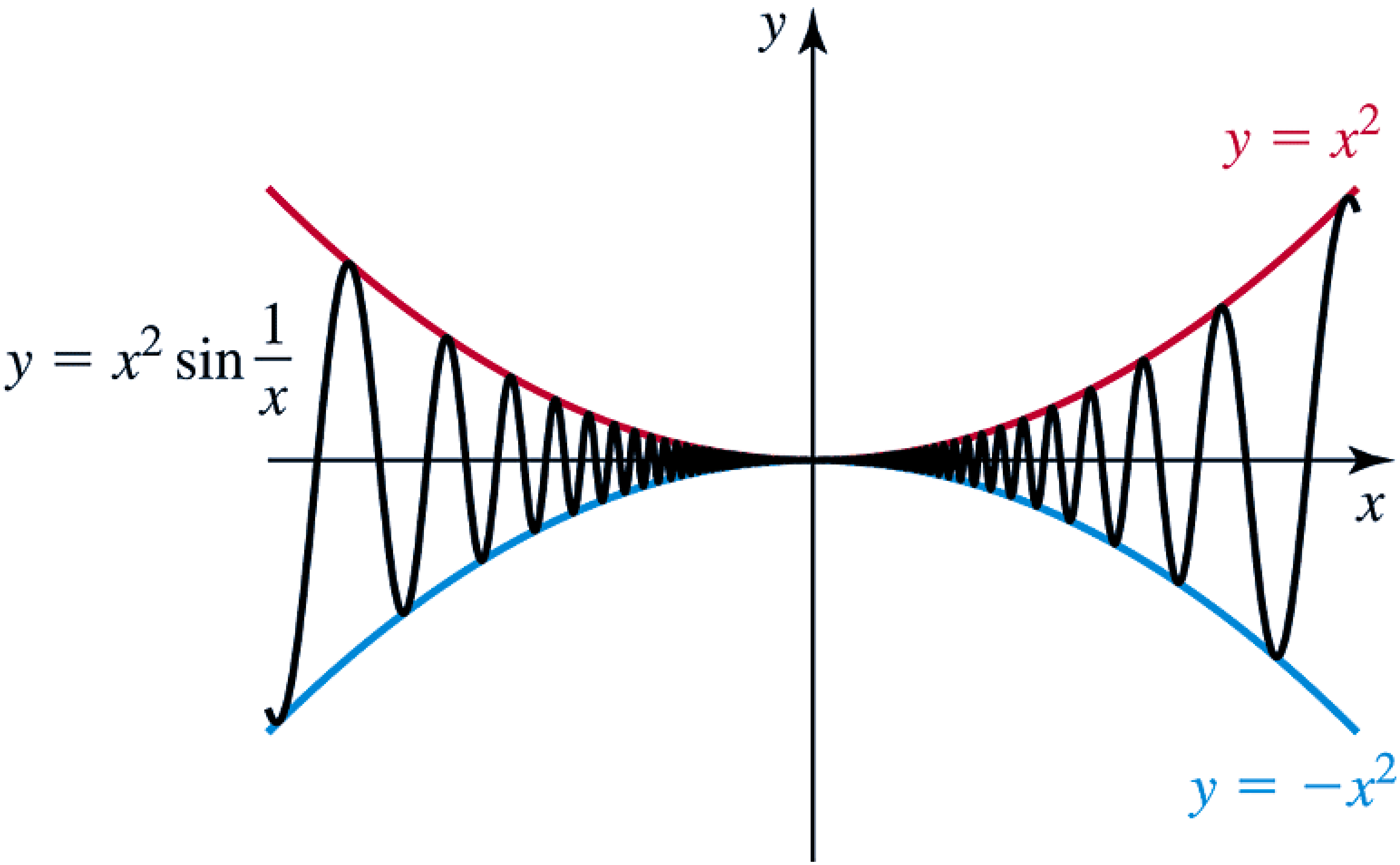
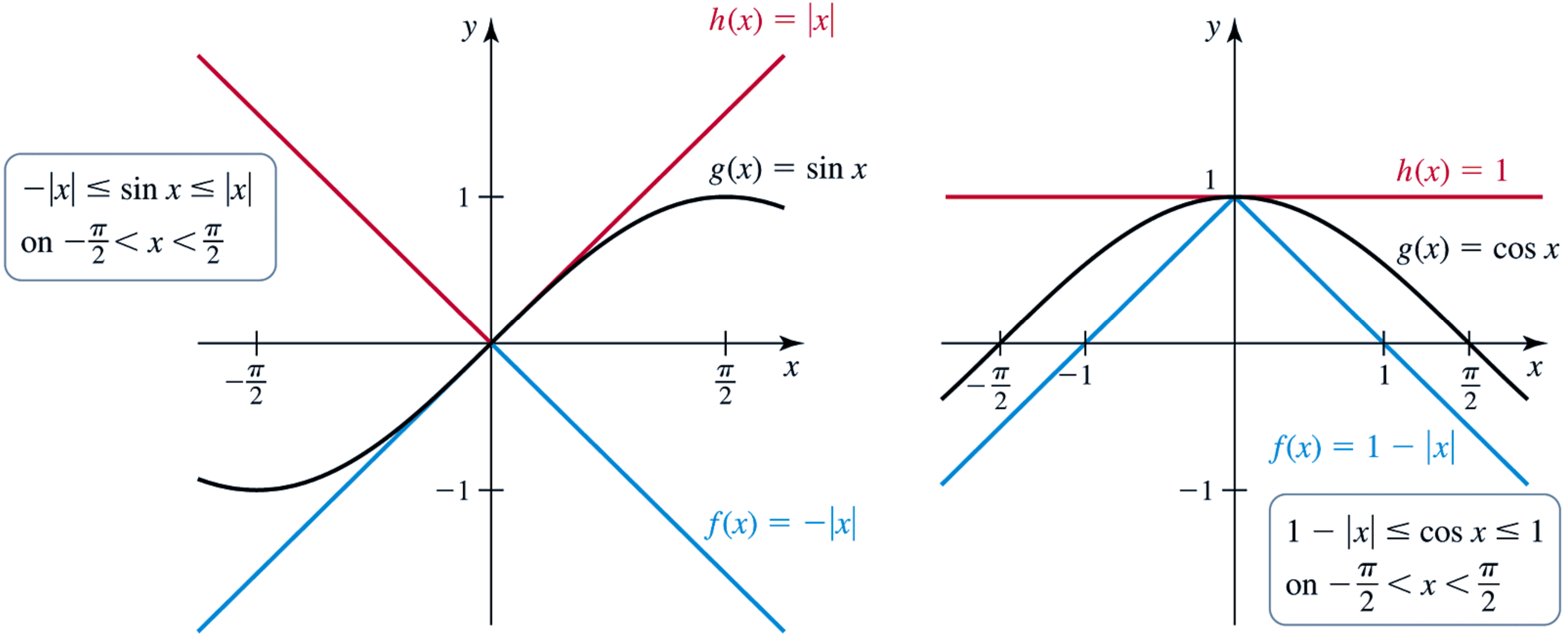


Figure 2.22



$-|x| \le \sin x \le |x|$
on $-\frac{\pi}{2} < x < \frac{\pi}{2}$
$h(x) = |x|$
$g(x) = \sin x$
$f(x) = -|x|$
y
x
1
−1
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$h(x) = 1$
$g(x) = \cos x$
$f(x) = 1 - |x|$
$1 - |x| \le \cos x \le 1$
on $-\frac{\pi}{2} < x < \frac{\pi}{2}$

2.4

Infinite Limits

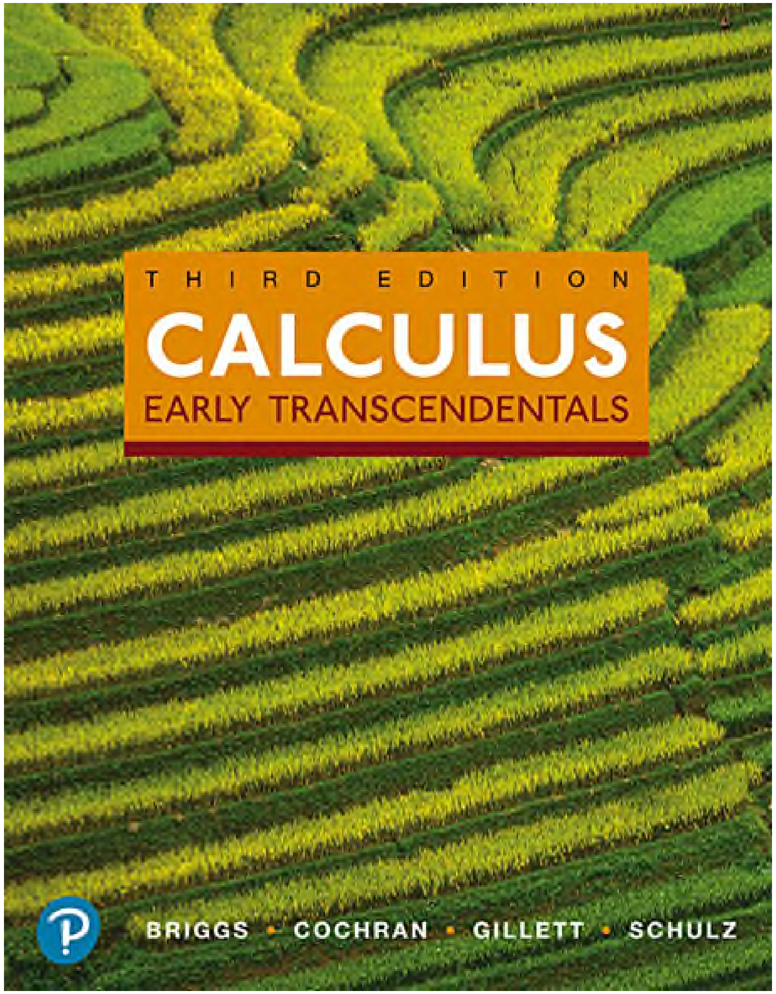


Table 2.6

x	$f(x) = 1/x^2$
± 0.1	100
± 0.01	10,000
± 0.001	1,000,000
$\downarrow$	$\downarrow$
0	∞

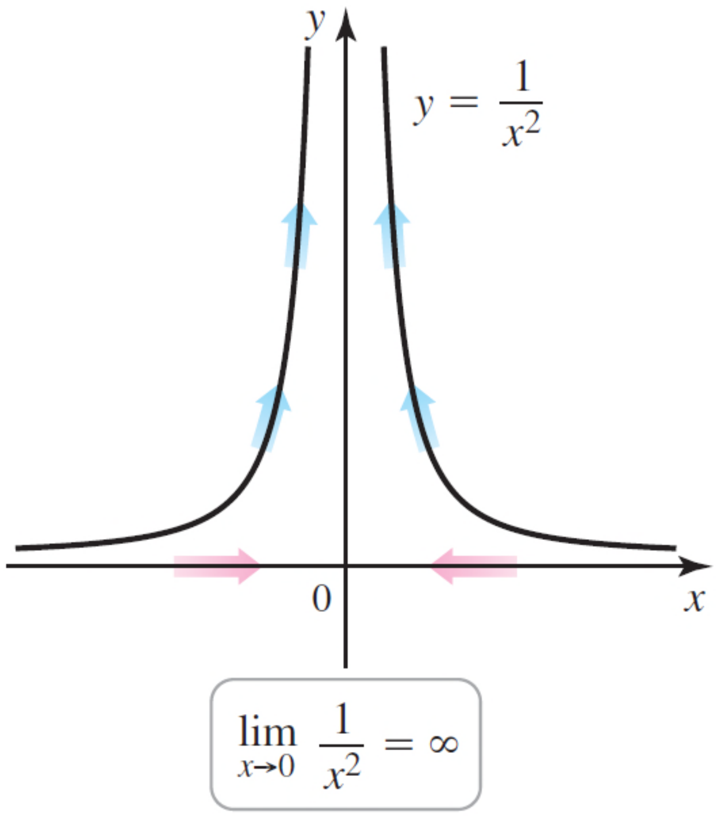


Table 2.7

x	$f(x) = 1/x^2$
10	0.01
100	0.0001
1000	0.000001
↓	↓
∞	0

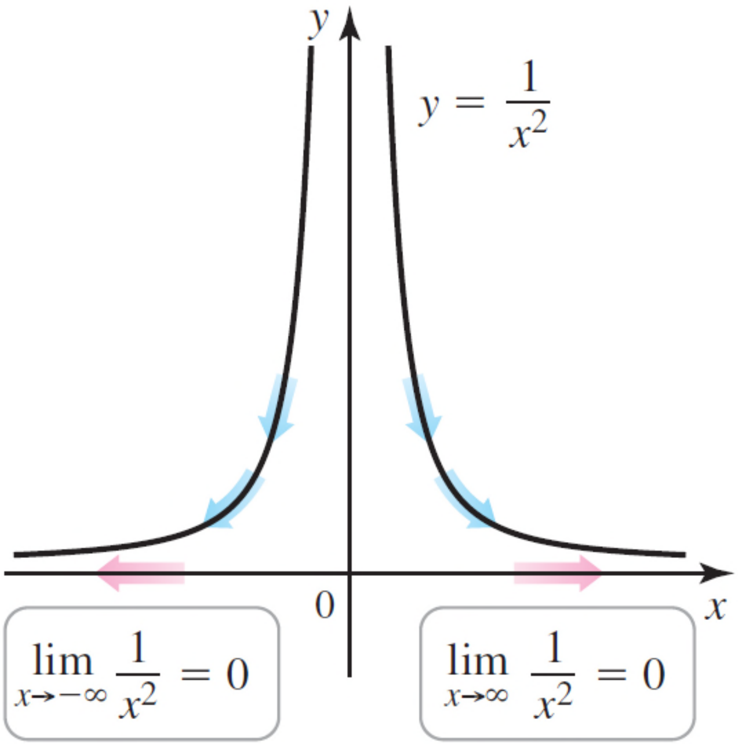
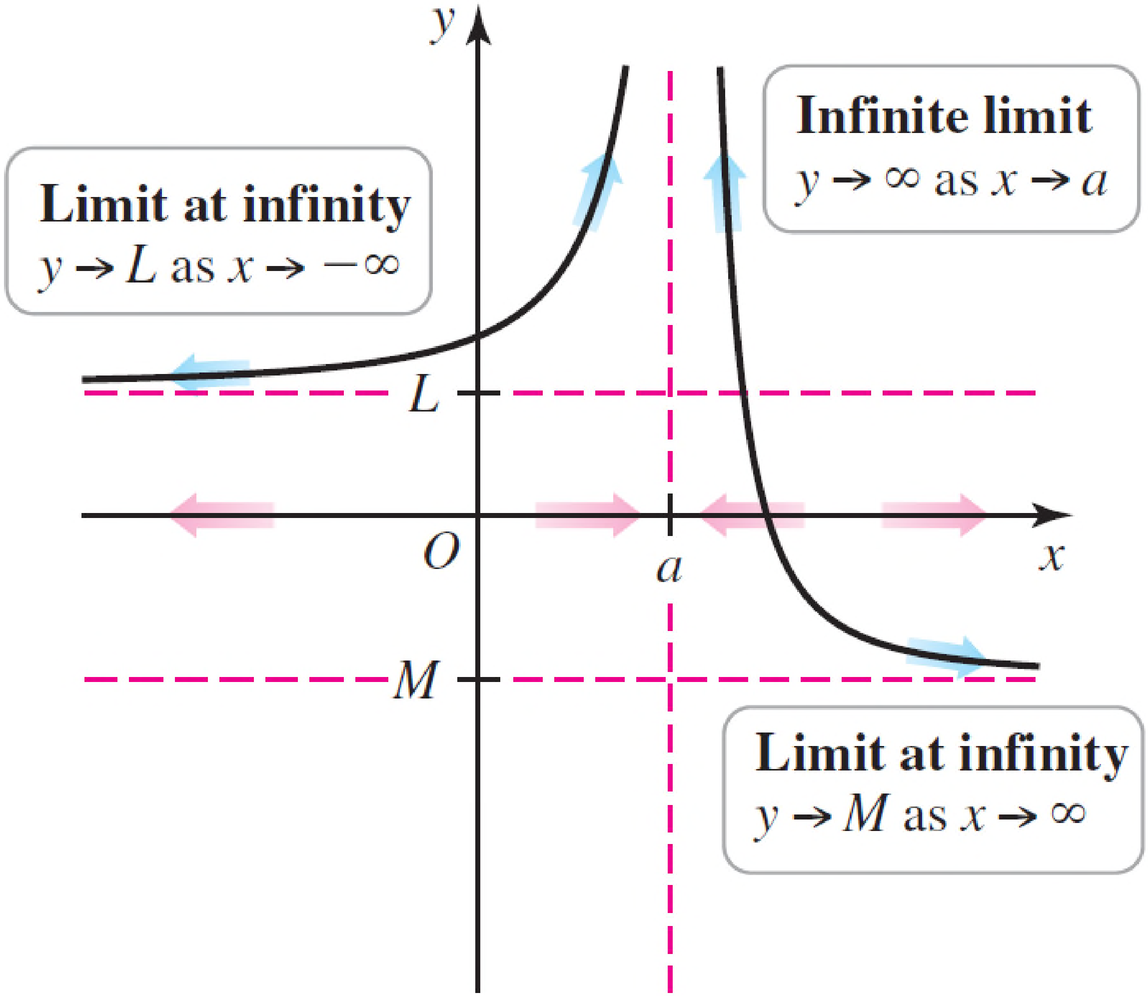


Figure 2.23



y
x
O
a
L
M
Limit at infinity
y → L as x → −∞
Infinite limit
y → ∞ as x → a
Limit at infinity
y → M as x → ∞

DEFINITION Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a (Figure 2.24a), we write

$$\lim_{x \to a} f(x) = \infty$$

and say the limit of $f(x)$ as x approaches a is infinity.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a (Figure 2.24b), we write

$$\lim_{x \to a} f(x) = -\infty$$

and say the limit of $f(x)$ as x approaches a is negative infinity. *In both cases, the limit does not exist.*

Figure 2.24 (a)

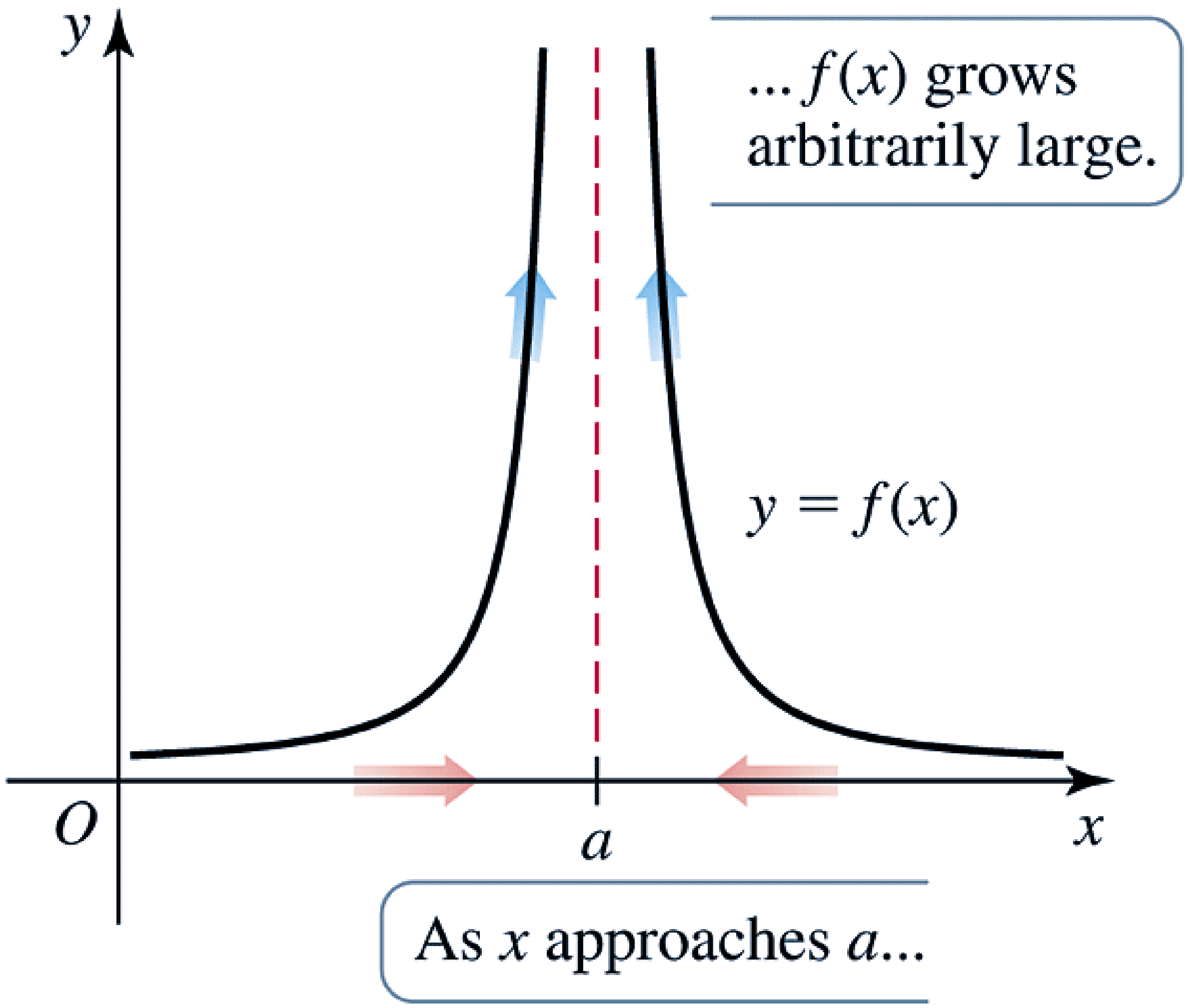


Figure 2.24 (b)

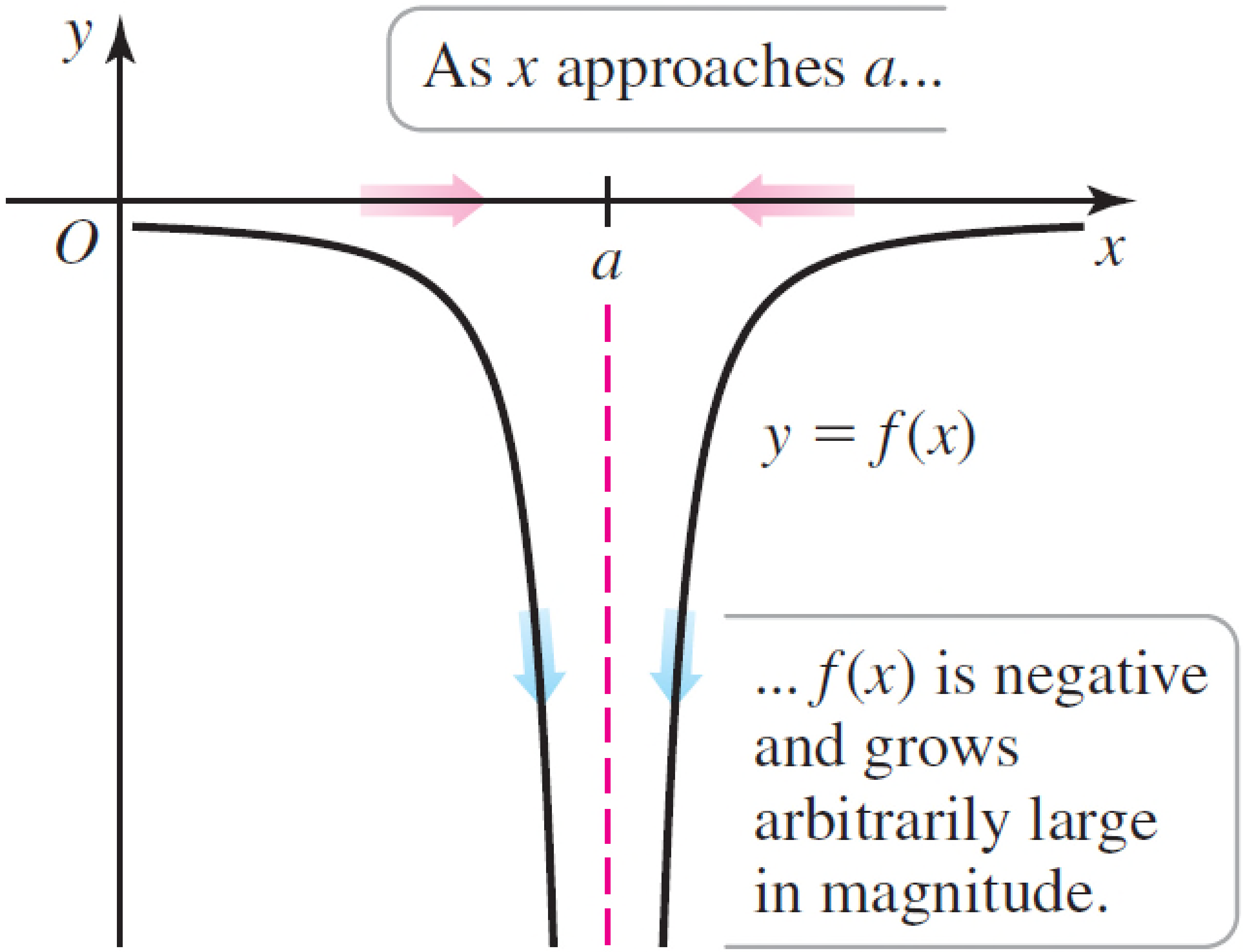
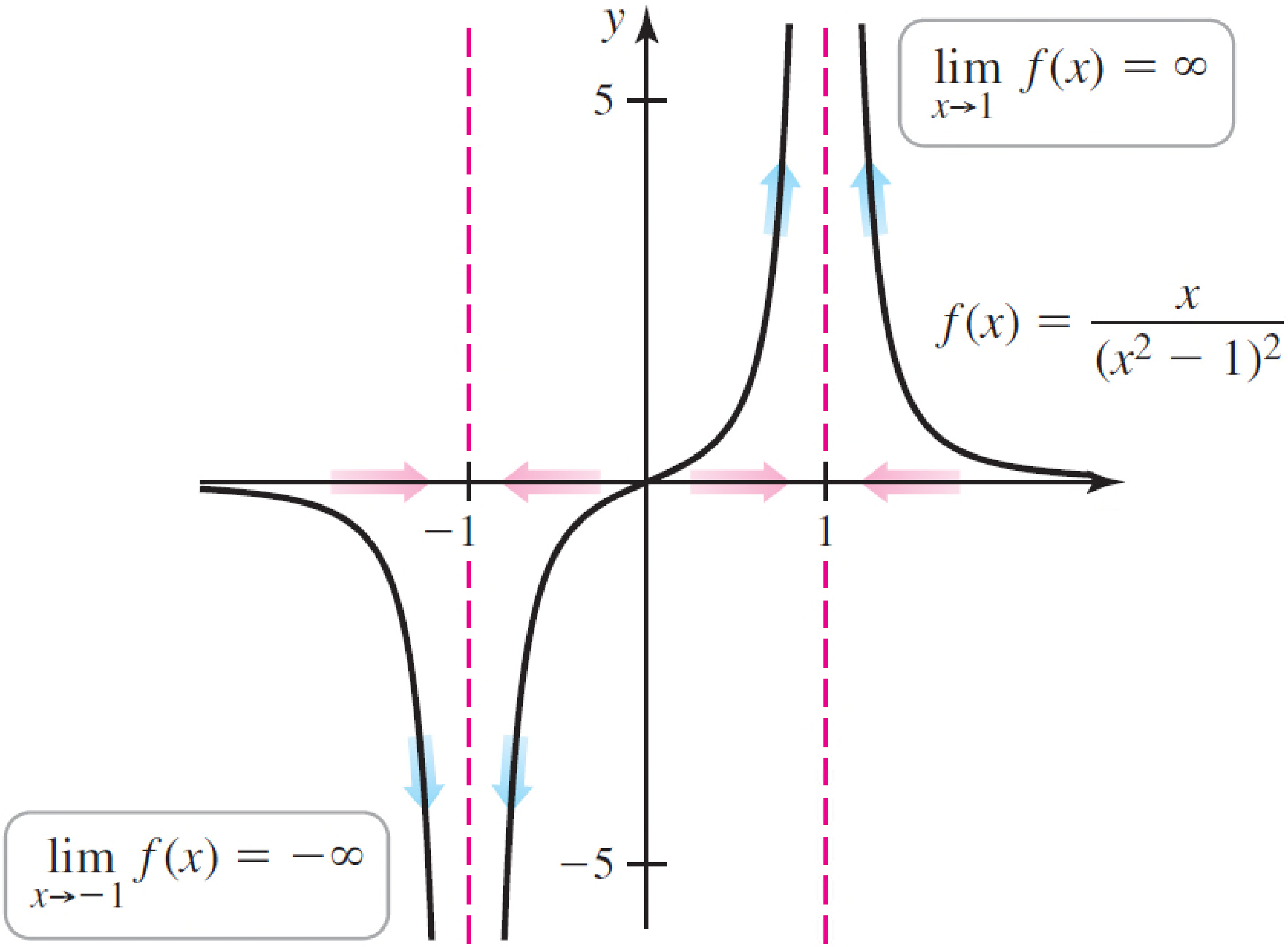

y
As x approaches a...
O
a
x
y = f(x)
... f(x) is negative
and grows
arbitrarily large
in magnitude.

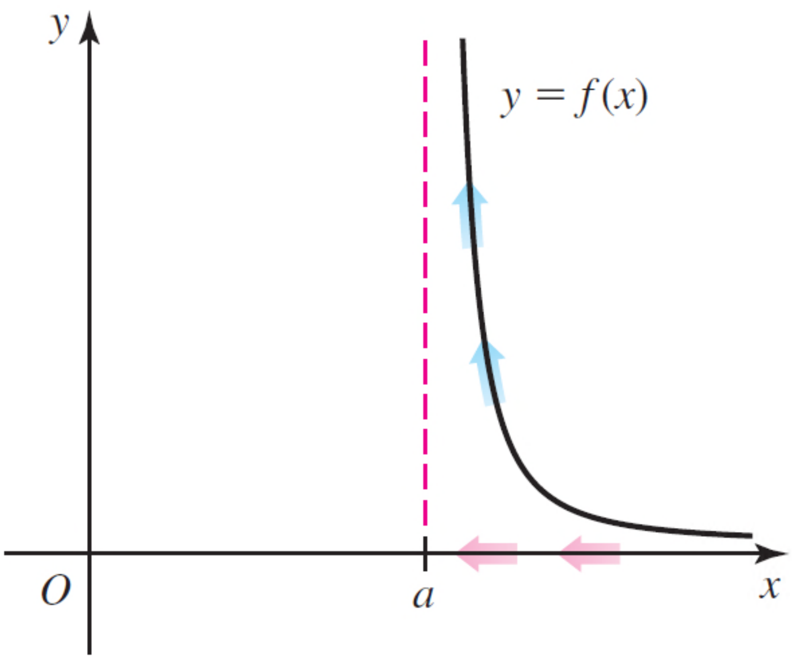
Figure 2.25



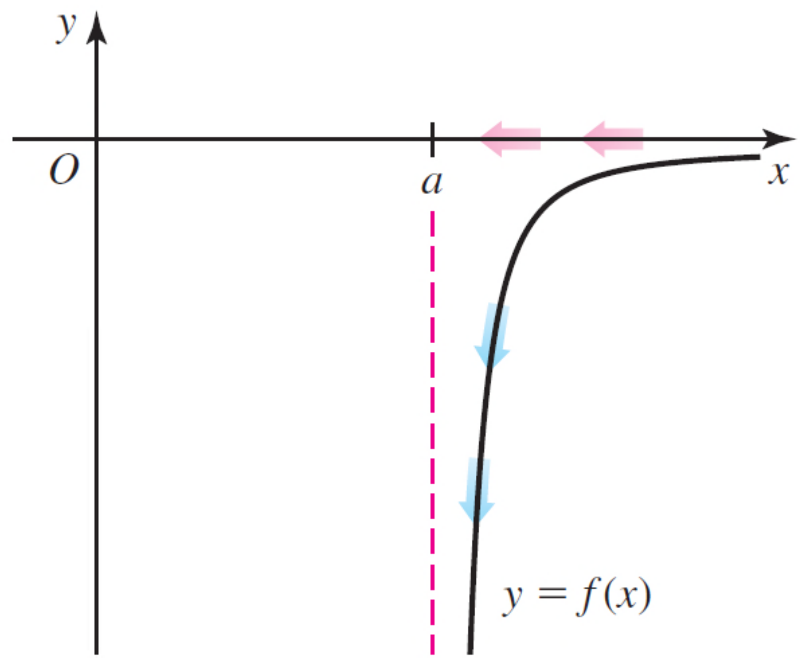
DEFINITION One-Sided Infinite Limits

Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write $\lim_{x \to a^+} f(x) = \infty$ (Figure 2.26a). The one-sided infinite limits $\lim_{x \to a^+} f(x) = -\infty$ (Figure 2.26b), $\lim_{x \to a^-} f(x) = \infty$ (Figure 2.26c), and $\lim_{x \to a^-} f(x) = -\infty$ (Figure 2.26d) are defined analogously.

Figure 2.26 (a & b) continued…

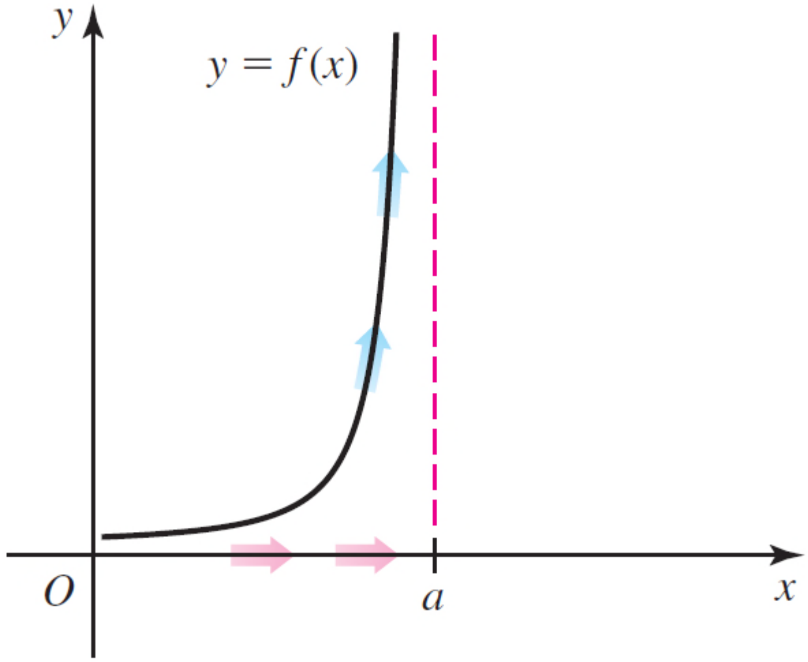


$$\lim_{x \to a^+} f(x) = \infty$$

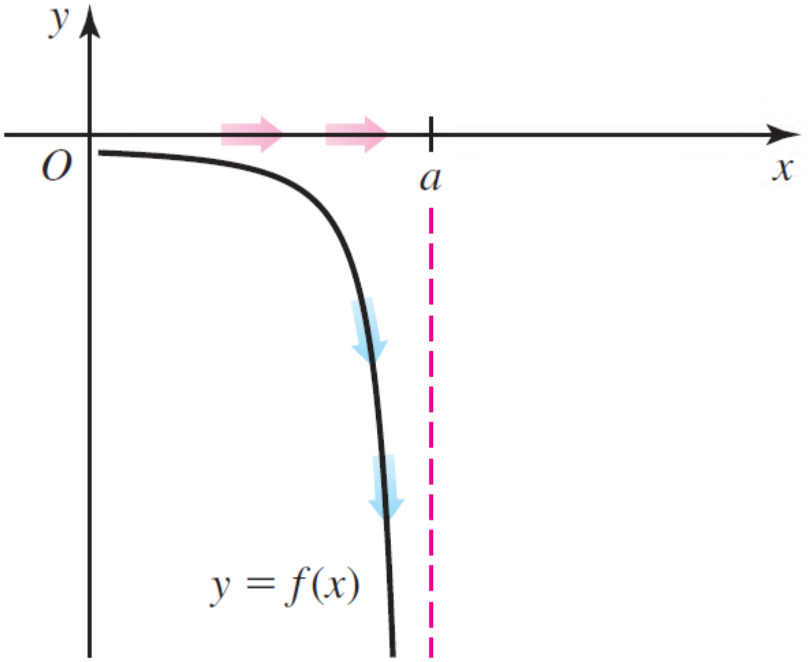


$$\lim_{x \to a^+} f(x) = -\infty$$

Figure 2.26 (c & d)



$$\lim_{x \to a^-} f(x) = \infty$$



$$\lim_{x \to a^-} f(x) = -\infty$$

DEFINITION Vertical Asymptote

If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a^+} f(x) = \pm\infty$, or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

Figure 2.27

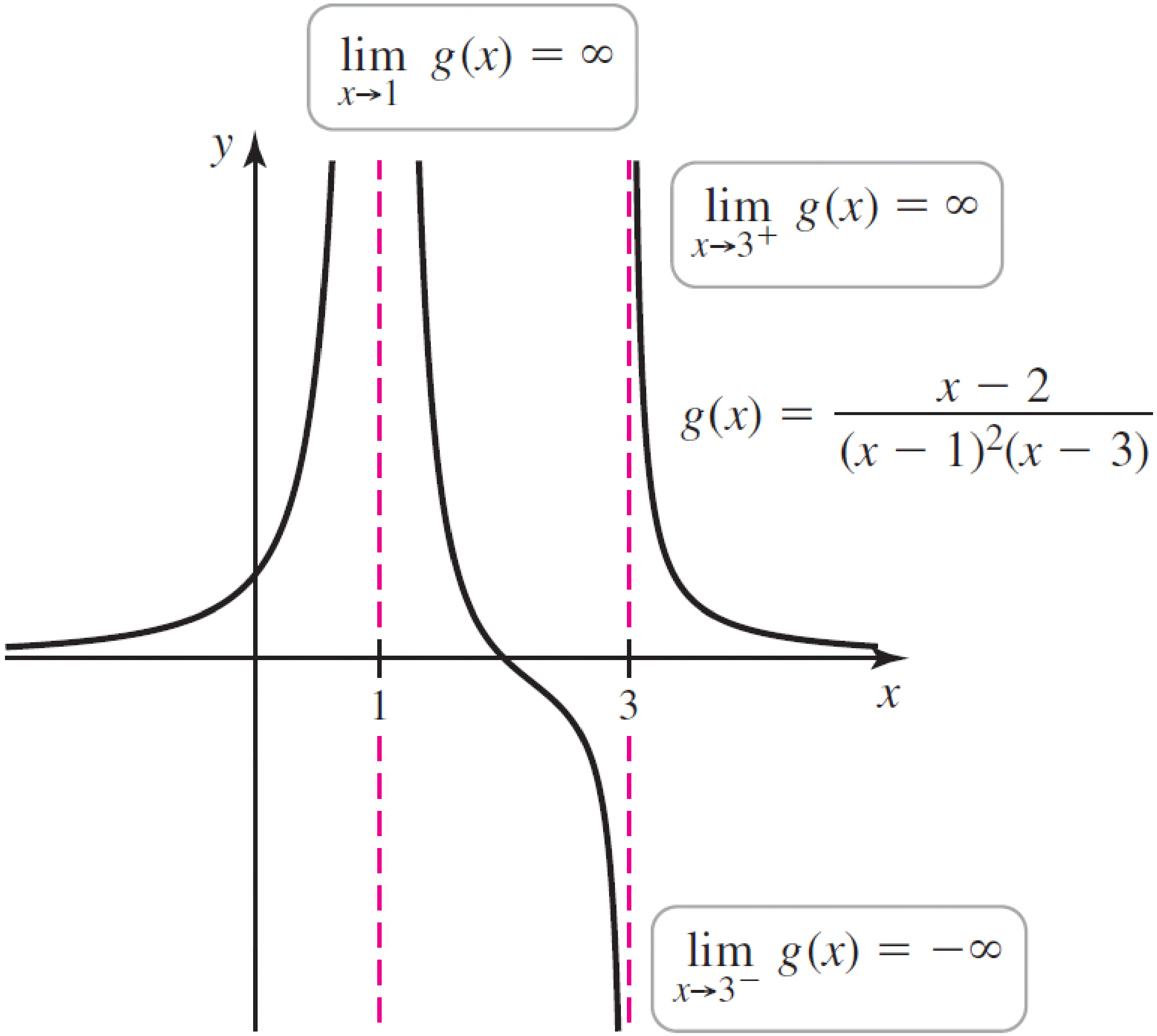

$\lim_{x \to 1} g(x) = \infty$
$\lim_{x \to 3^+} g(x) = \infty$
$g(x) = \dfrac{x-2}{(x-1)^2(x-3)}$
$\lim_{x \to 3^-} g(x) = -\infty$
y
x
1
3

Table 2.8

x	$\dfrac{5+x}{x}$
0.01	$\dfrac{5.01}{0.01} = 501$
0.001	$\dfrac{5.001}{0.001} = 5001$
0.0001	$\dfrac{5.0001}{0.0001} = 50{,}001$
$\downarrow$	$\downarrow$
0^+	∞

Figure 2.28

Two versions of the graph of $y = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

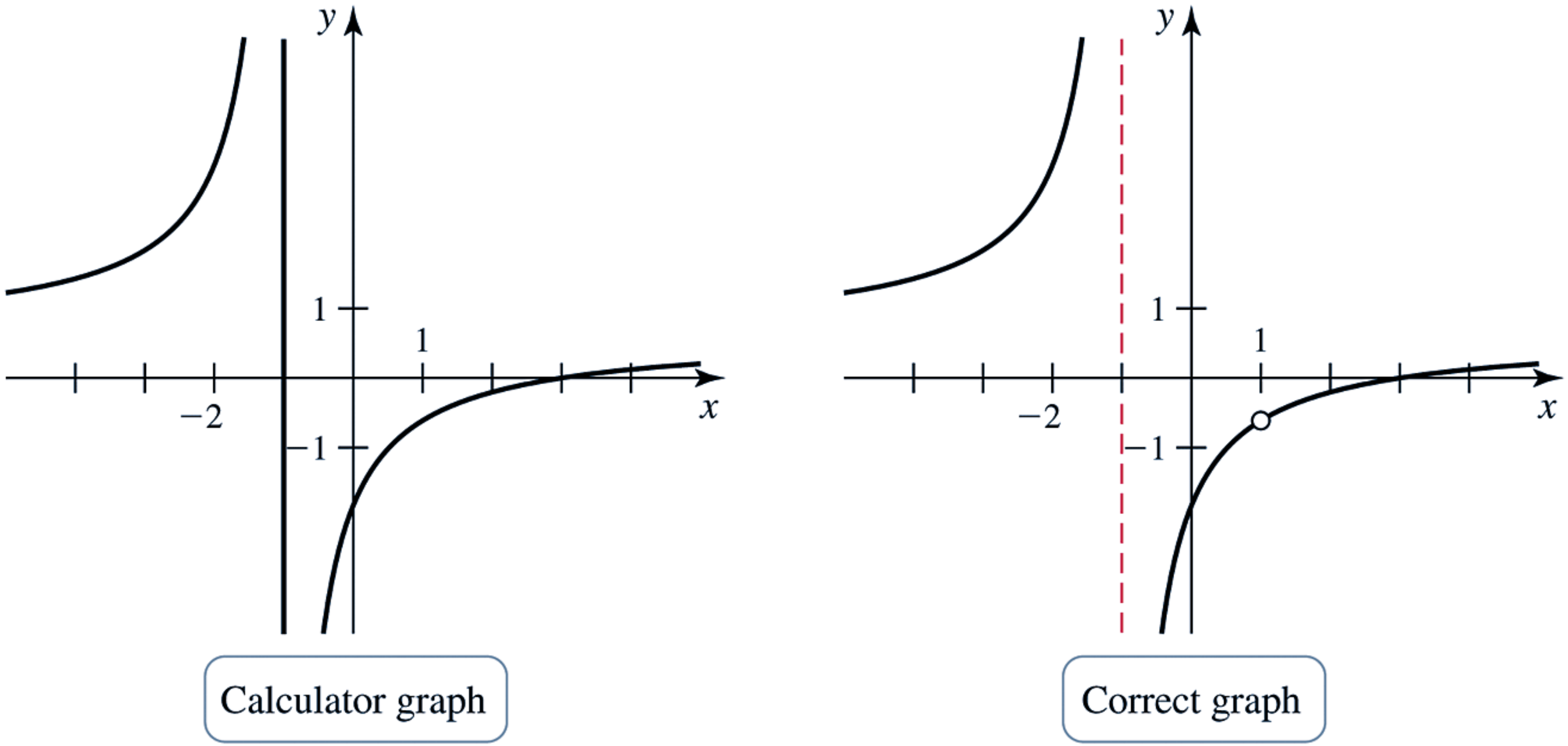
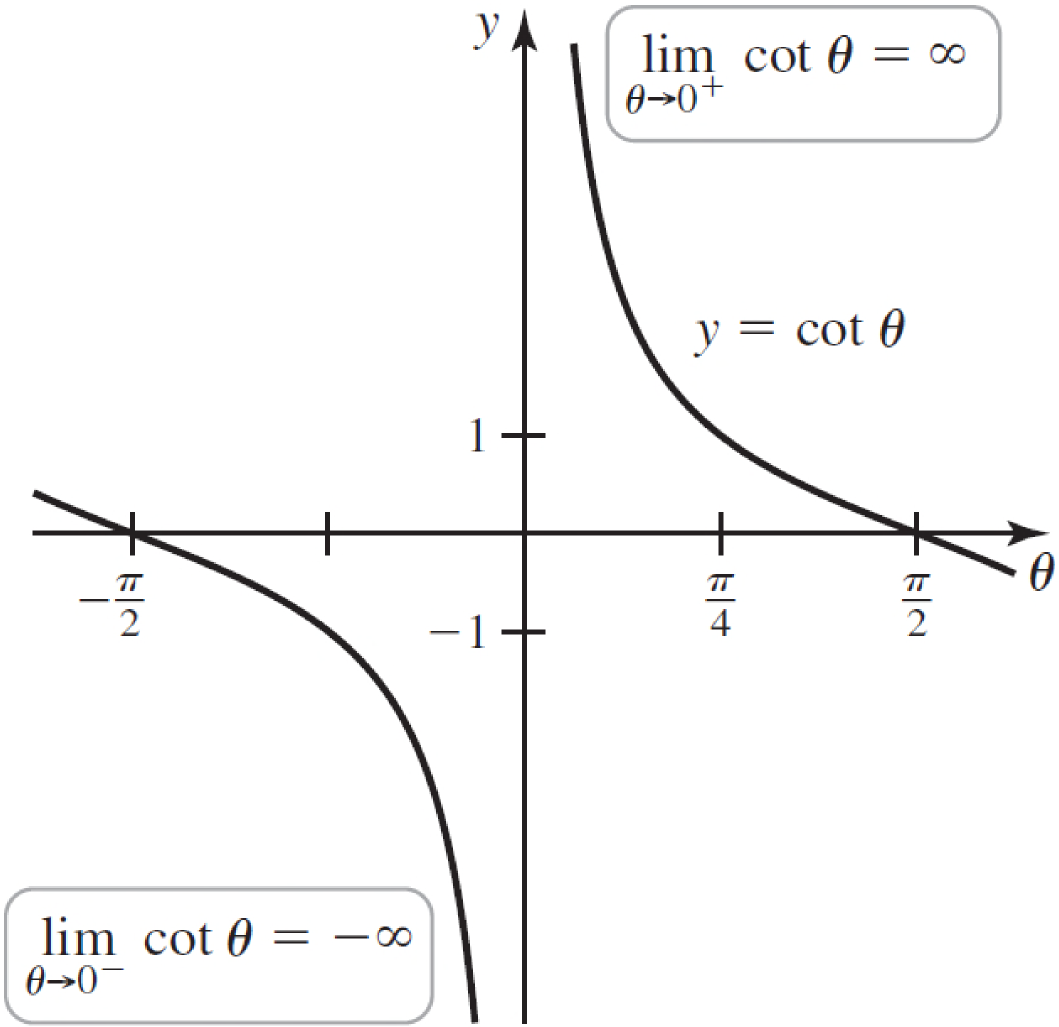


Figure 2.29



y
$\lim_{\theta \to 0^+} \cot \theta = \infty$
$y = \cot \theta$
1
−1
$-\frac{\pi}{2}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
θ
$\lim_{\theta \to 0^-} \cot \theta = -\infty$

2.5

Limits at Infinity

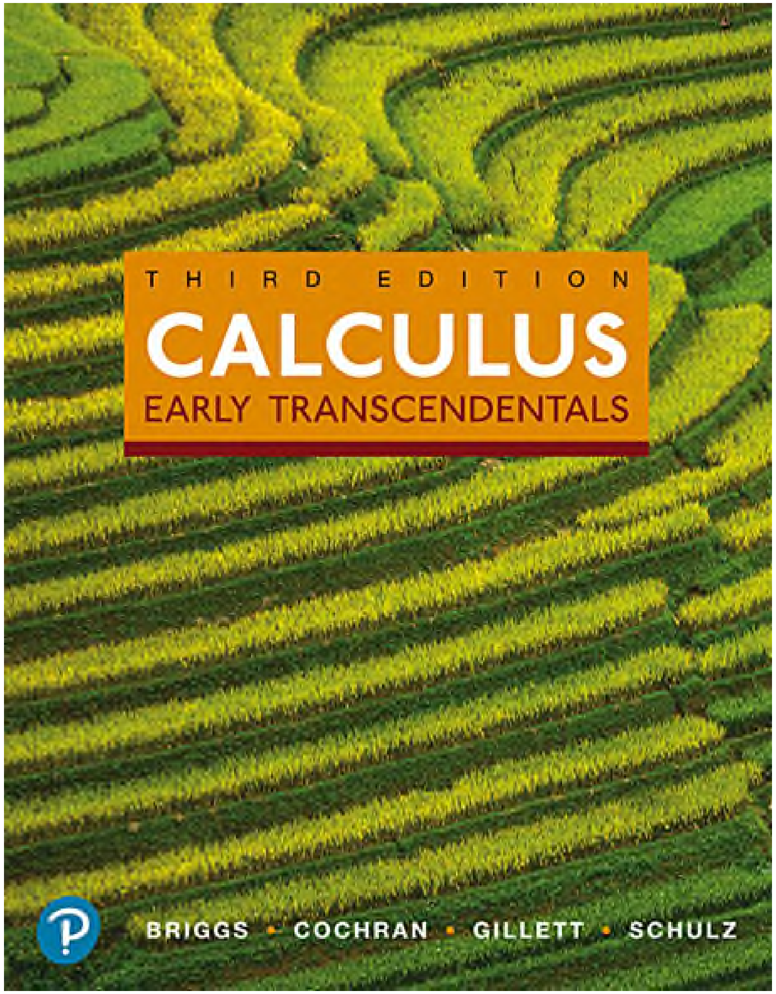


Figure 2.30

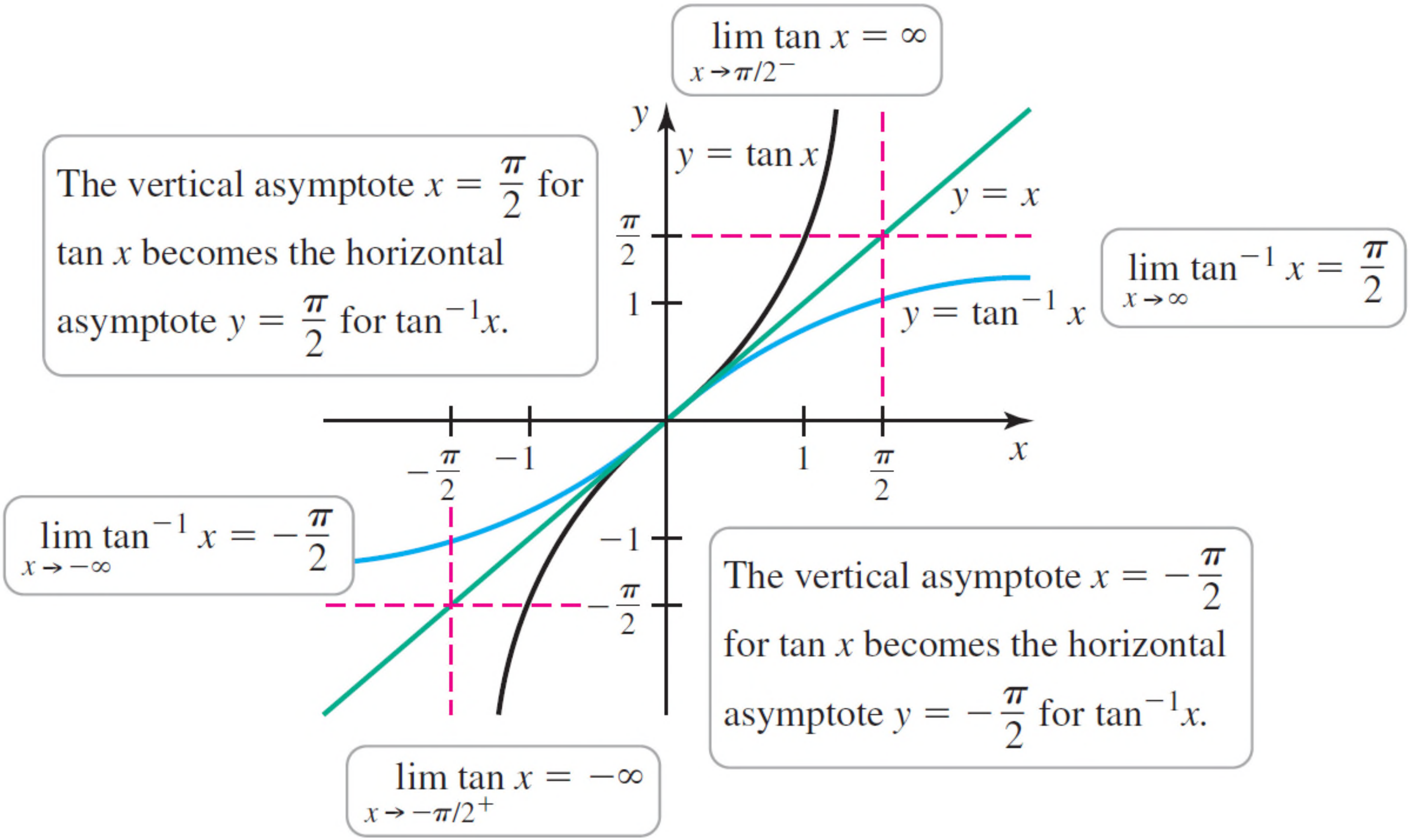
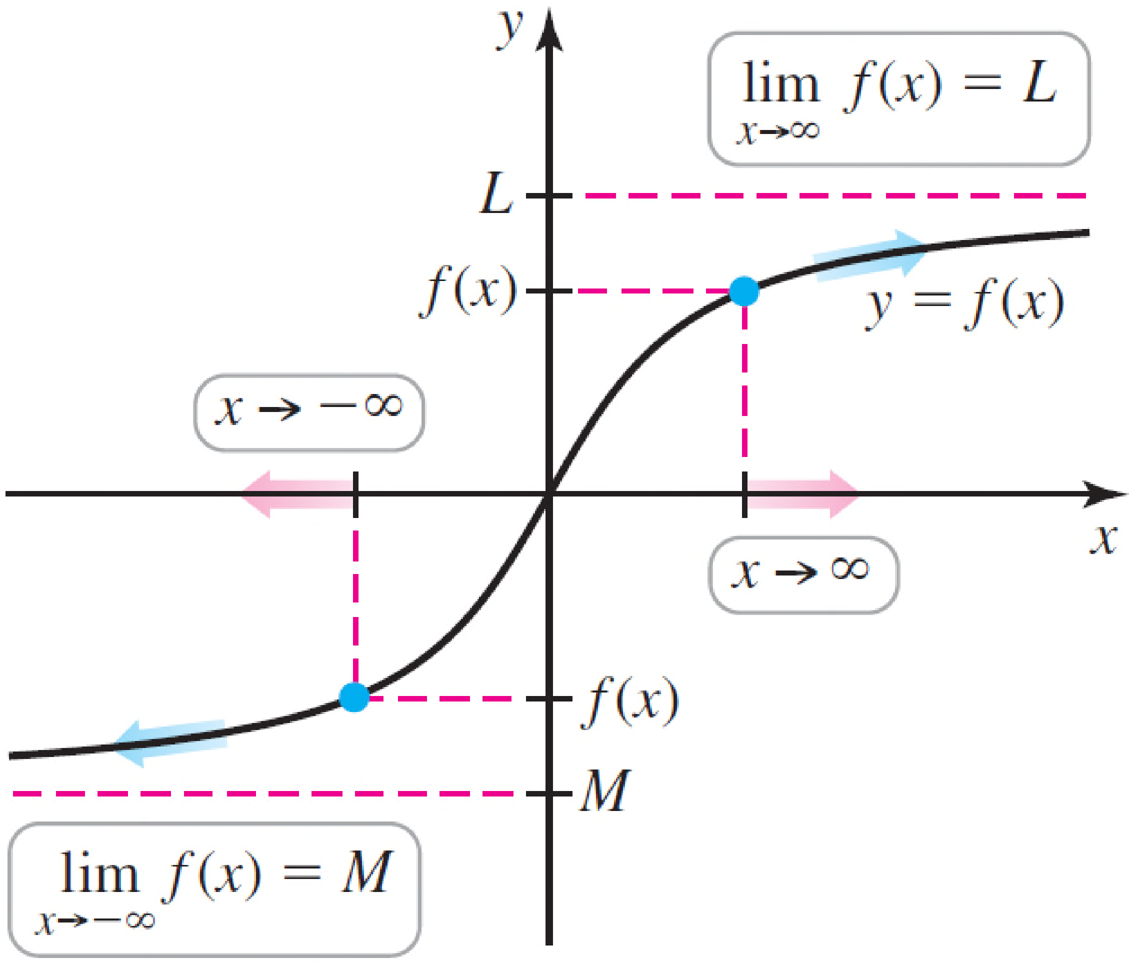

$\lim_{x \to \pi/2^-} \tan x = \infty$
The vertical asymptote $x = \frac{\pi}{2}$ for $\tan x$ becomes the horizontal asymptote $y = \frac{\pi}{2}$ for $\tan^{-1}x$.
y
$y = \tan x$
$y = x$
$\frac{\pi}{2}$
1
$y = \tan^{-1} x$
$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$
$-\frac{\pi}{2}$
-1
1
$\frac{\pi}{2}$
x
$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$
-1
$-\frac{\pi}{2}$
The vertical asymptote $x = -\frac{\pi}{2}$ for $\tan x$ becomes the horizontal asymptote $y = -\frac{\pi}{2}$ for $\tan^{-1}x$.
$\lim_{x \to -\pi/2^+} \tan x = -\infty$

Figure 2.31



DEFINITION Limits at Infinity and Horizontal Asymptotes

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x \to \infty} f(x) = L.$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f (Figure 2.31). The limit at negative infinity, $\lim_{x \to -\infty} f(x) = M$, is defined analogously. When this limit exists, $y = M$ is a horizontal asymptote.

Figure 2.32

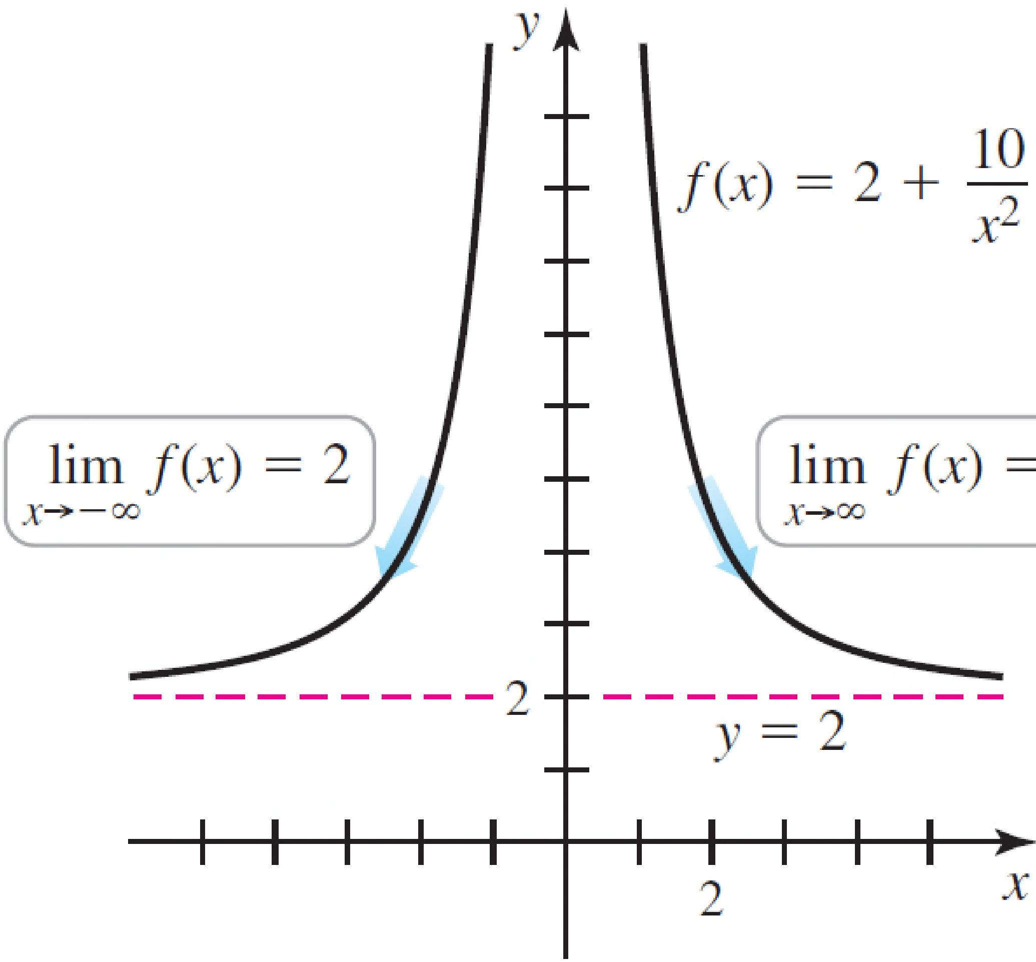
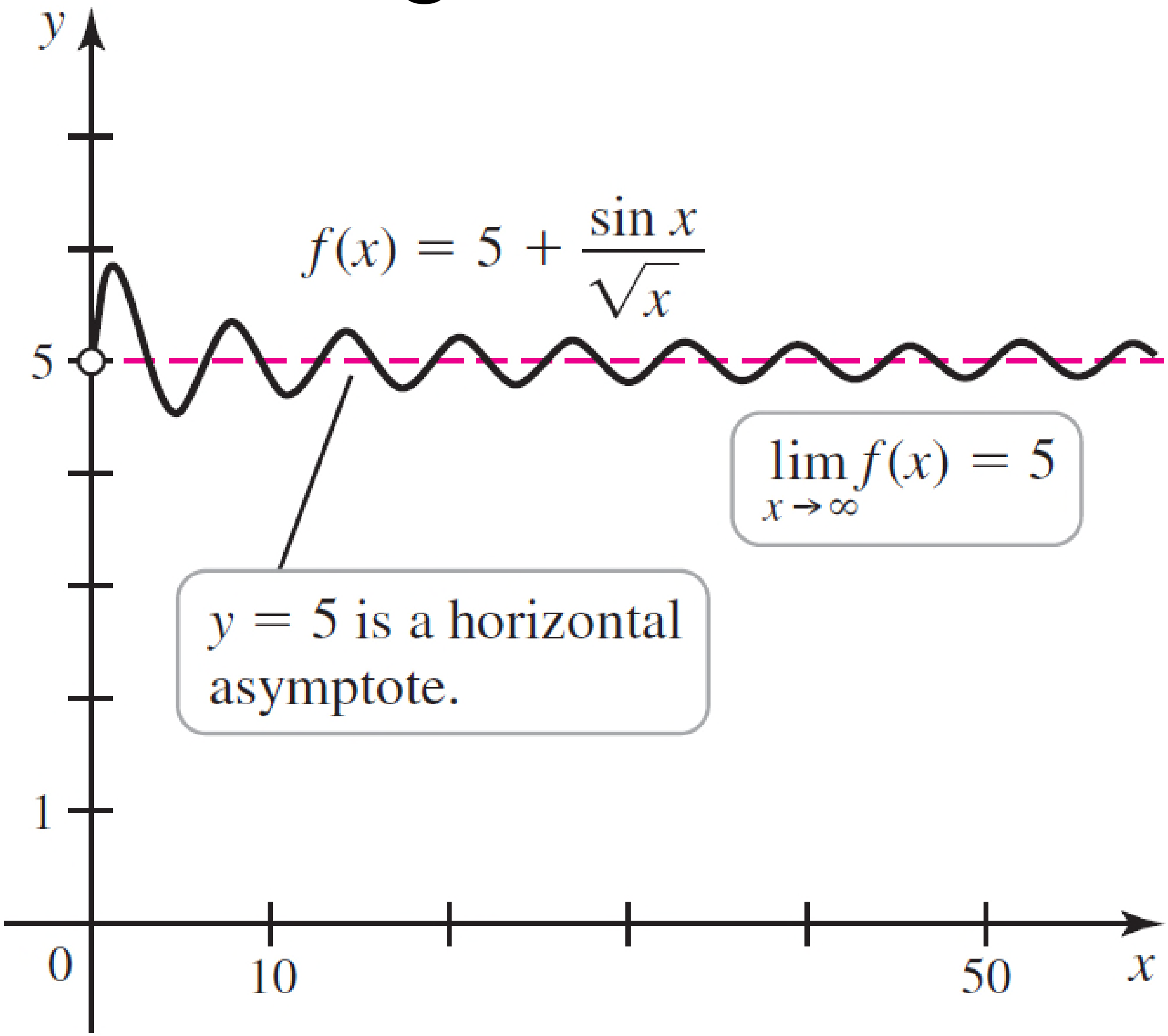
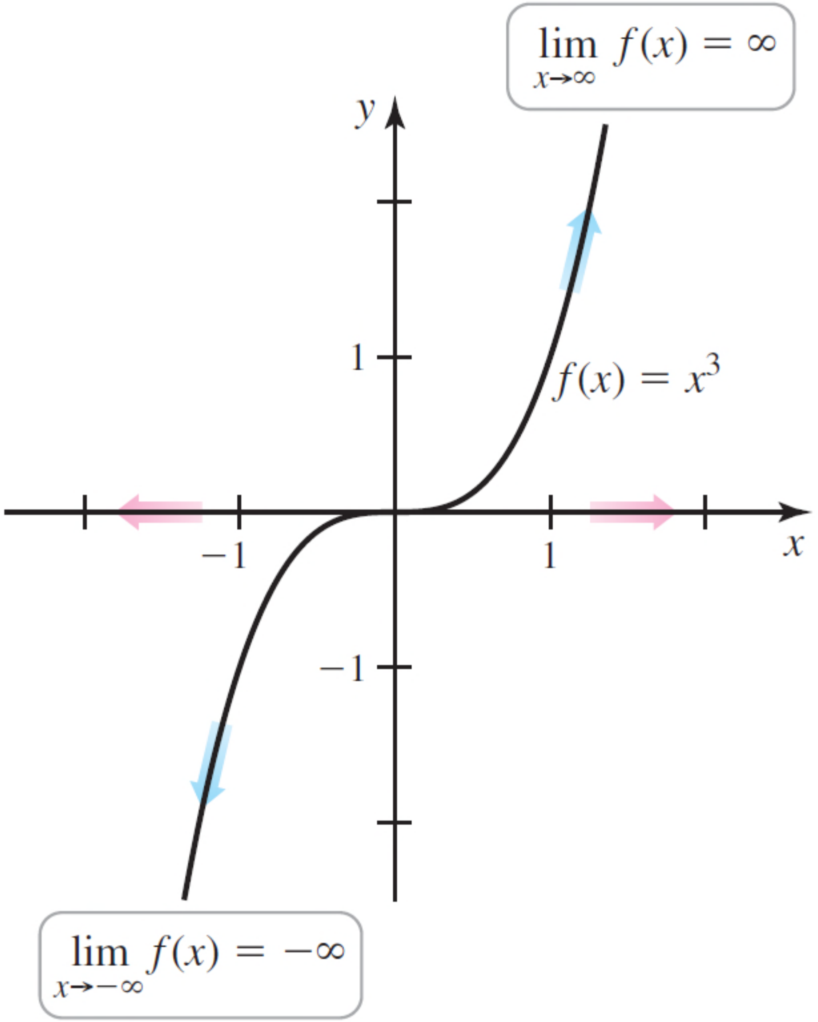


Figure 2.33



y
5
1
0
10
50
x
$f(x) = 5 + \frac{\sin x}{\sqrt{x}}$
$\lim_{x \to \infty} f(x) = 5$
y = 5 is a horizontal asymptote.

Figure 2.34



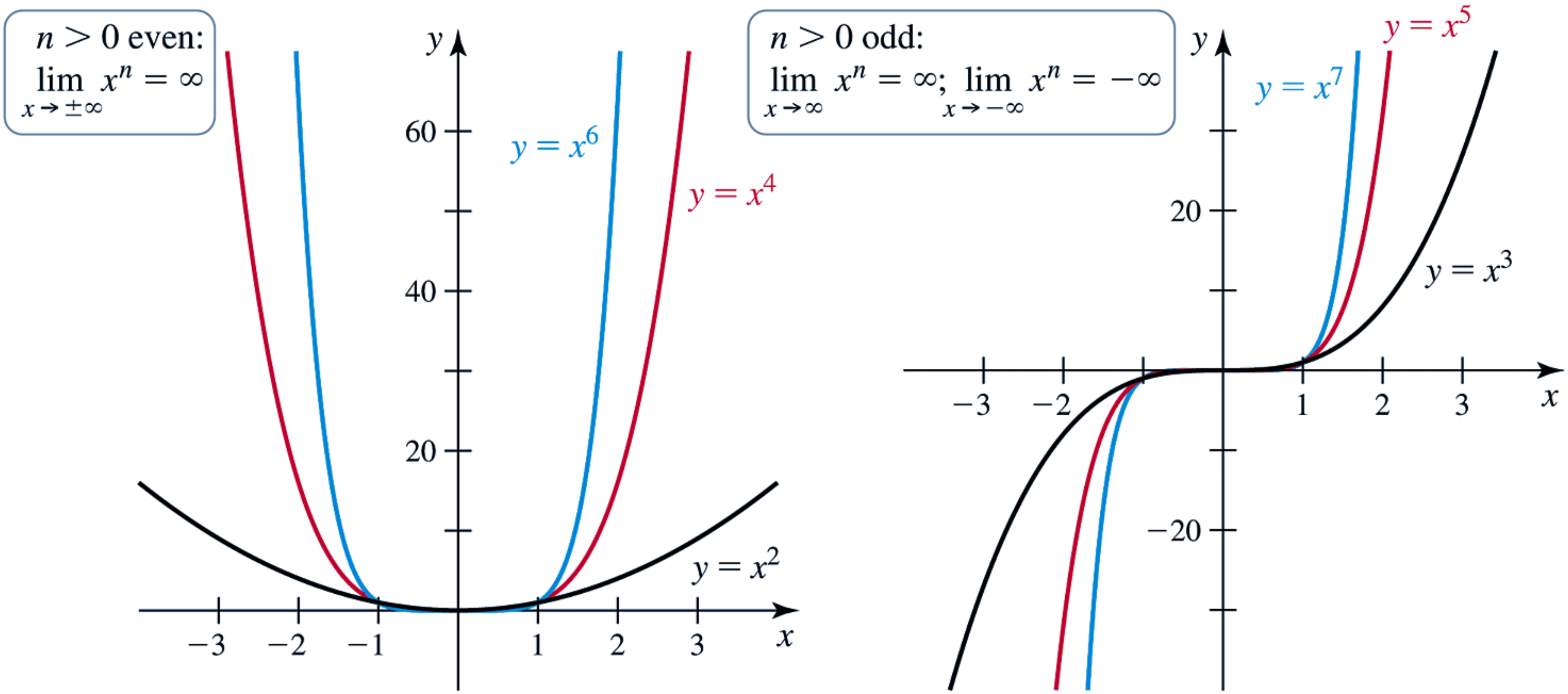
DEFINITION Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x \to \infty} f(x) = \infty.$$

The limits $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$ are defined similarly.

Figure 2.35



$n > 0$ even:
$\lim_{x \to \pm\infty} x^n = \infty$
$y = x^6$
$y = x^4$
$y = x^2$
y
x
60
40
20
−3
−2
−1
1
2
3
$n > 0$ odd:
$\lim_{x \to \infty} x^n = \infty$; $\lim_{x \to -\infty} x^n = -\infty$
$y = x^7$
$y = x^5$
$y = x^3$
20
−20

THEOREM 2.6 **Limits at Infinity of Powers and Polynomials**

Let n be a positive integer and let p be the polynomial
$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$.

1. $\lim_{x \to \pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$.
4. $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Figure 2.36

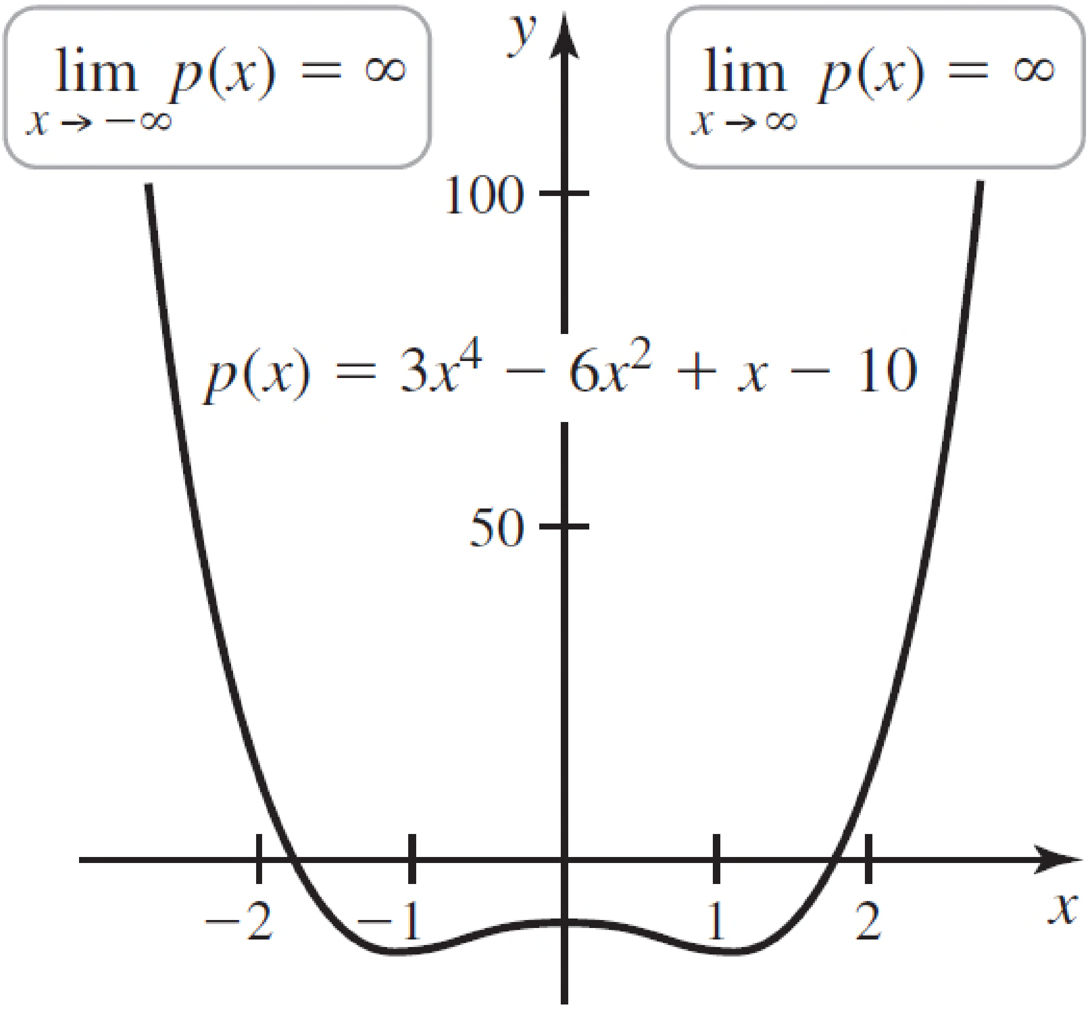

$\lim_{x \to -\infty} p(x) = \infty$
$\lim_{x \to \infty} p(x) = \infty$
y
100
50
$p(x) = 3x^4 - 6x^2 + x - 10$
−2
−1
1
2
x

Figure 2.37

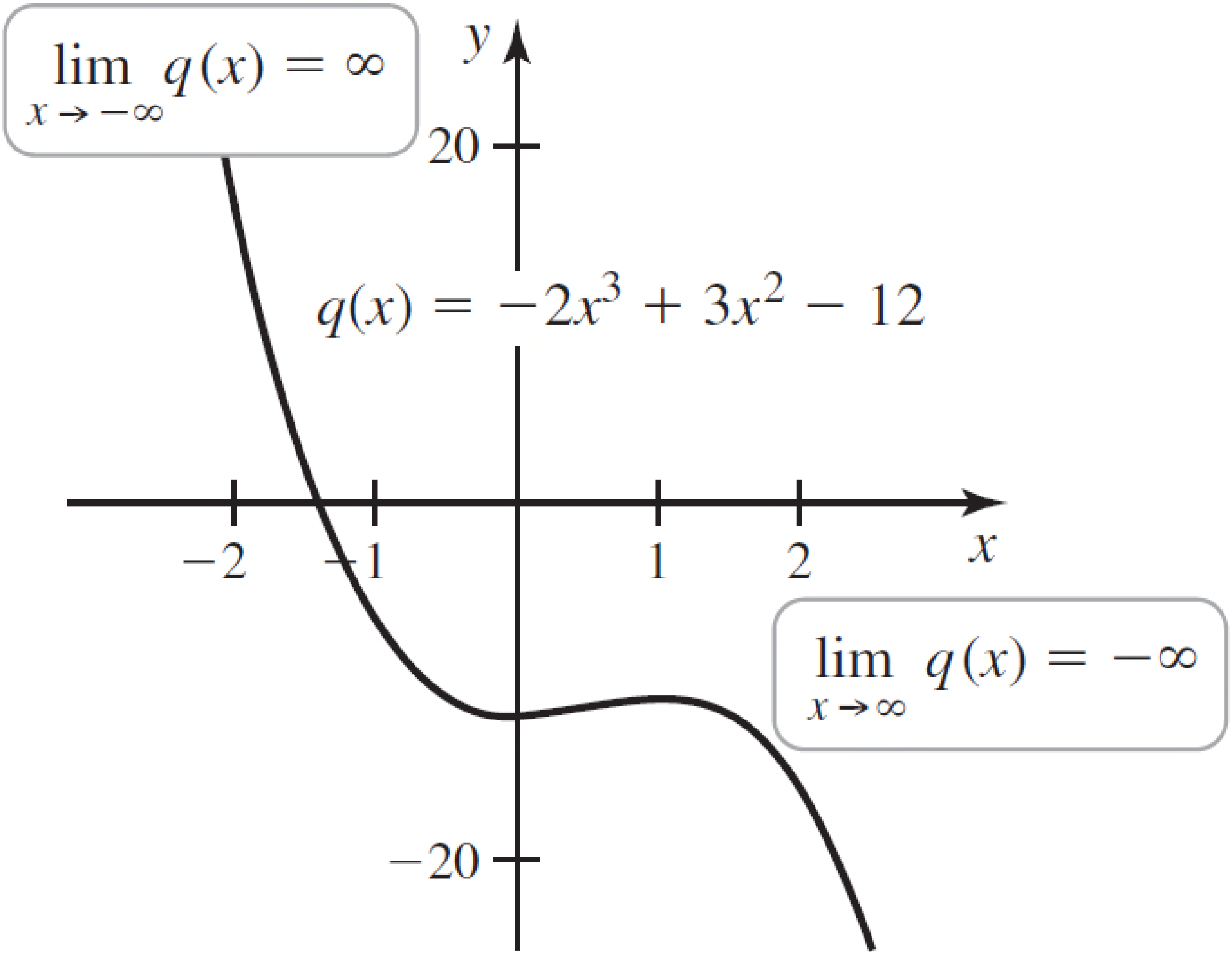

$\lim_{x \to -\infty} q(x) = \infty$
y
20
$q(x) = -2x^3 + 3x^2 - 12$
−2
−1
1
2
x
$\lim_{x \to \infty} q(x) = -\infty$
−20

Figure 2.38

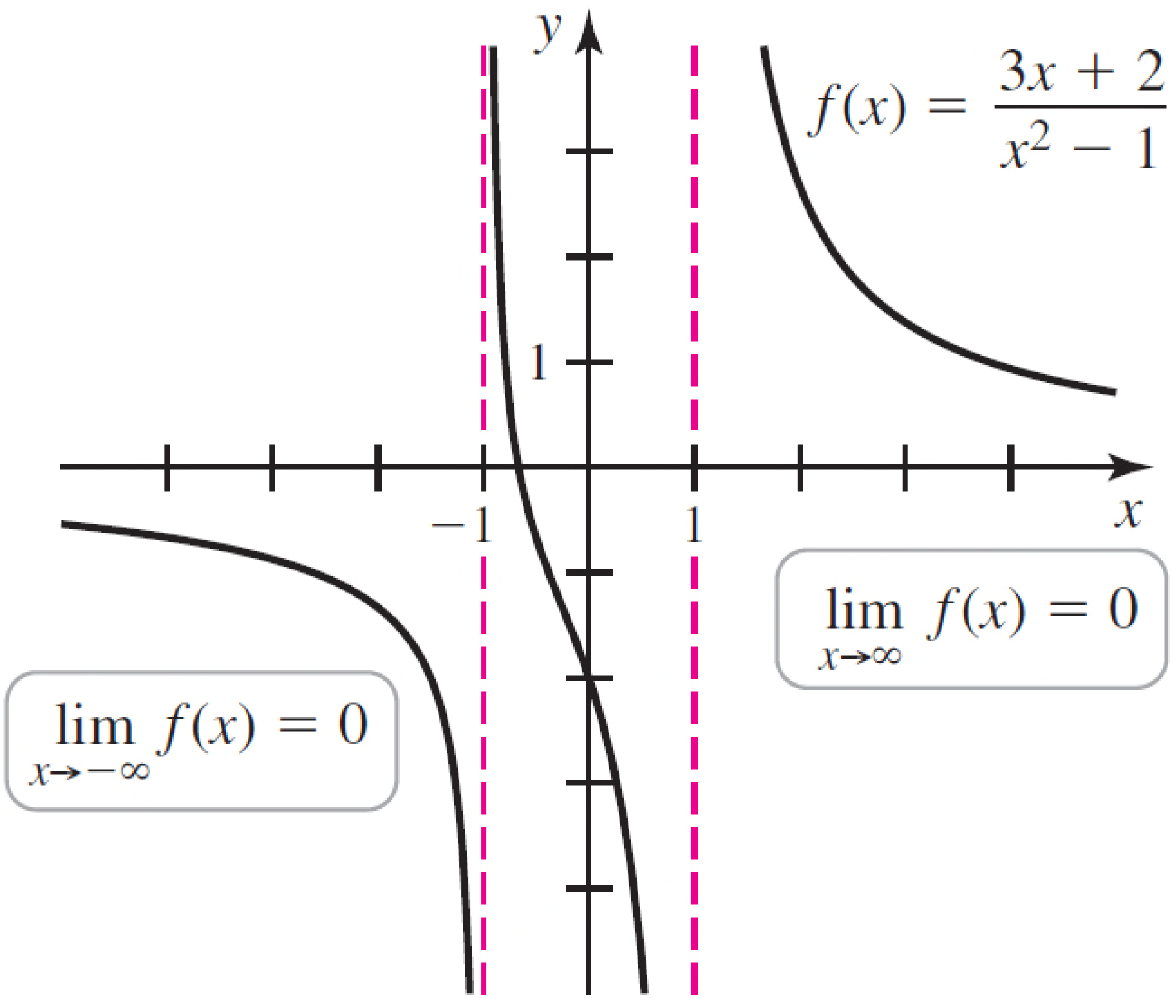
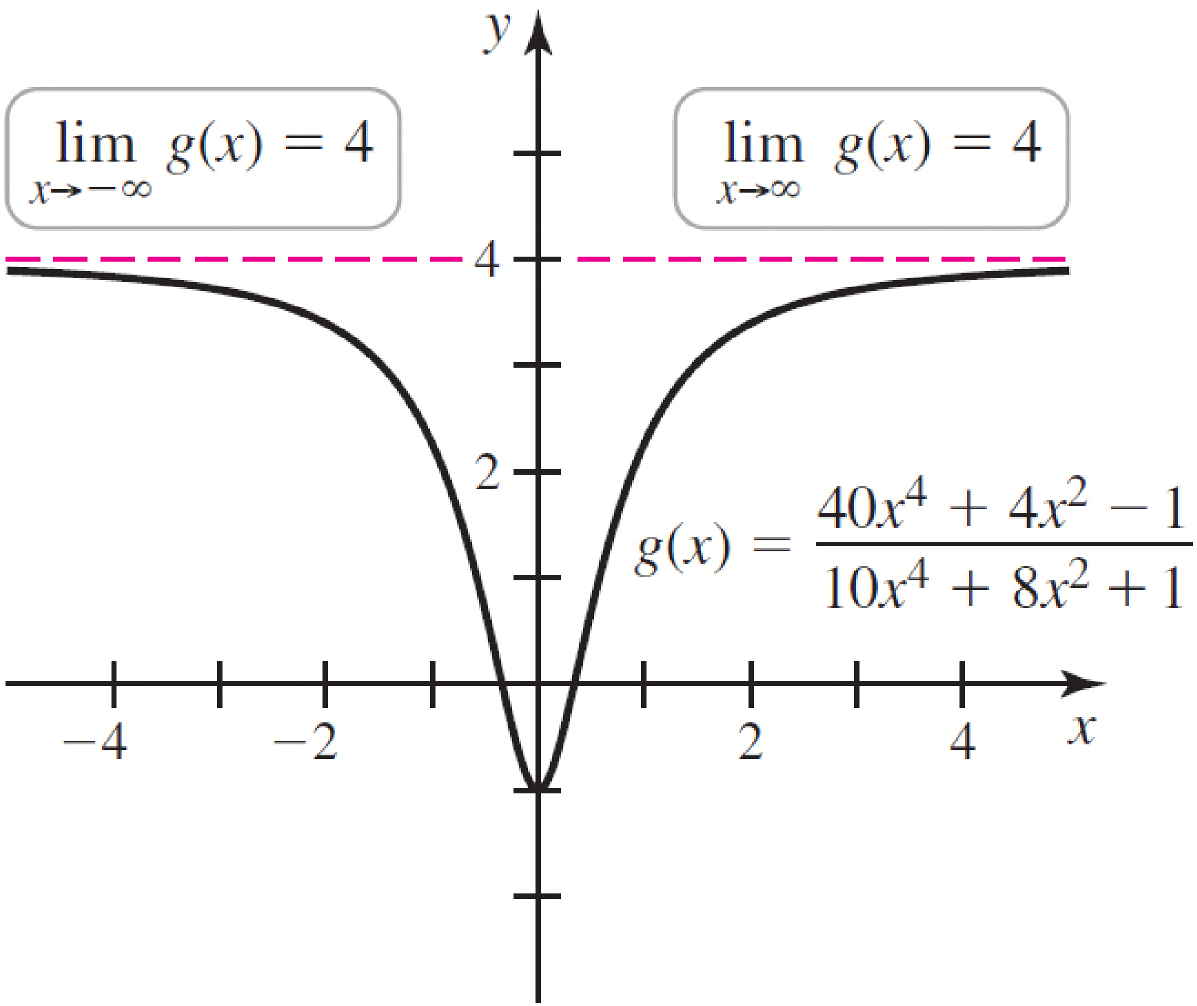
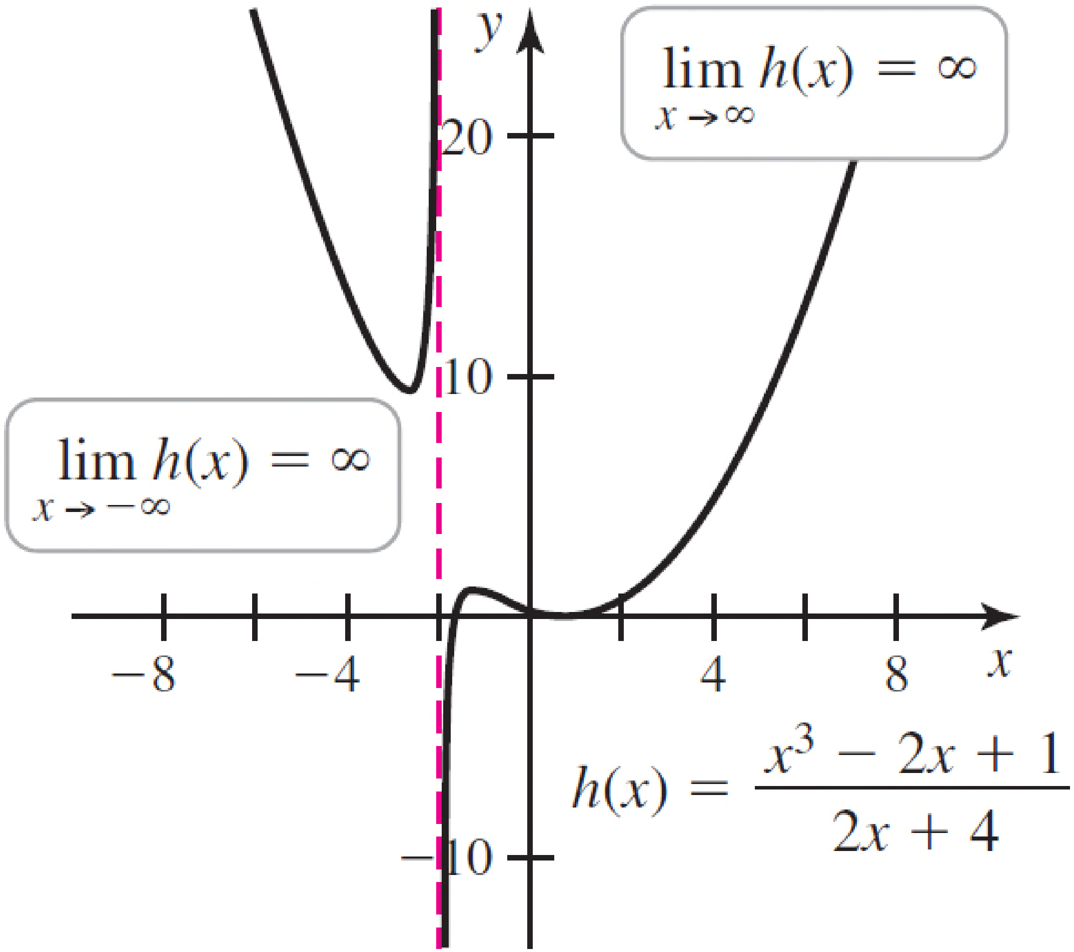


Figure 2.39



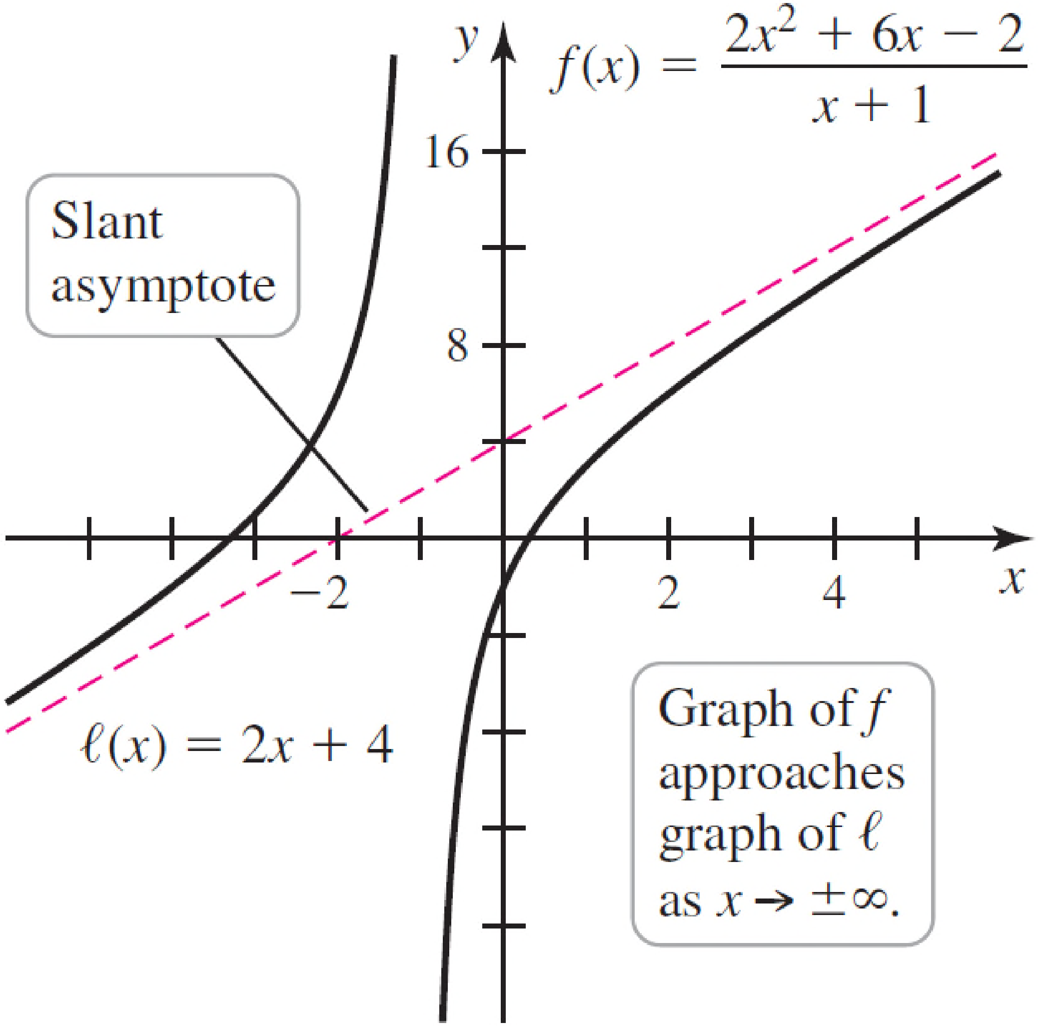
$\lim_{x \to -\infty} g(x) = 4$
$\lim_{x \to \infty} g(x) = 4$
$g(x) = \frac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1}$
y
x
4
2
−4
−2
2
4

Figure 2.40



y
20
10
−10
$\lim_{x \to \infty} h(x) = \infty$
$\lim_{x \to -\infty} h(x) = \infty$
−8
−4
4
8
x
$h(x) = \dfrac{x^3 - 2x + 1}{2x + 4}$

Figure 2.41



y
$f(x) = \frac{2x^2 + 6x - 2}{x + 1}$
16
8
Slant
asymptote
−2
2
4
x
$\ell(x) = 2x + 4$
Graph of f
approaches
graph of ℓ
as $x \to \pm\infty$.

THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0 \quad \text{and}$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0,$$

with $a_m \neq 0$ and $b_n \neq 0$.

a. Degree of numerator less than degree of denominator If $m < n$, then $\lim_{x \to \pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

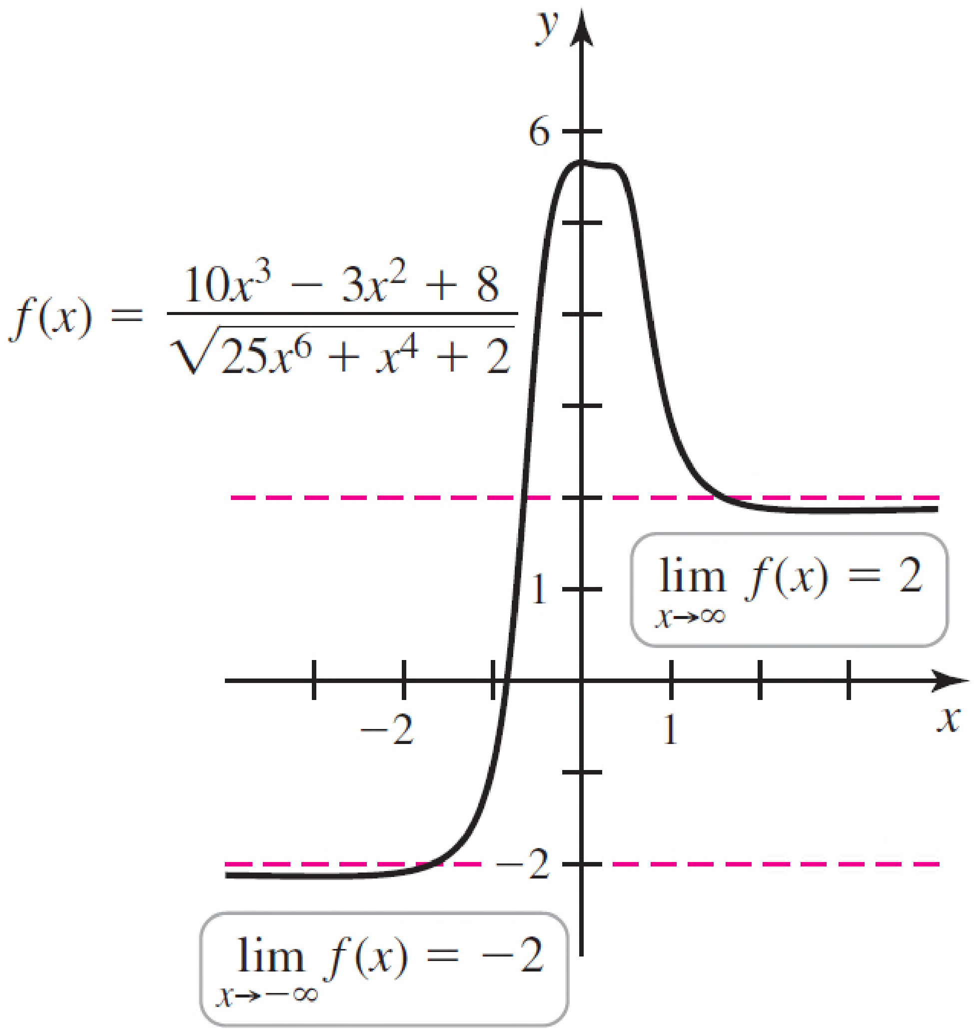
b. Degree of numerator equals degree of denominator If $m = n$, then $\lim_{x \to \pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote of f.

c. Degree of numerator greater than degree of denominator If $m > n$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

d. Slant asymptote If $m = n + 1$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.

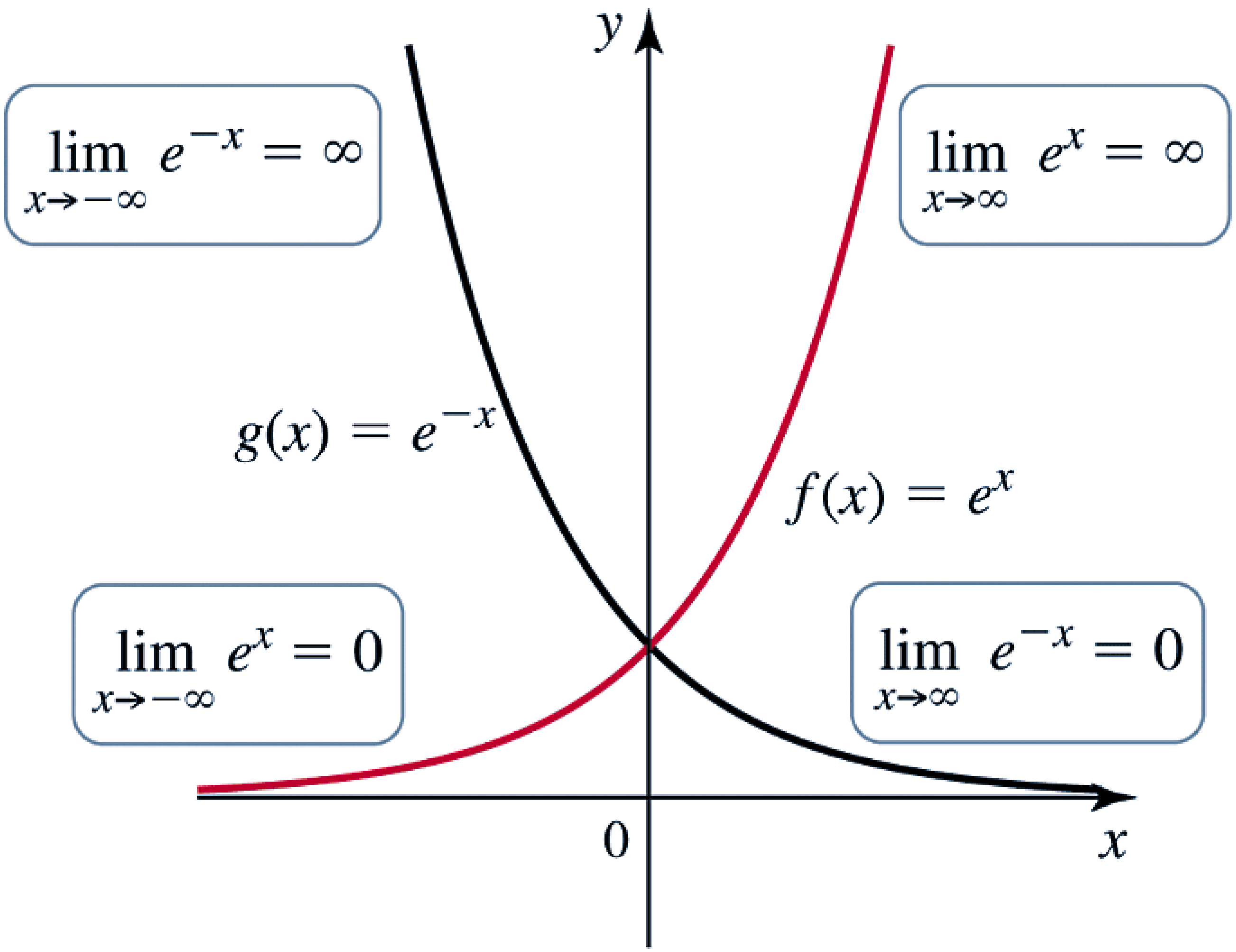
e. Vertical asymptotes Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Figure 2.42



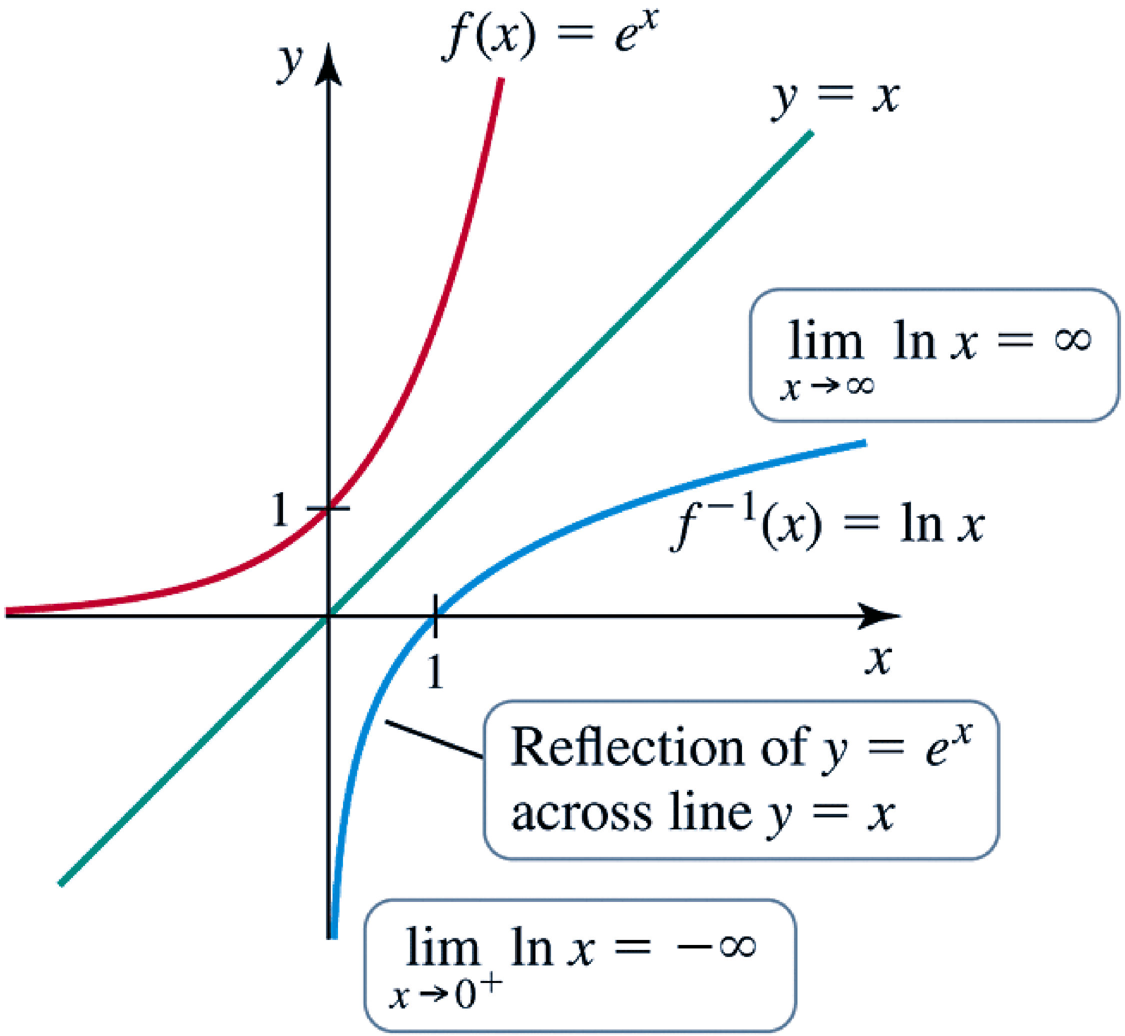
y
6
$f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$
1
$\lim_{x \to \infty} f(x) = 2$
−2
1
x
−2
$\lim_{x \to -\infty} f(x) = -2$

Figure 2.43



$\lim_{x\to-\infty} e^{-x} = \infty$
$\lim_{x\to\infty} e^{x} = \infty$
$g(x) = e^{-x}$
$f(x) = e^{x}$
$\lim_{x\to-\infty} e^{x} = 0$
$\lim_{x\to\infty} e^{-x} = 0$
y
x
0

Figure 2.44


$f(x) = e^x$
y
$y = x$
$\lim_{x \to \infty} \ln x = \infty$
1
$f^{-1}(x) = \ln x$
1
x
Reflection of $y = e^x$ across line $y = x$
$\lim_{x \to 0^+} \ln x = -\infty$

THEOREM 2.8 End Behavior of e^x, e^{-x}, and $\ln x$

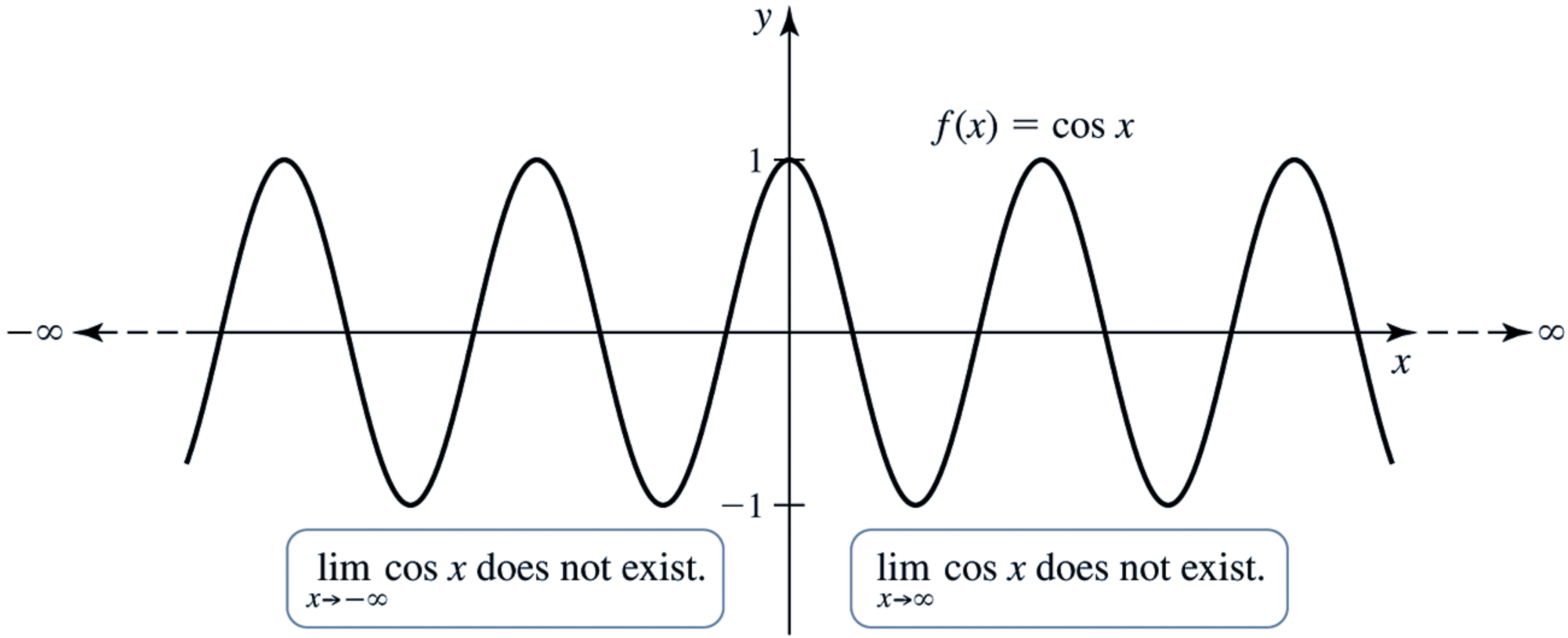
The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0,$$

$$\lim_{x \to \infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-x} = \infty,$$

$$\lim_{x \to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln x = \infty.$$

Figure 2.45



y
f(x) = cos x
1
−1
−∞
∞
x
lim cos x does not exist.
x→−∞
lim cos x does not exist.
x→∞

2.6

Continuity

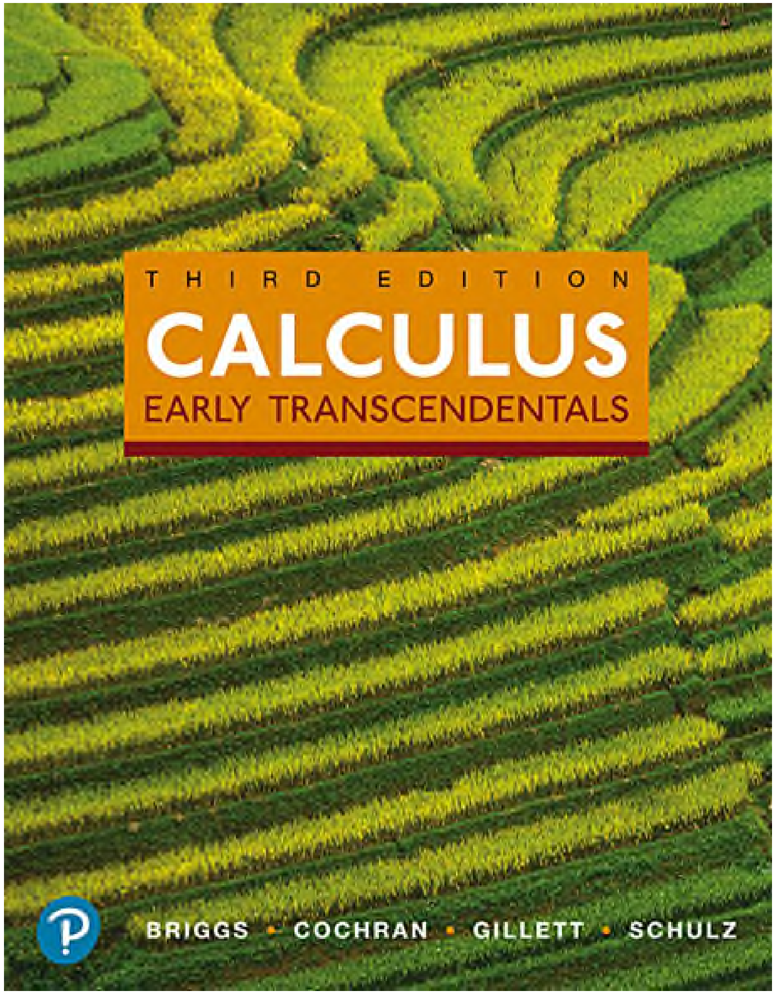


Figure 2.46 (a)

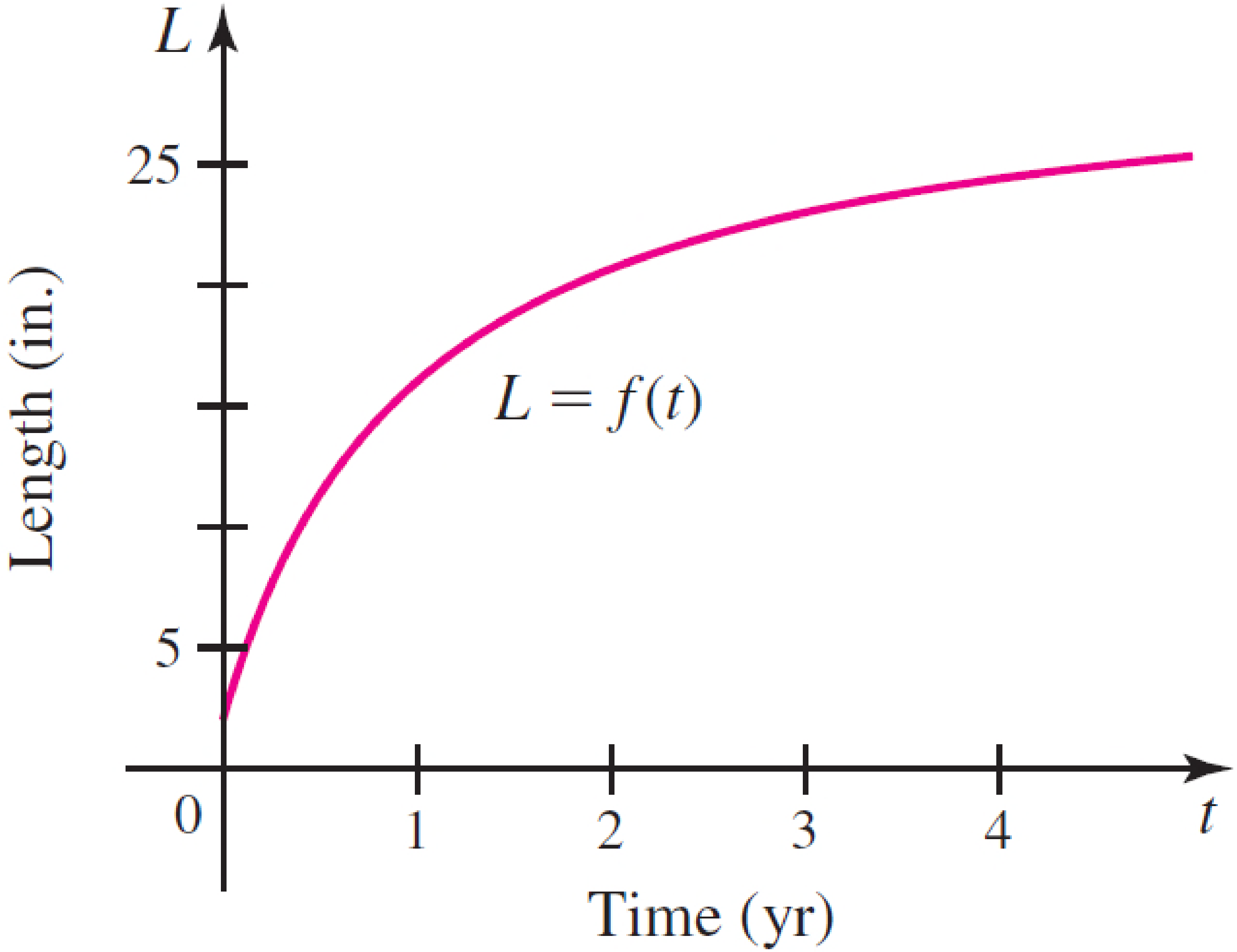


Figure 2.46 (b)

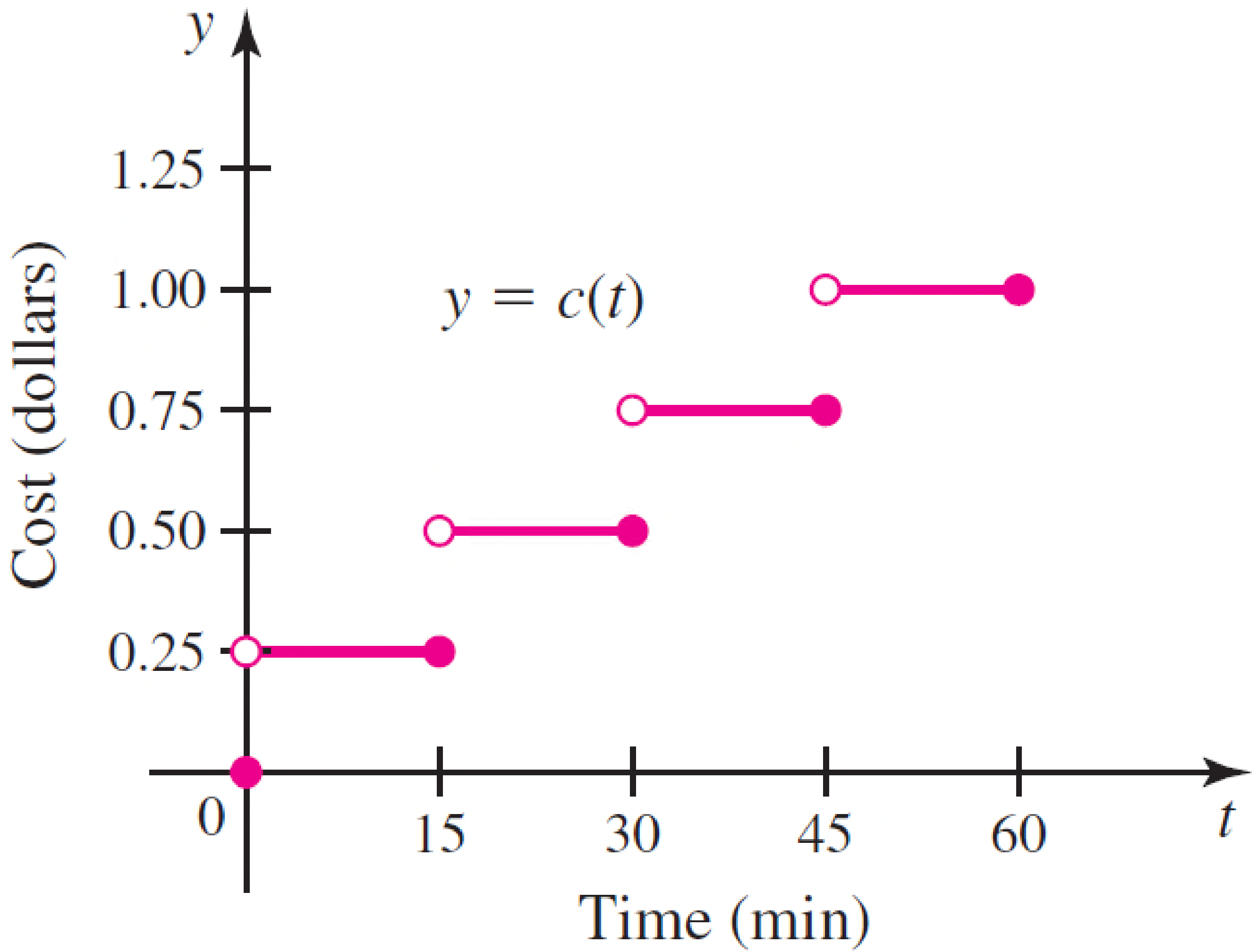
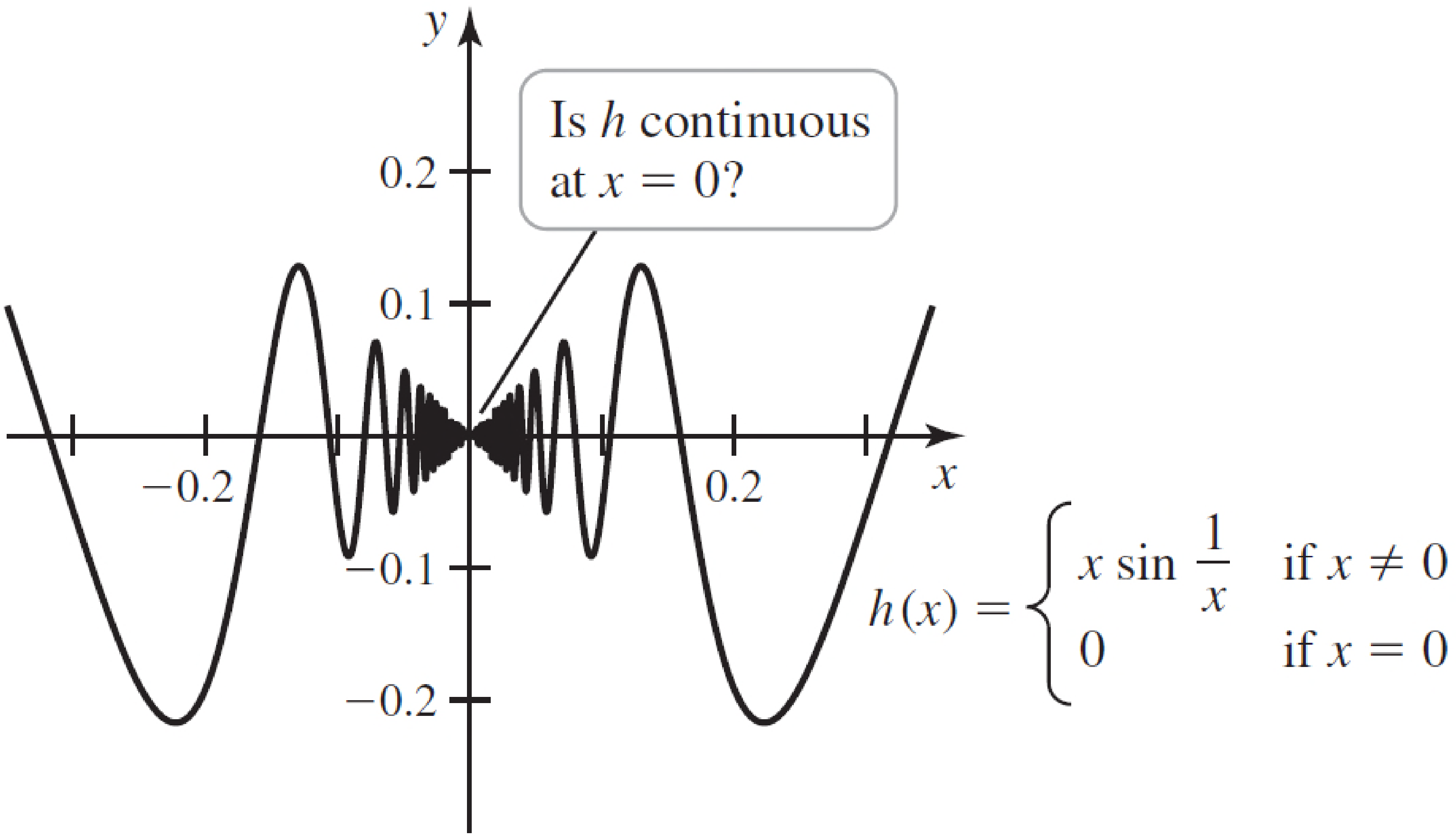


Figure 2.47



y
Is h continuous at $x = 0$?
0.2
0.1
−0.1
−0.2
−0.2
0.2
x
$h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

DEFINITION Continuity at a Point

A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

Continuity Checklist

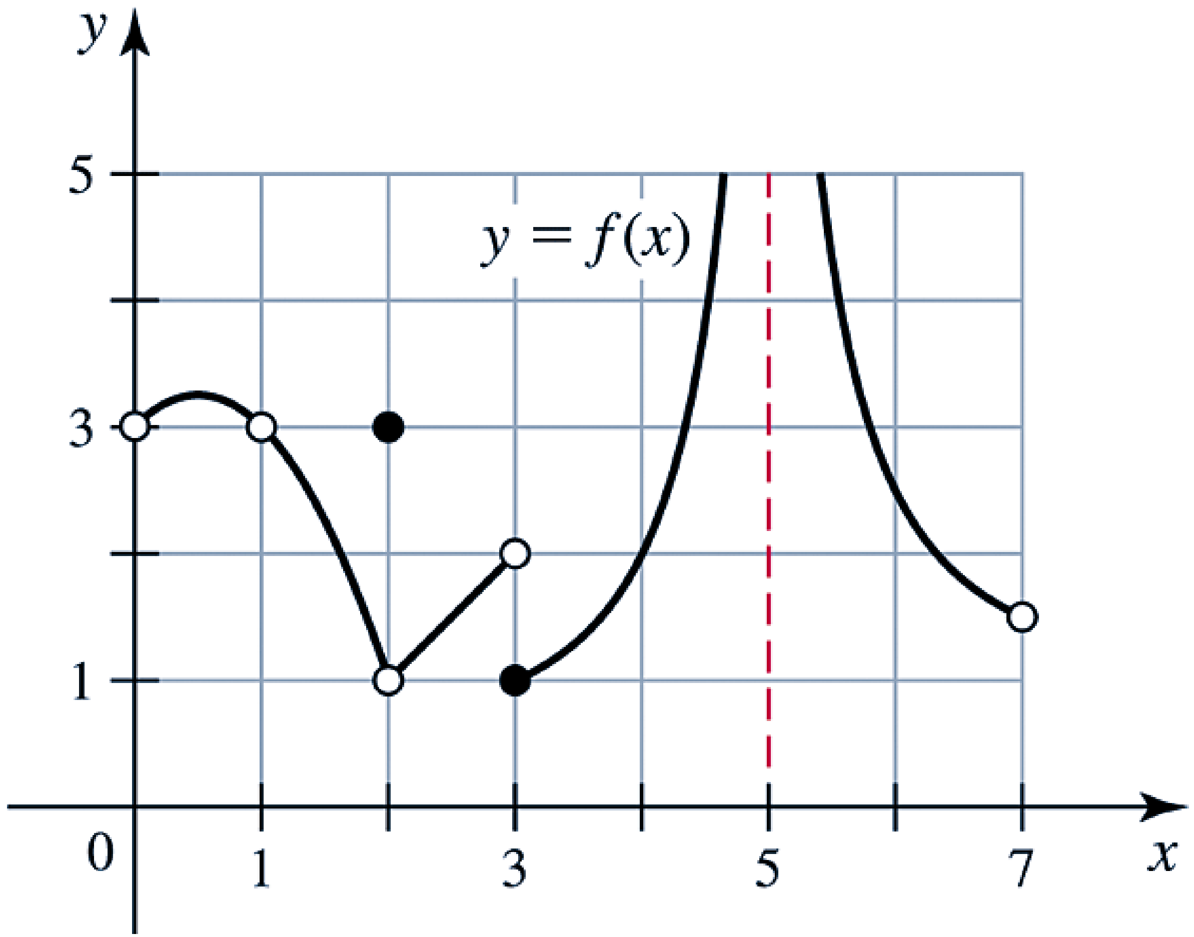
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f).

2. $\lim_{x \to a} f(x)$ exists.

3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Figure 2.48



THEOREM 2.9 Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$

b. $f - g$

c. cf

d. fg

e. f/g, provided $g(a) \neq 0$

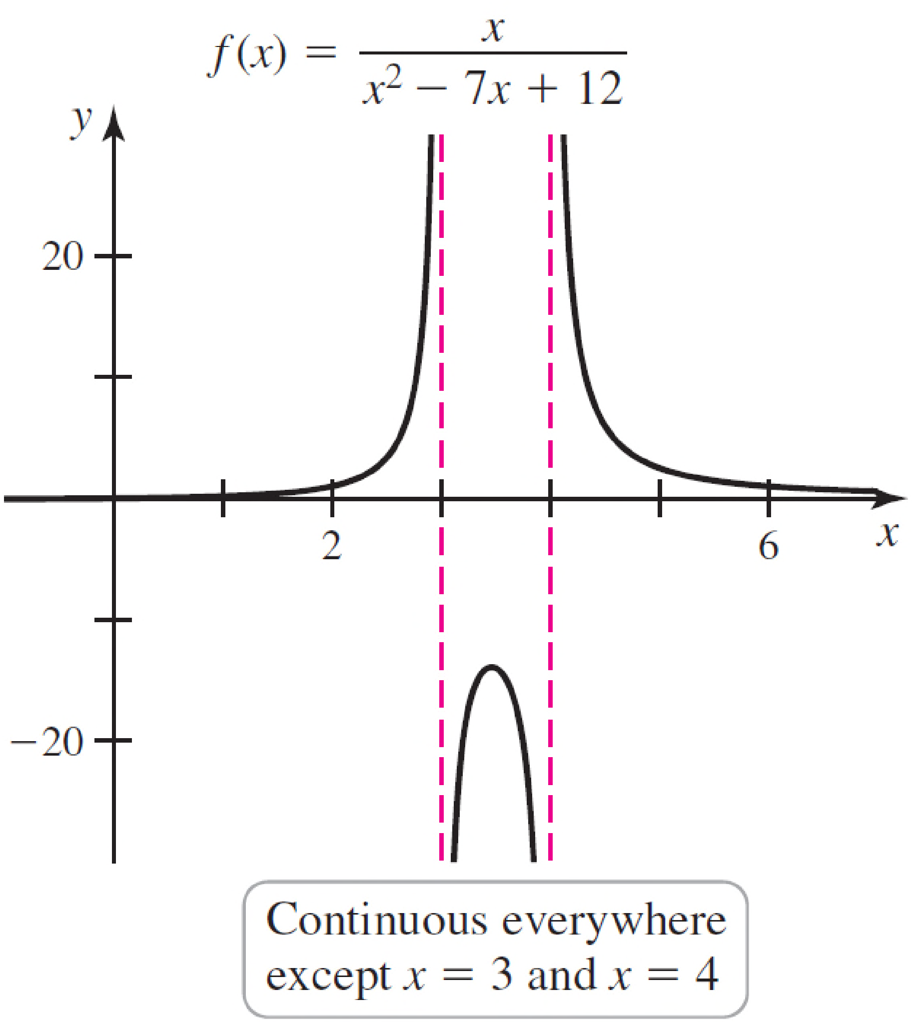
f. $(f(x))^n$

THEOREM 2.10 Polynomial and Rational Functions

a. A polynomial function is continuous for all x.

b. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Figure 2.49


$f(x) = \dfrac{x}{x^2 - 7x + 12}$
y
20
−20
2
6
x
Continuous everywhere except $x = 3$ and $x = 4$

THEOREM 2.11 Continuity of Composite Functions at a Point

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

THEOREM 2.12 Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Figure 2.50 (a)

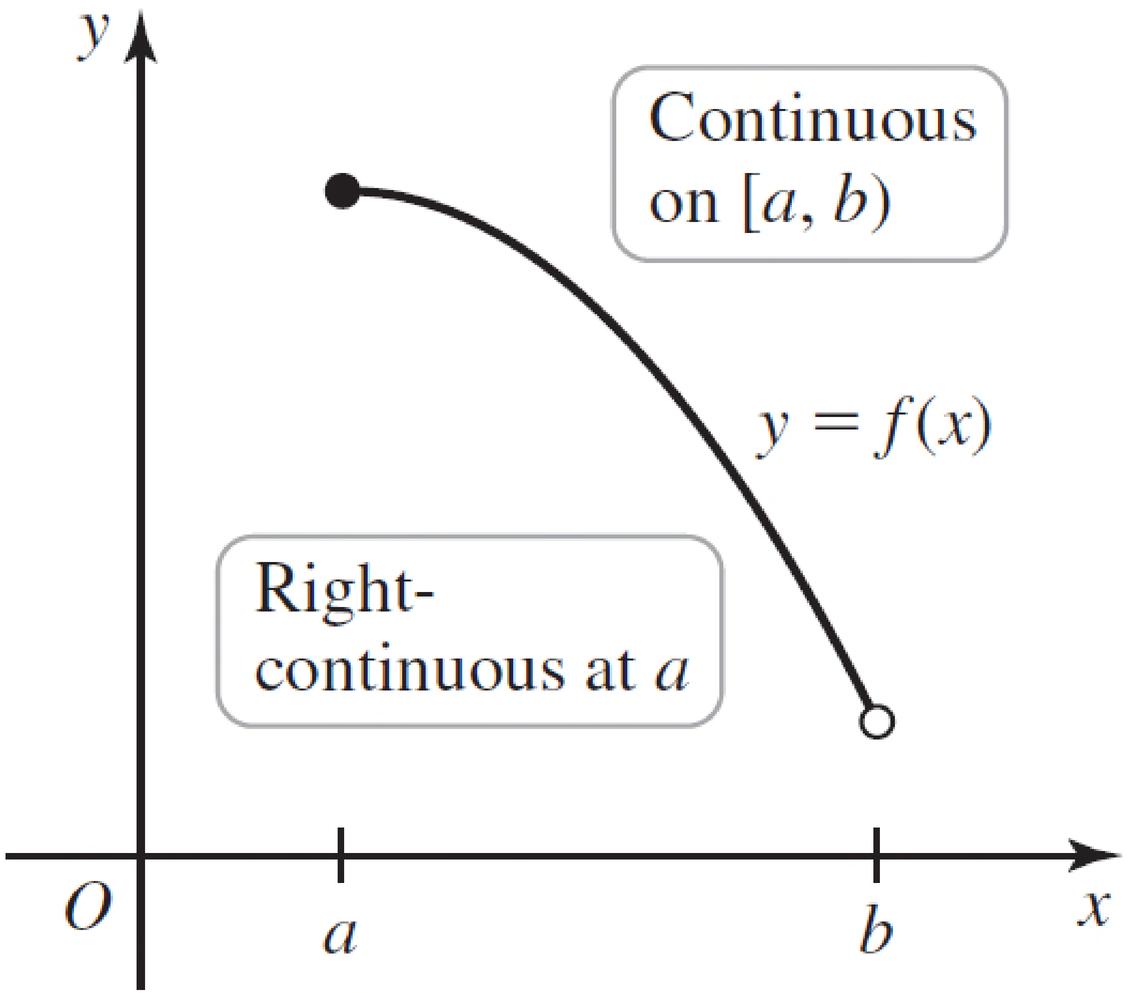
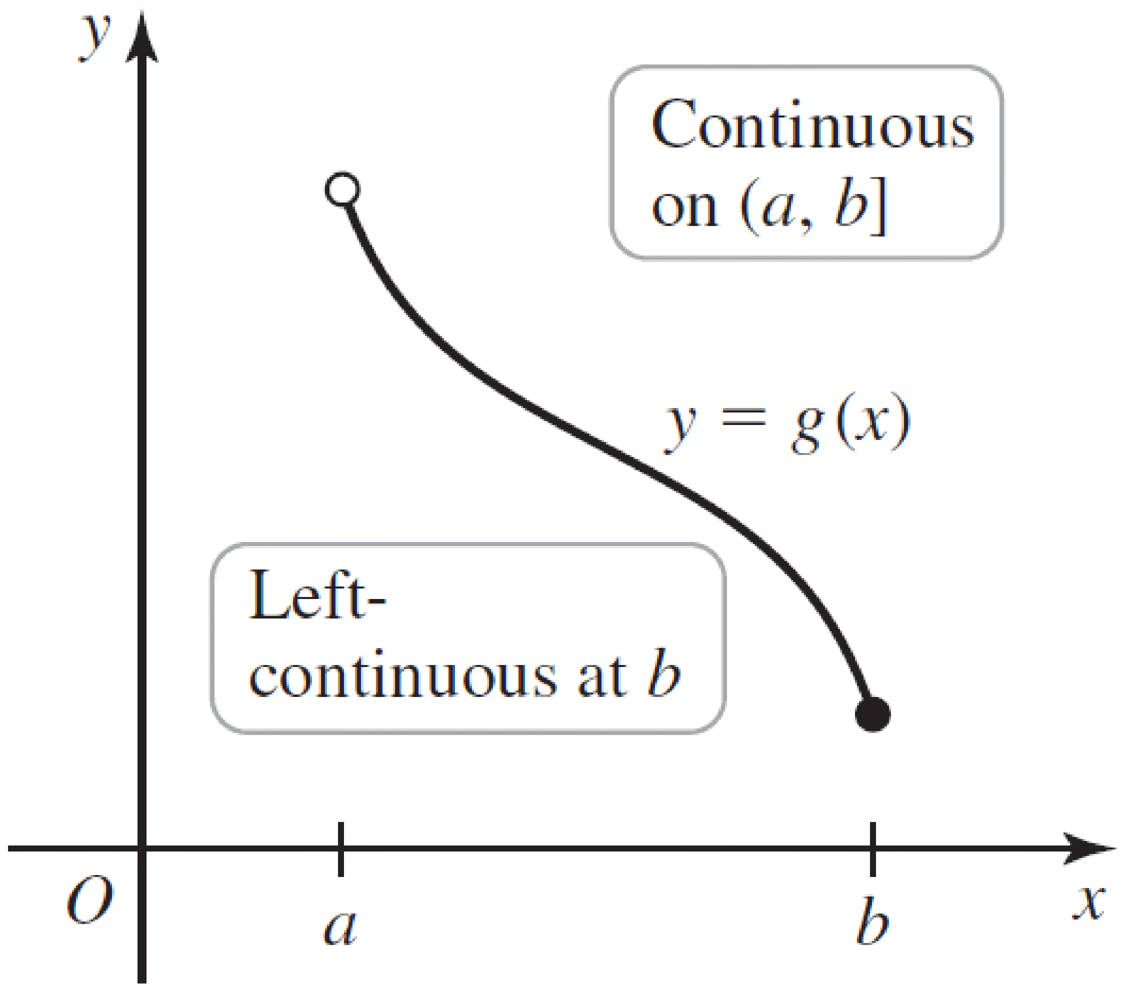


Figure 2.50 (b)



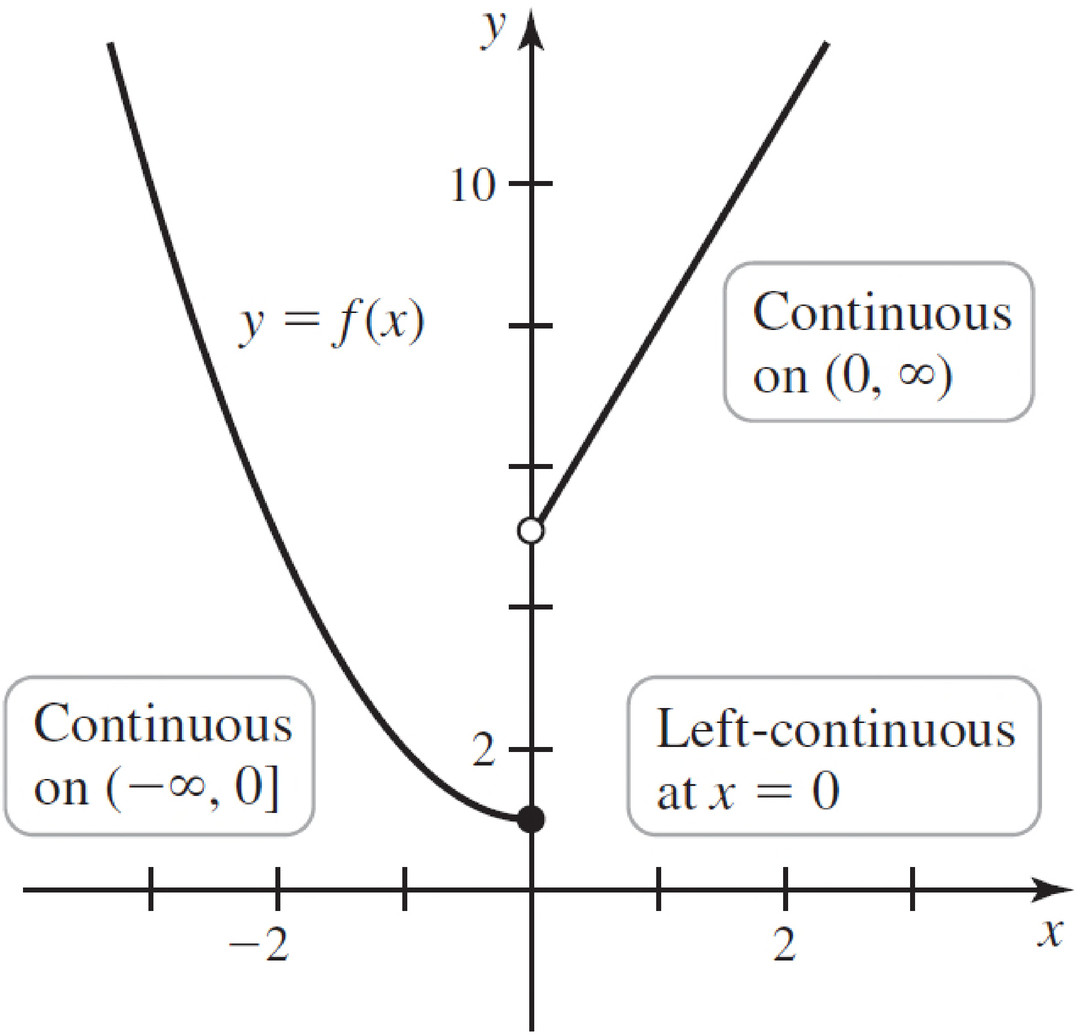
DEFINITION Continuity at Endpoints

A function f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$, and f is **continuous from the left** (or **left-continuous**) at b if $\lim_{x \to b^-} f(x) = f(b)$.

DEFINITION Continuity on an Interval

A function f is **continuous on an interval** I if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

Figure 2.51



y
10
2
y = f(x)
Continuous on (0, ∞)
Continuous on (−∞, 0]
Left-continuous at x = 0
−2
2
x

THEOREM 2.13 Continuity of Functions with Roots

Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous.

If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous and $f(a) > 0$.

Figure 2.52

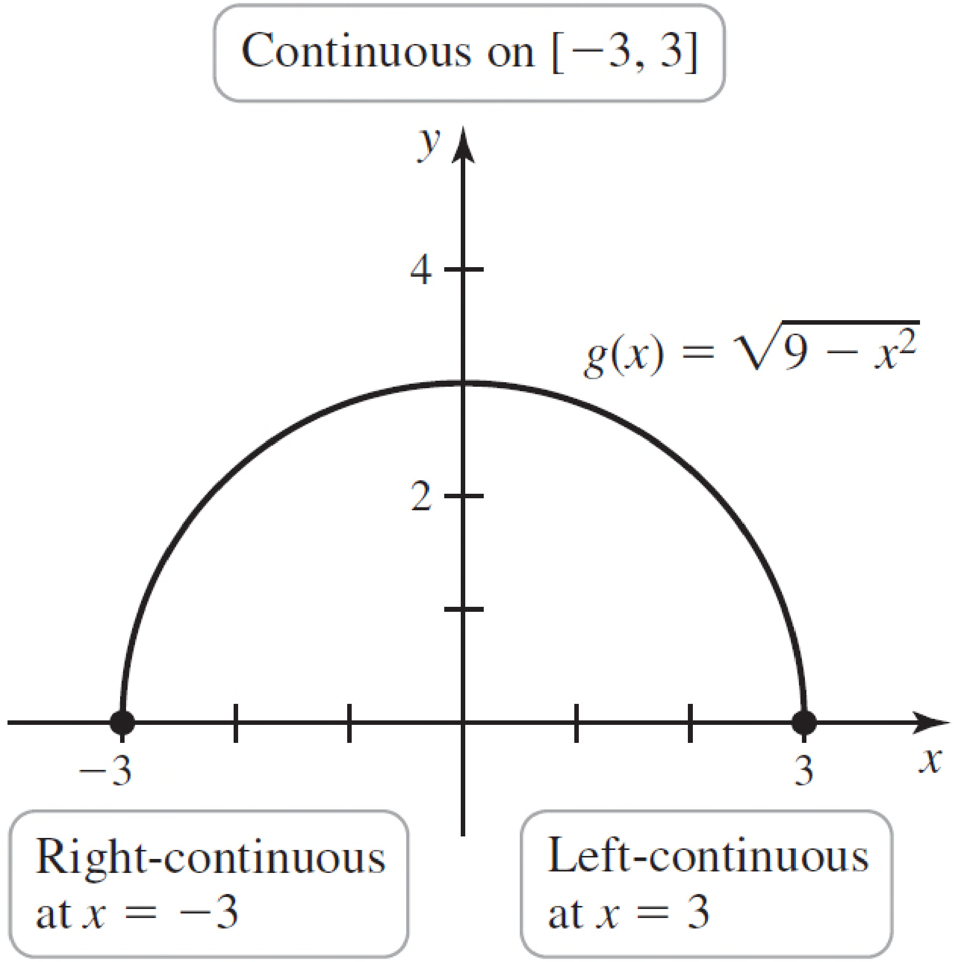

Continuous on [−3, 3]
y
4
2
$g(x) = \sqrt{9 - x^2}$
−3
3
x
Right-continuous at $x = -3$
Left-continuous at $x = 3$

Figure 2.53

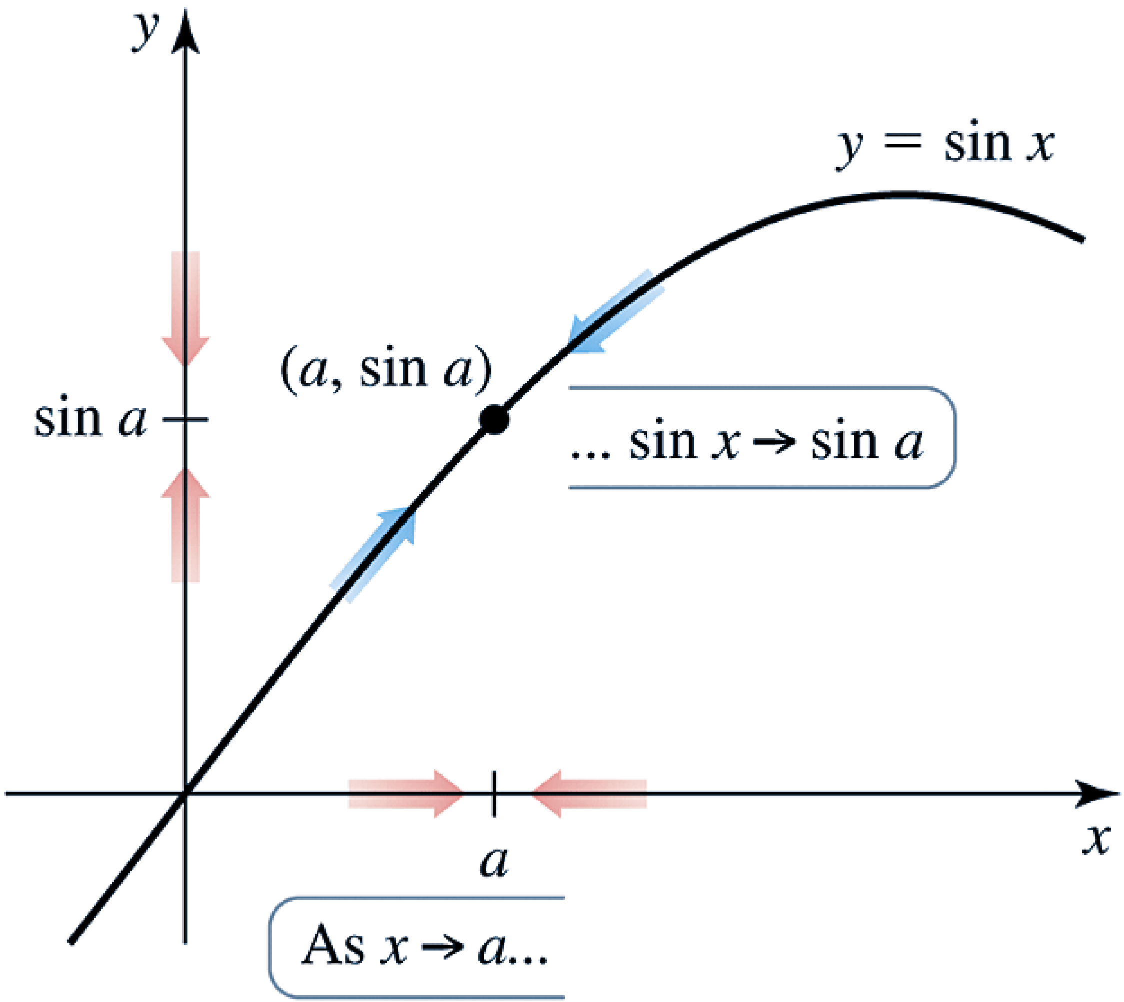
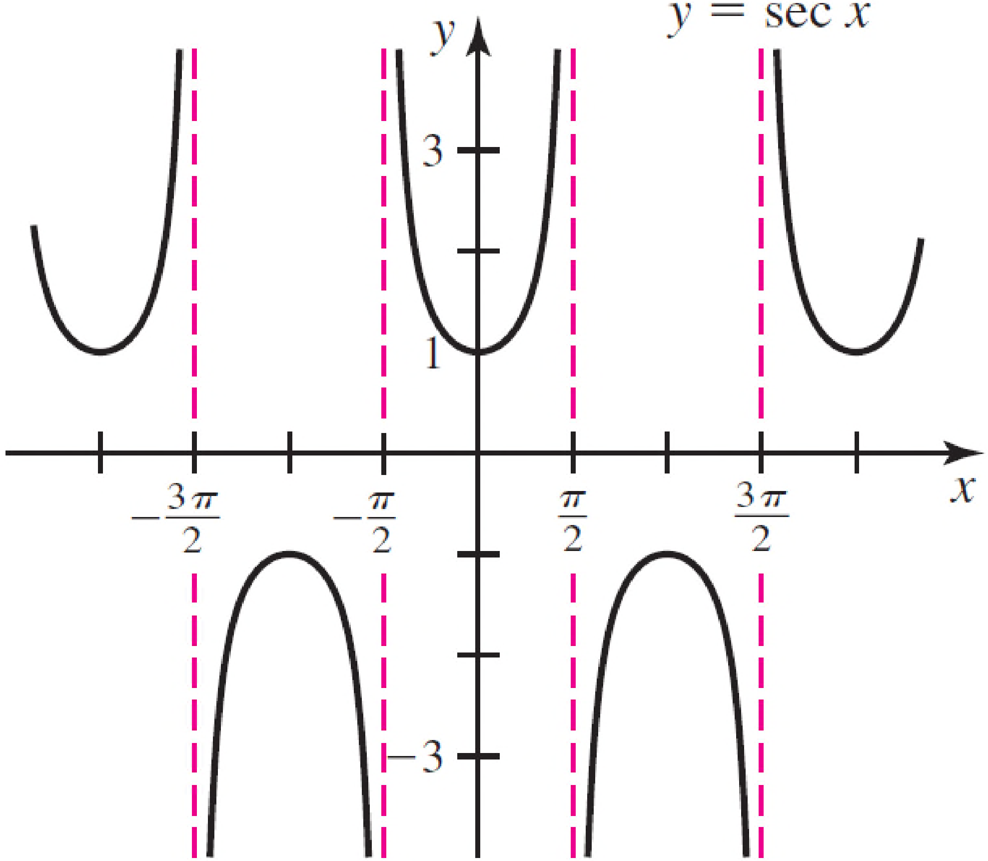
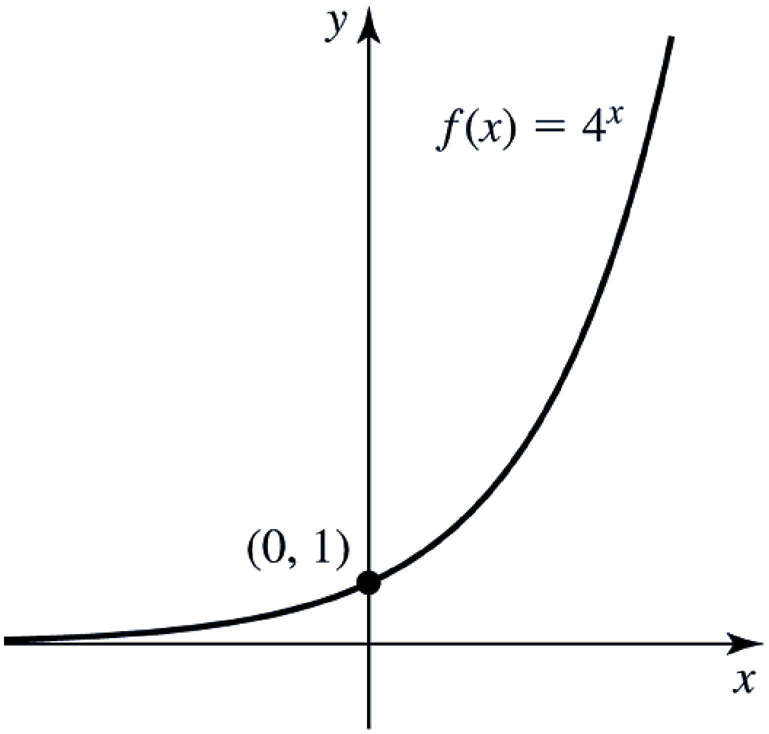


Figure 2.54



sec x is continuous at all points of its domain.

Figure 2.55



Exponential functions are defined for all real numbers and are continuous on $(-\infty, \infty)$, as shown in Chapter 7.

THEOREM 2.14 Continuity of Inverse Functions

If a function f is continuous on an interval I and has an inverse on I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

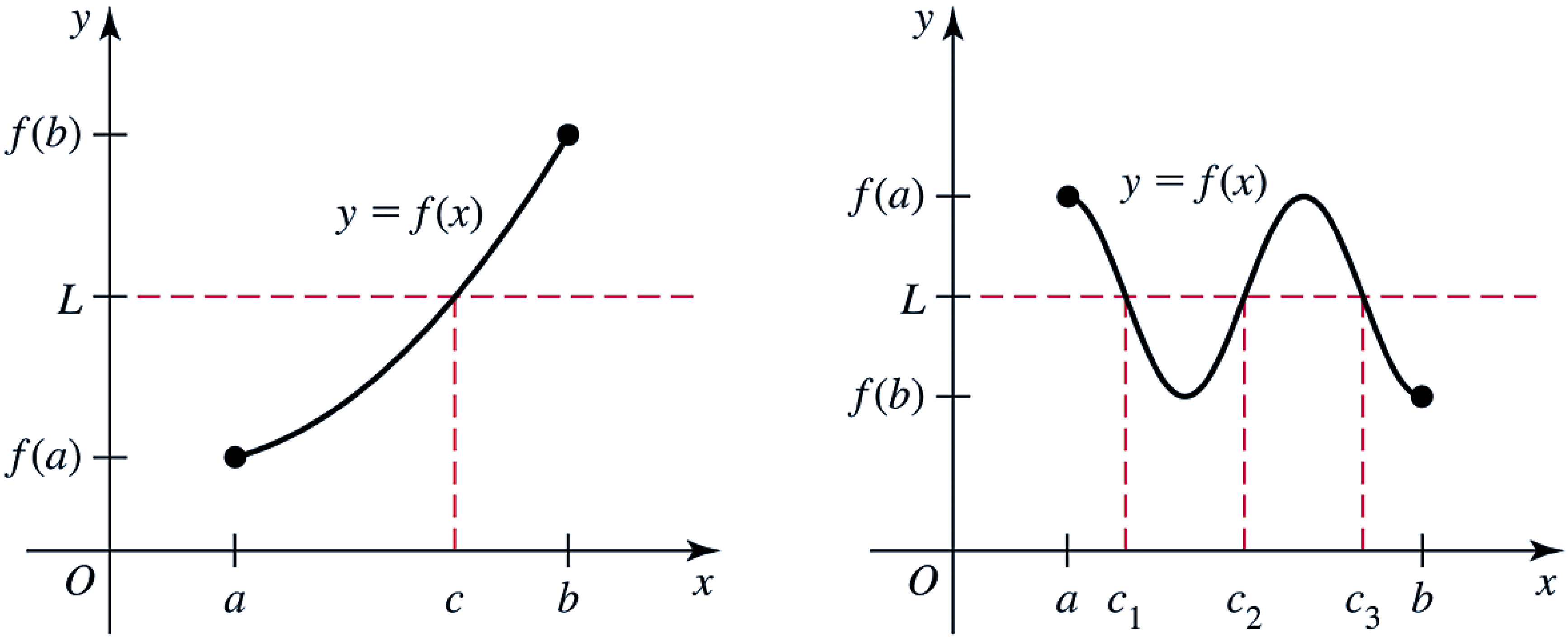
THEOREM 2.15 Continuity of Transcendental Functions

The following functions are continuous at all points of their domains.

Trigonometric		Inverse Trigonometric		Exponential	
$\sin x$	$\cos x$	$\sin^{-1} x$	$\cos^{-1} x$	b^x	e^x
$\tan x$	$\cot x$	$\tan^{-1} x$	$\cot^{-1} x$	**Logarithmic**	
$\sec x$	$\csc x$	$\sec^{-1} x$	$\csc^{-1} x$	$\log_b x$	$\ln x$

Figure 2.56

Intermediate Value Theorem

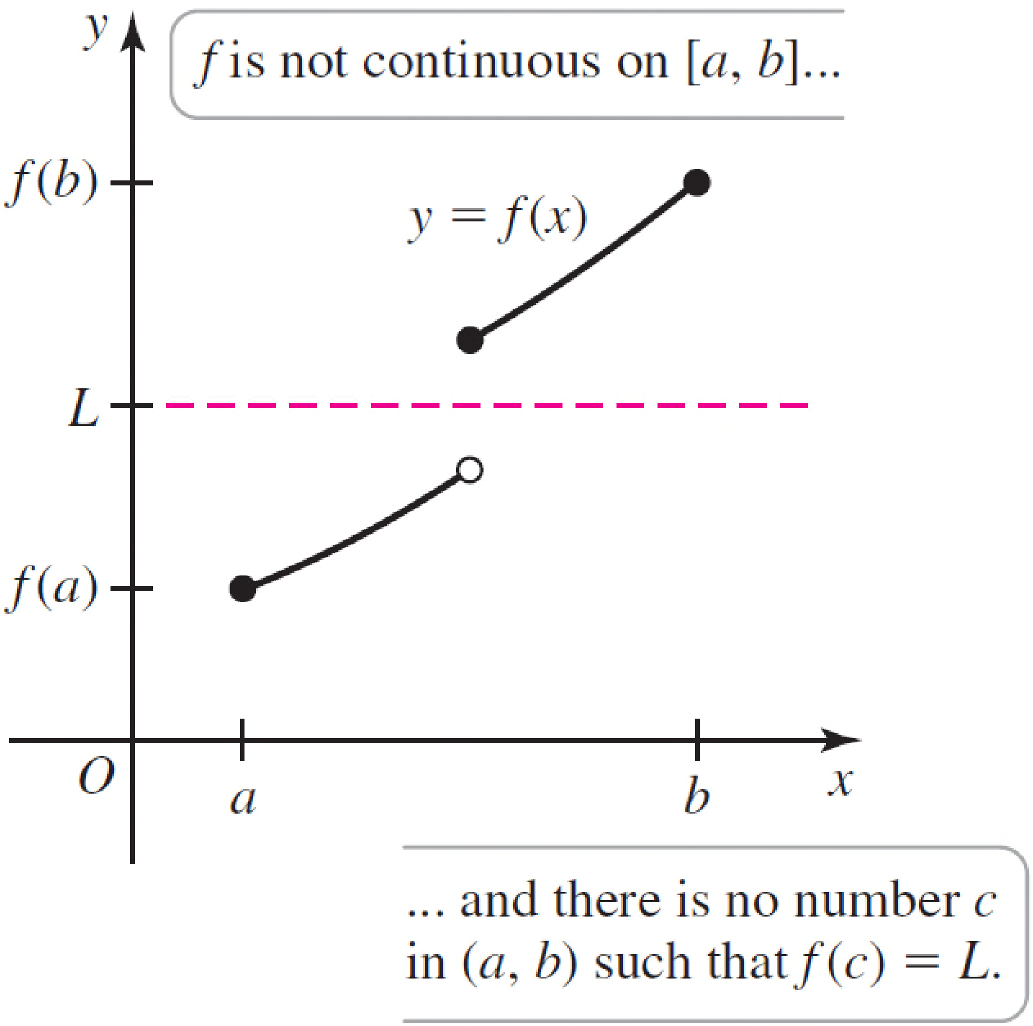


There is *at least* one number c in (a, b) such that $f(c) = L$, where L is between $f(a)$ and $f(b)$.

THEOREM 2.16 The Intermediate Value Theorem

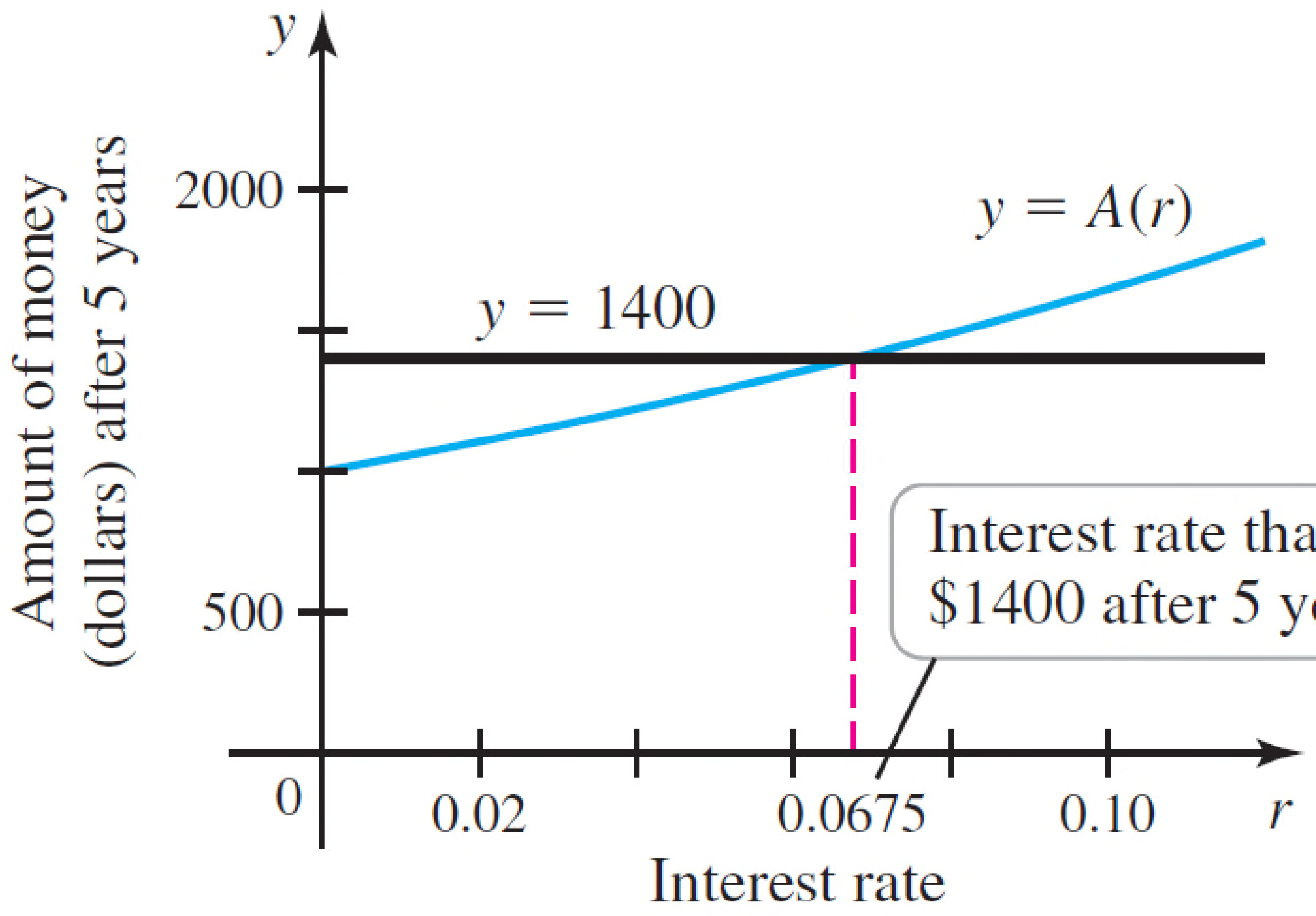
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

Figure 2.57



f is not continuous on [a, b]...
y
f(b)
y = f(x)
L
f(a)
O
a
b
x
... and there is no number c in (a, b) such that f(c) = L.

Figure 2.58


y
2000
500
0
Amount of money (dollars) after 5 years
y = A(r)
y = 1400
Interest rate that yields $1400 after 5 years
0.02
0.0675
0.10
r
Interest rate

2.7

Precise Definitions of Limits

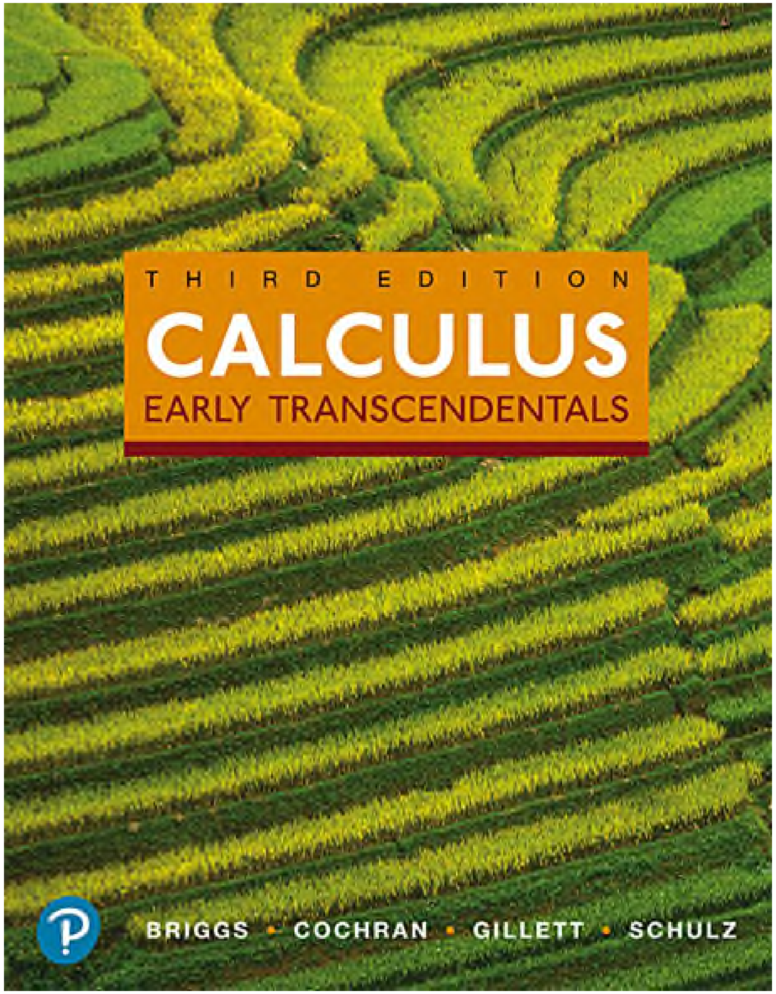


Figure 2.59

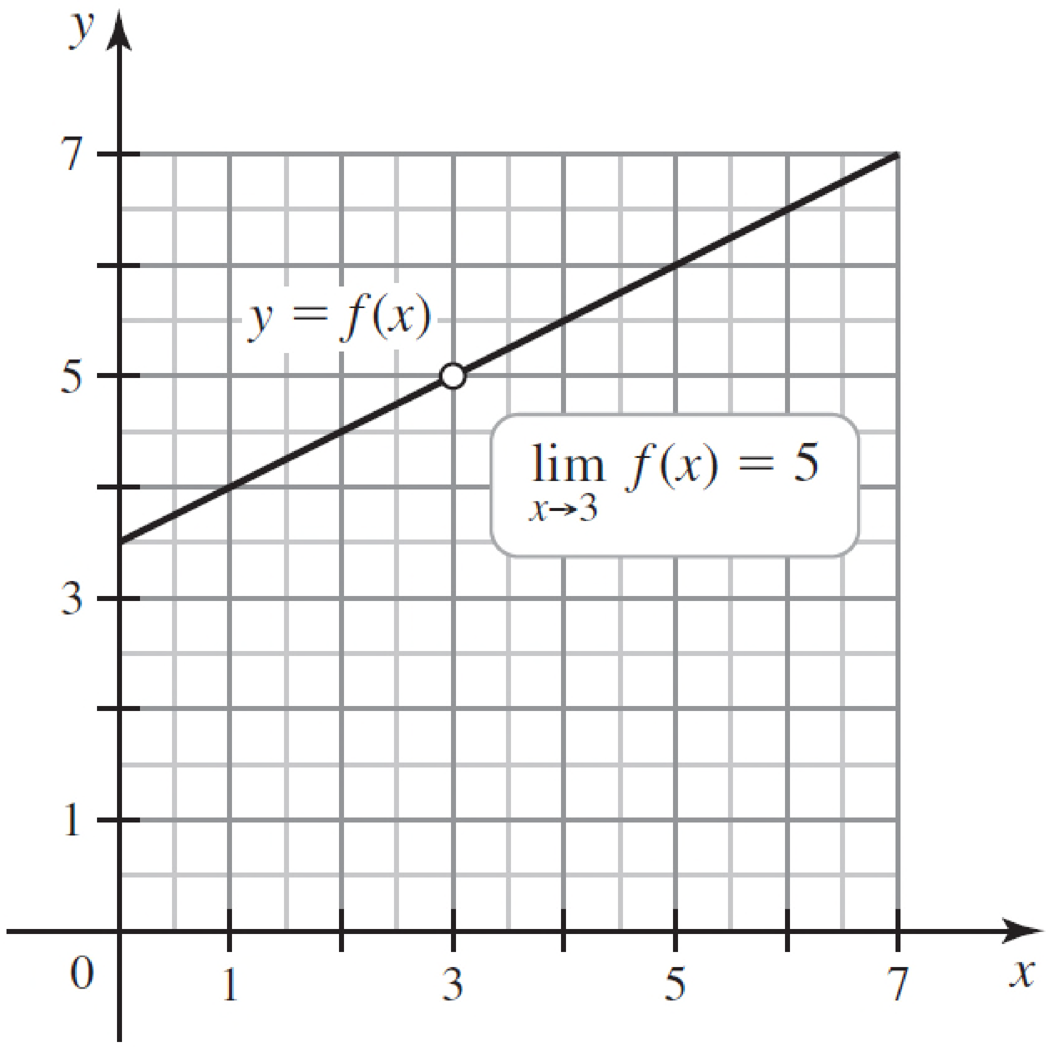


Figure 2.60 (a)

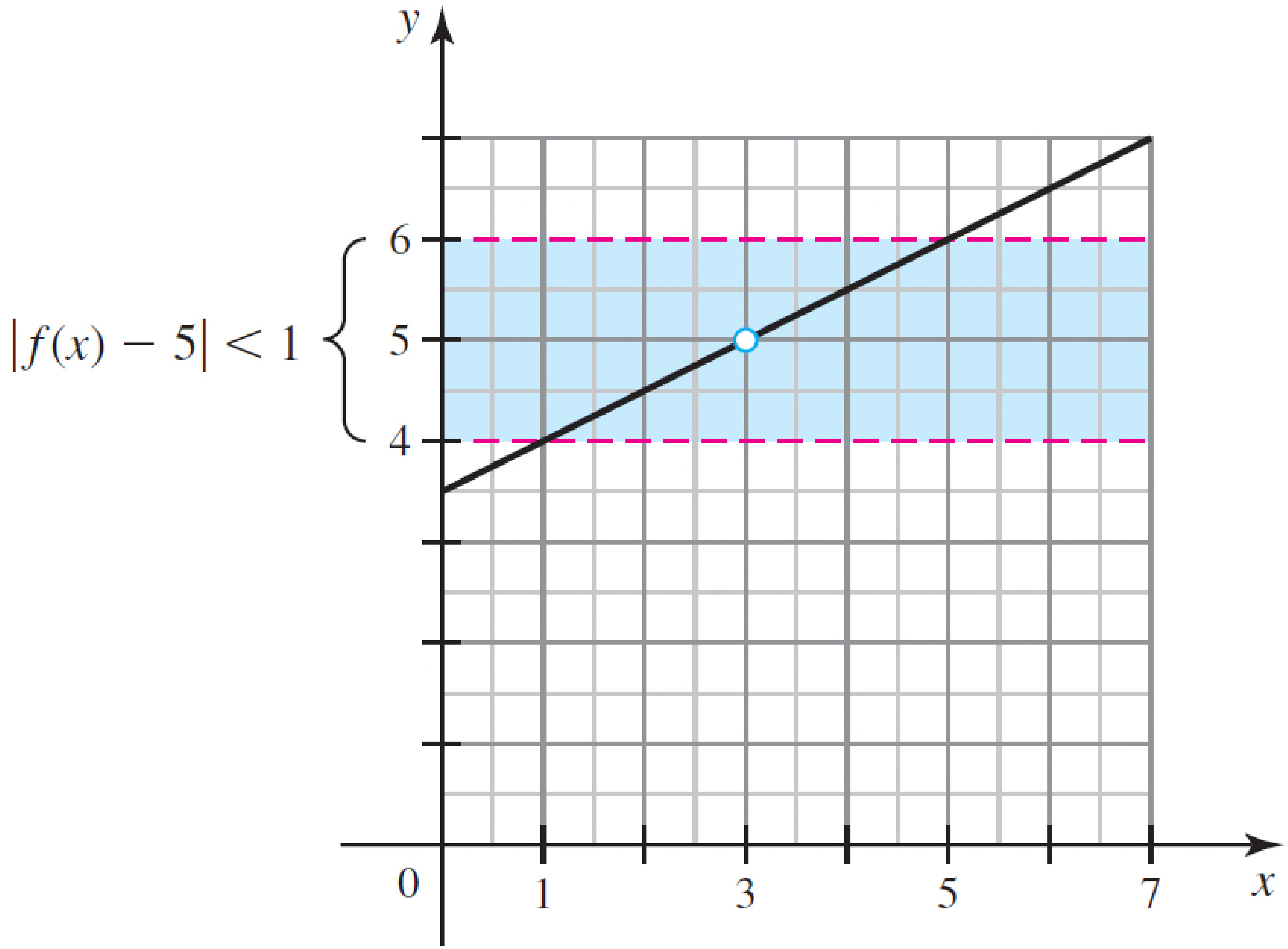


Figure 2.60 (b)

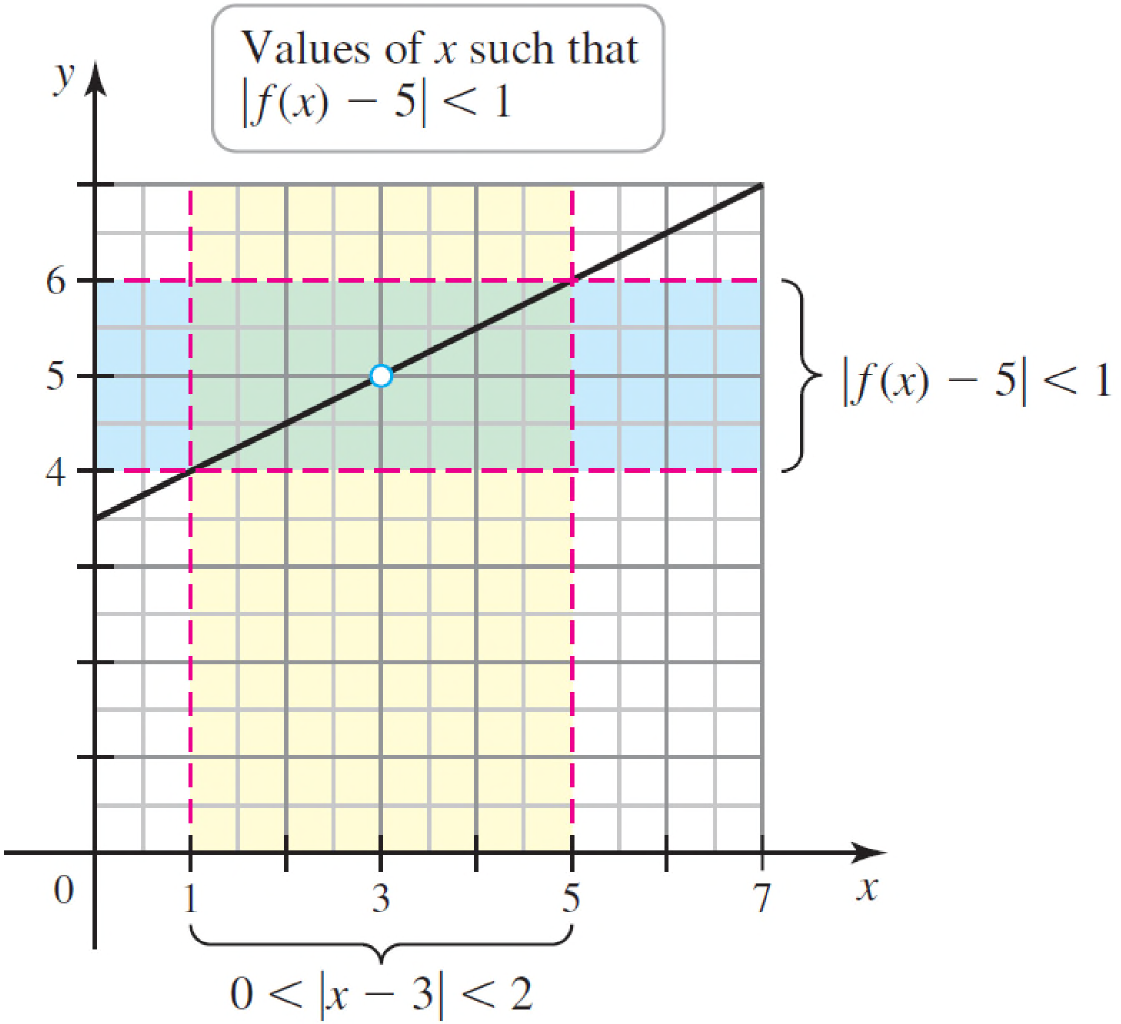


Figure 2.61 (a)

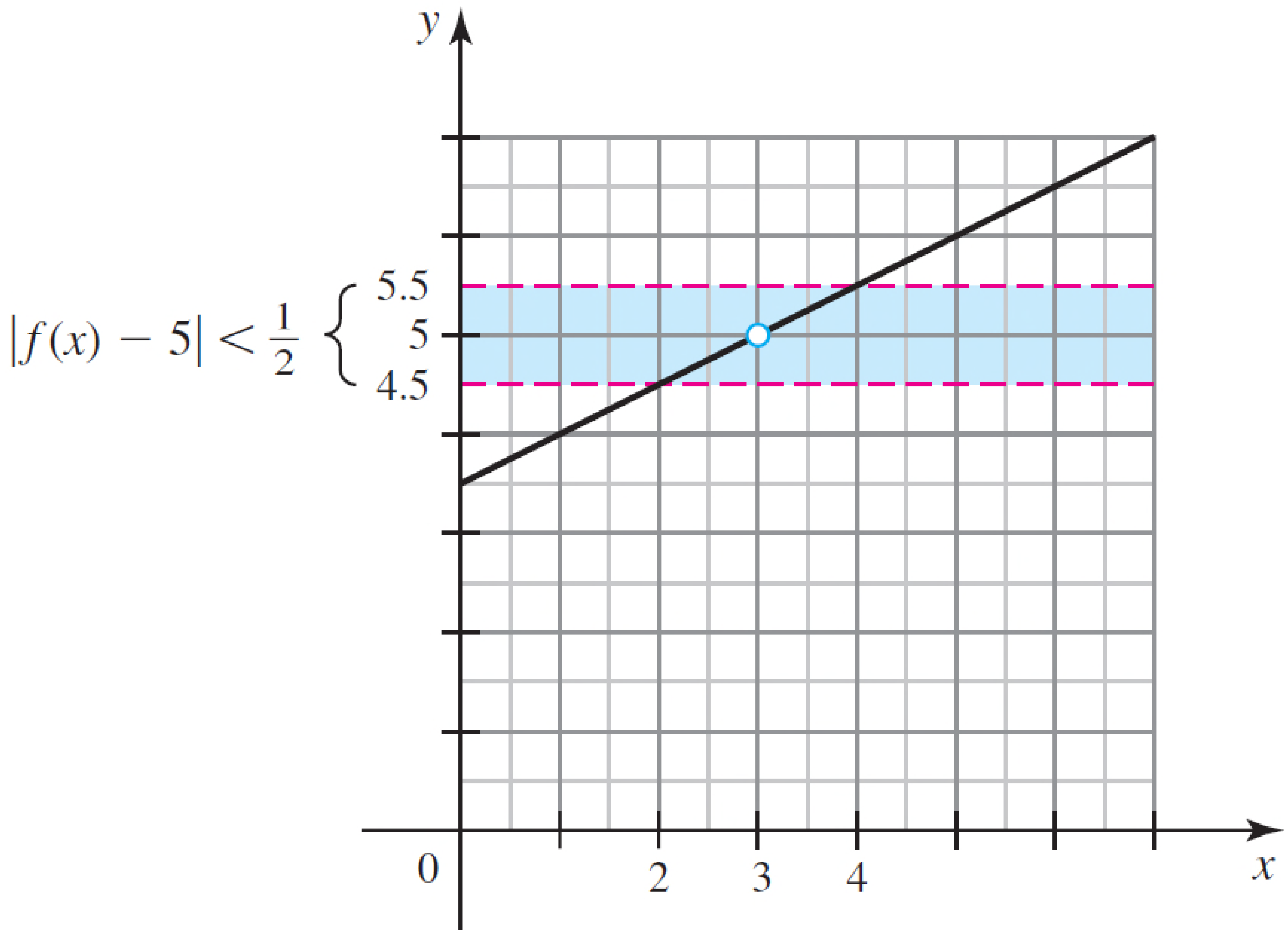


Figure 2.61 (b)

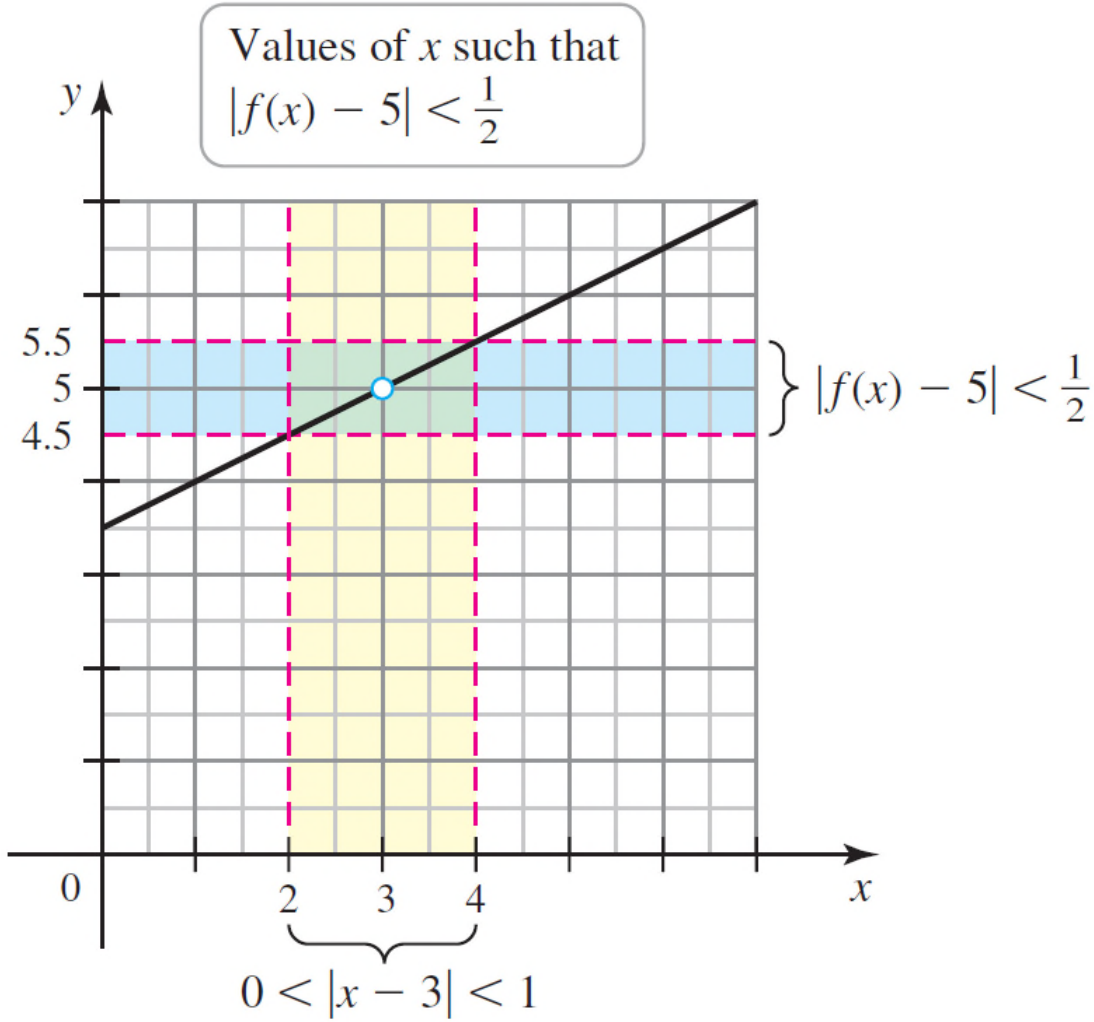

Values of x such that
$|f(x) - 5| < \frac{1}{2}$
y
5.5
5
4.5
0
2
3
4
x
$|f(x) - 5| < \frac{1}{2}$
$0 < |x - 3| < 1$

Figure 2.62

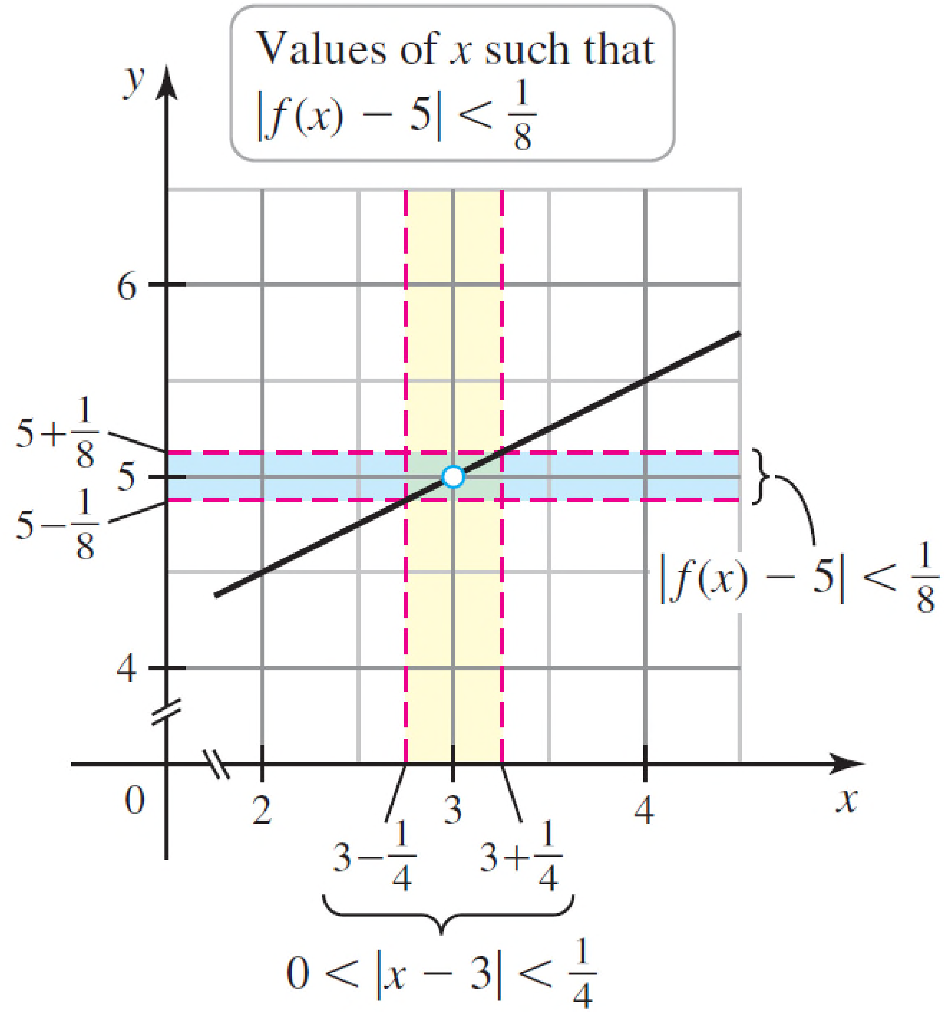
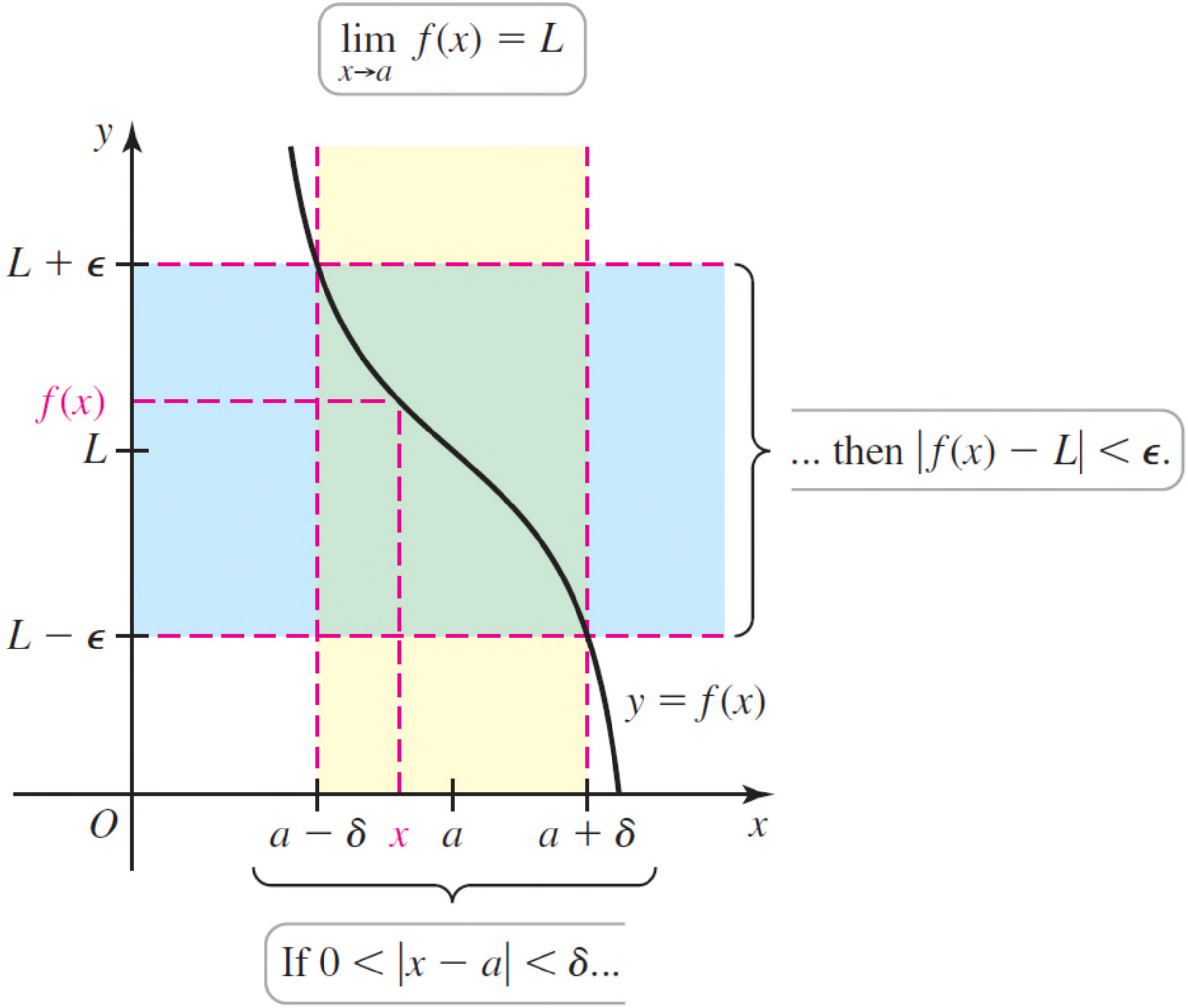

Values of x such that $|f(x) - 5| < \frac{1}{8}$
y
6
$5+\frac{1}{8}$
5
$5-\frac{1}{8}$
4
0
2
3
4
x
$|f(x) - 5| < \frac{1}{8}$
$3-\frac{1}{4}$
$3+\frac{1}{4}$
$0 < |x - 3| < \frac{1}{4}$

Figure 2.63



$\lim_{x \to a} f(x) = L$
y
$L + \epsilon$
$f(x)$
L
$L - \epsilon$
... then $|f(x) - L| < \epsilon$.
$y = f(x)$
O
$a - \delta$
x
a
$a + \delta$
x
If $0 < |x - a| < \delta$...

DEFINITION Limit of a Function

Assume $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say **the limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x \to a} f(x) = L,$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Figure 2.64

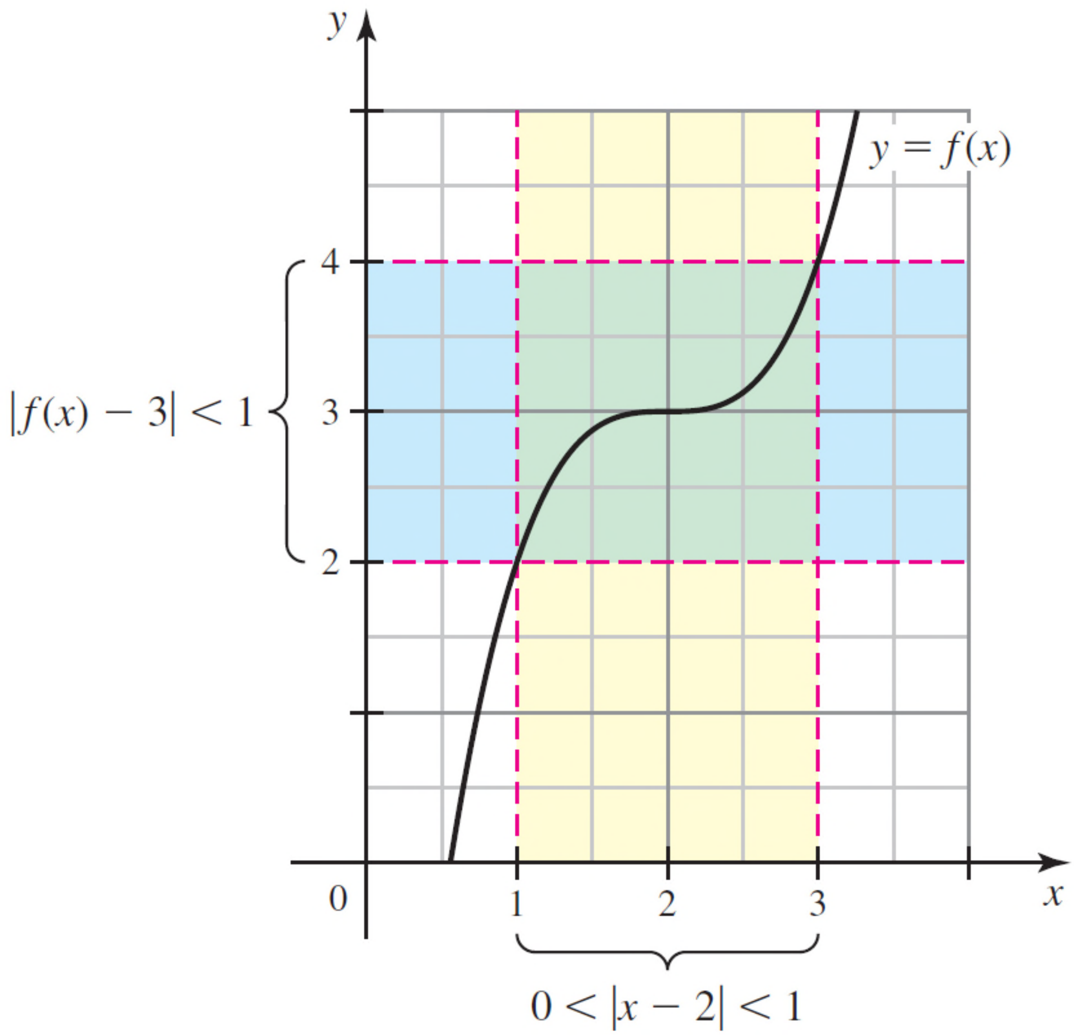


Figure 2.65

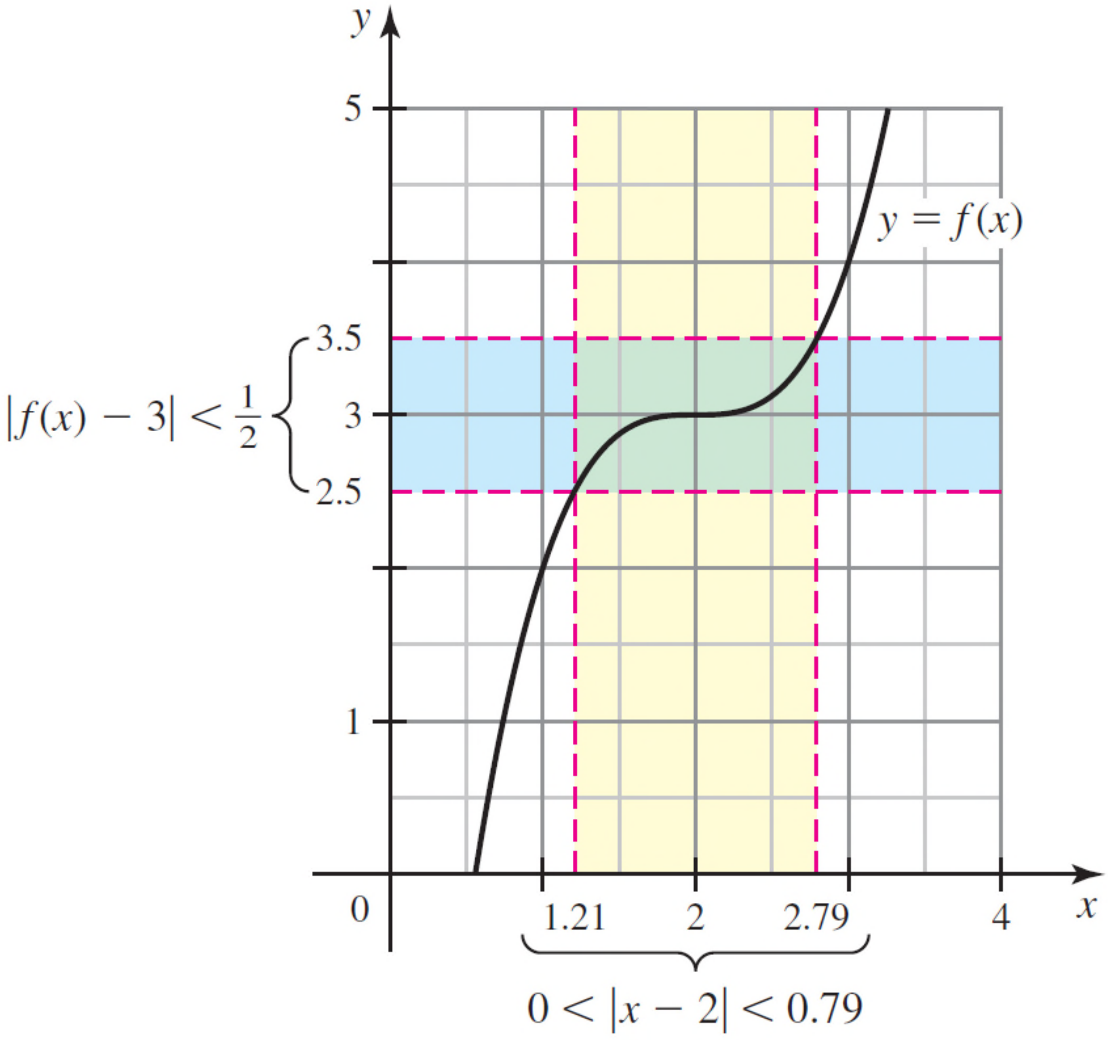


Figure 2.66

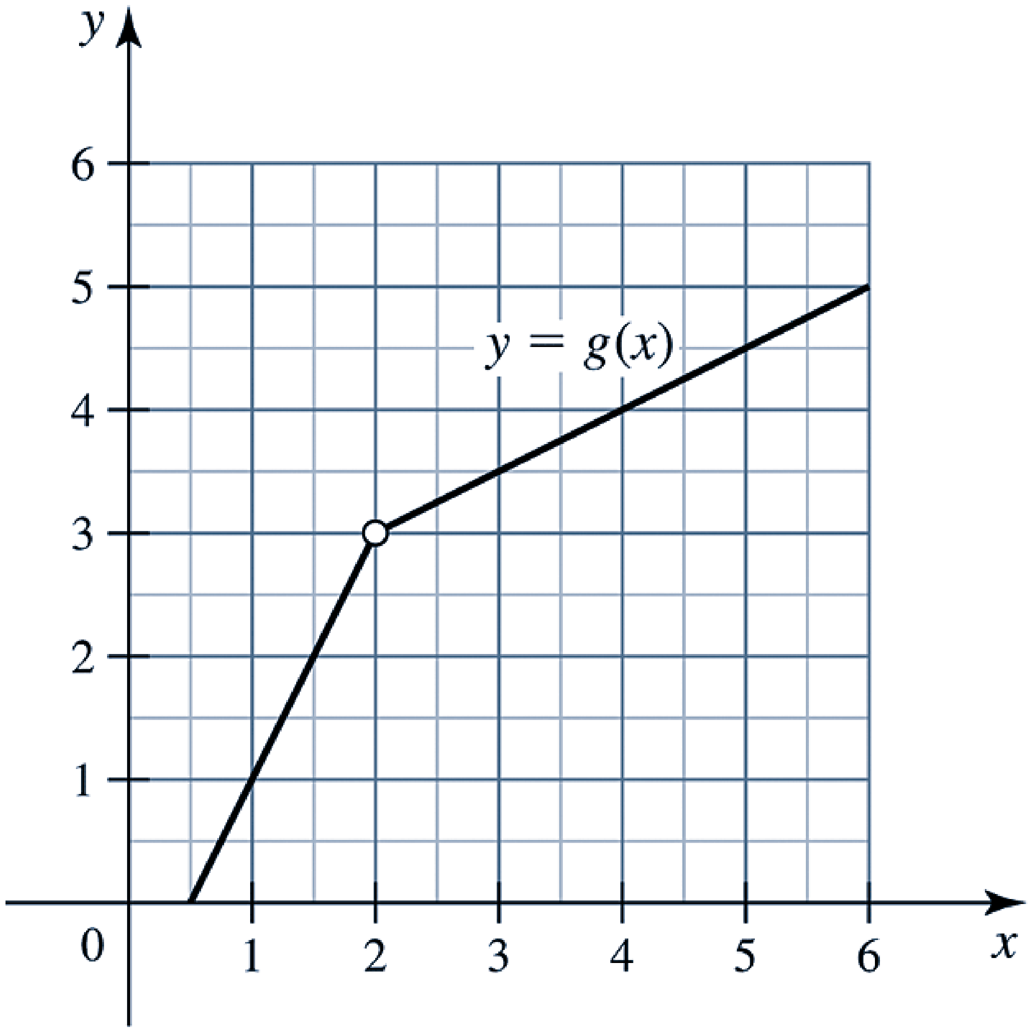


Figure 2.67 (a)

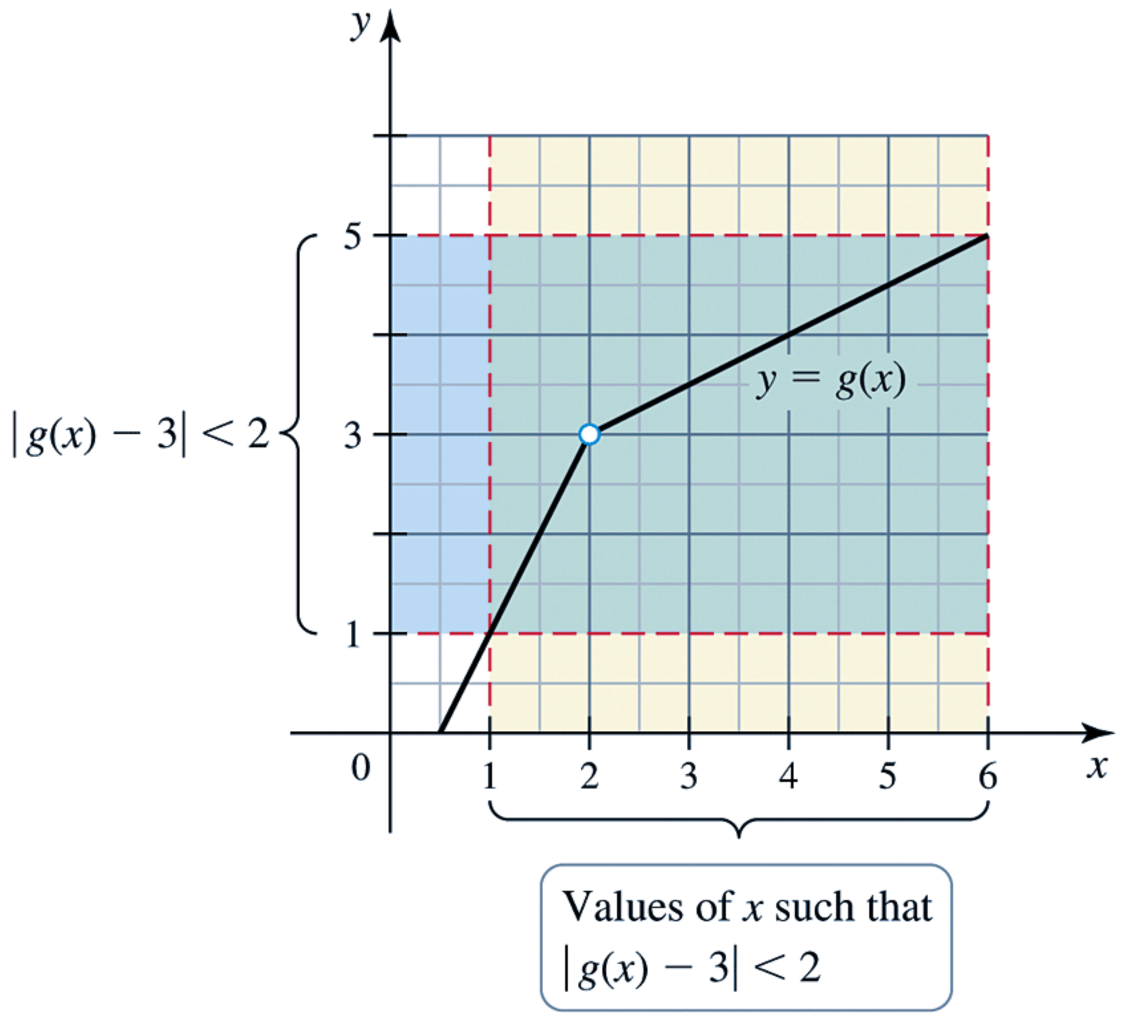


Figure 2.67 (b)

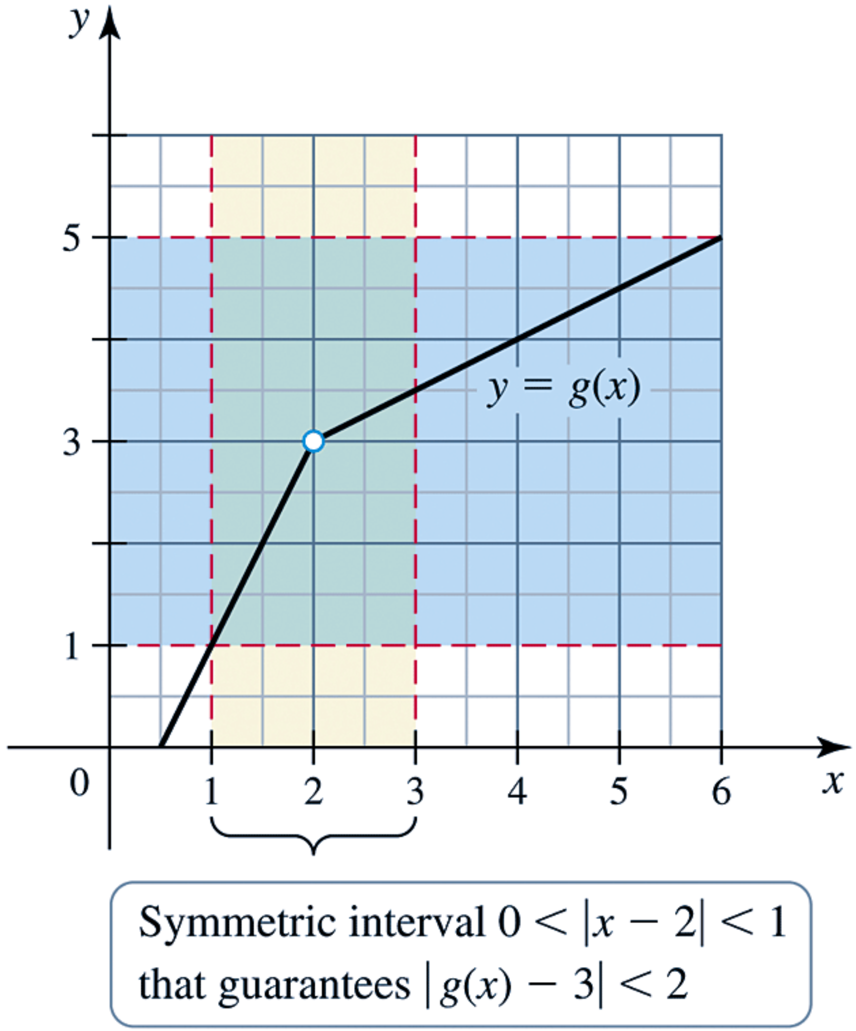
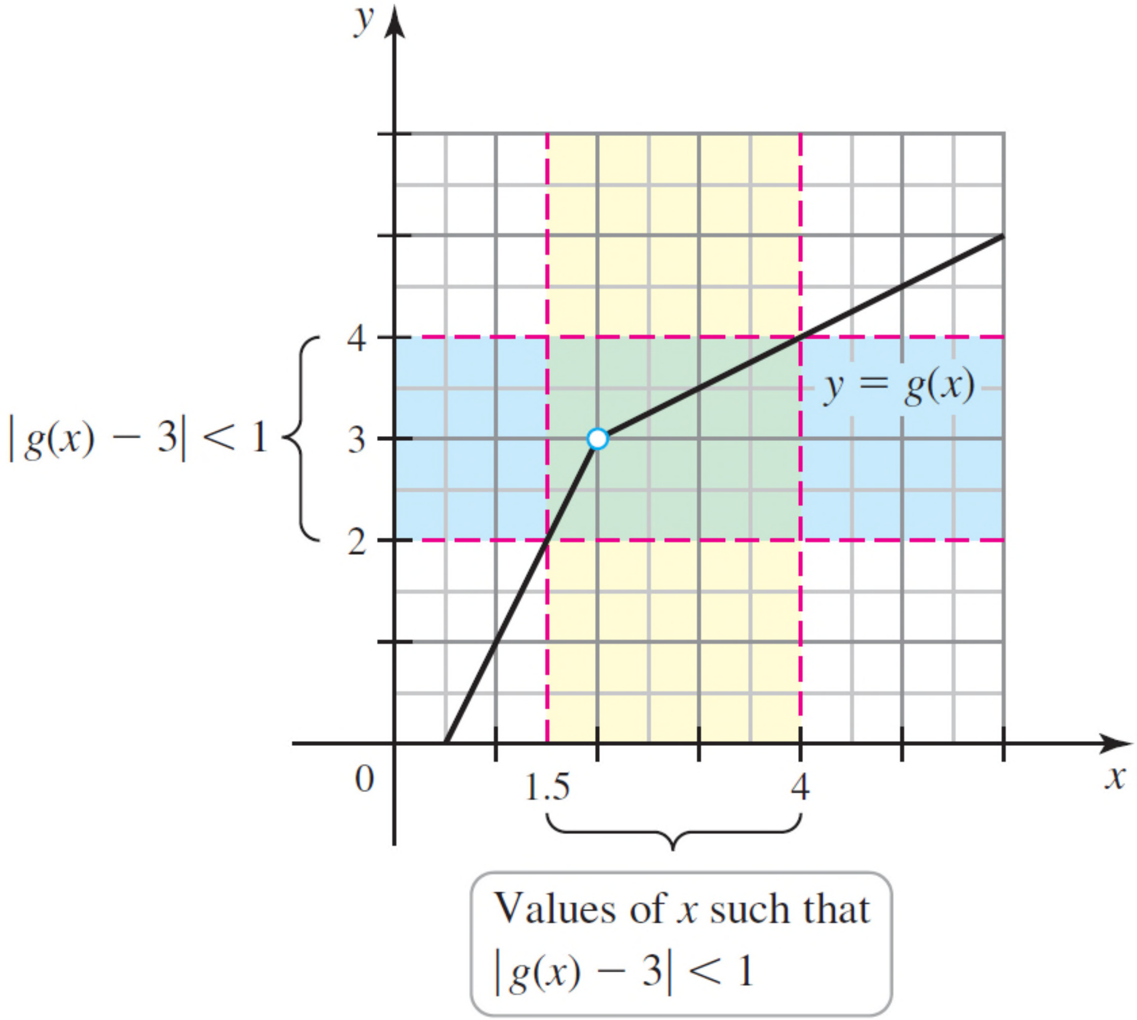
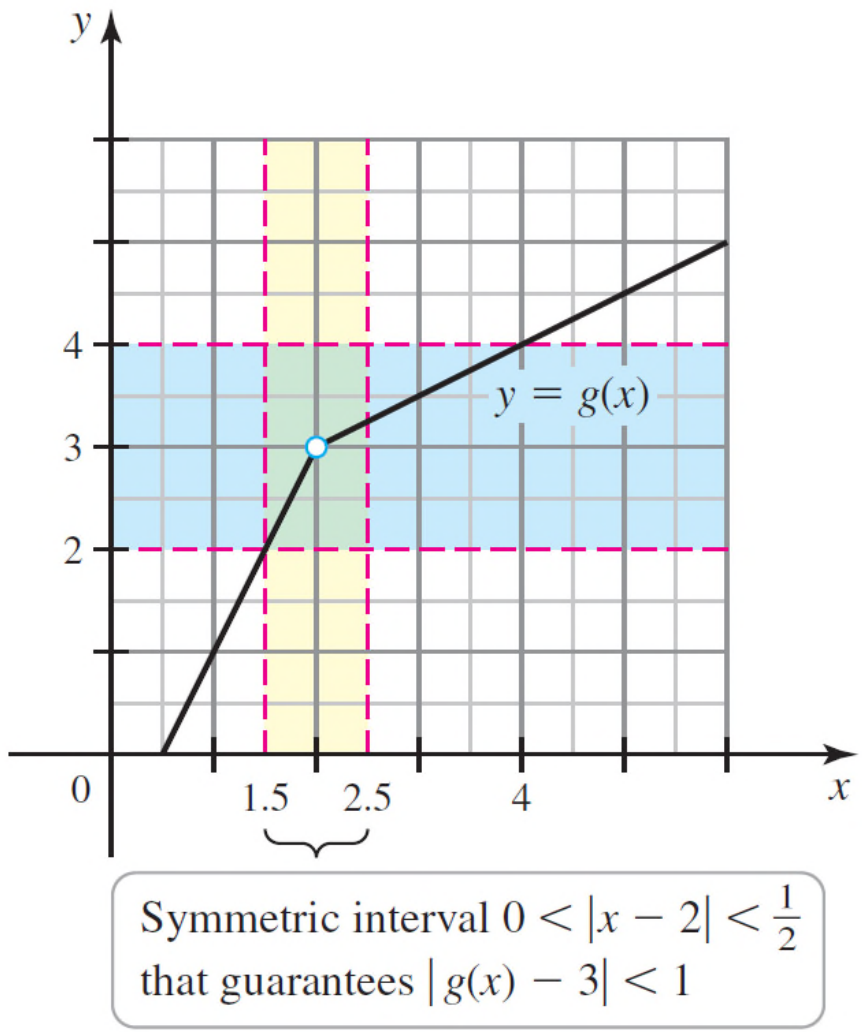

y
5
3
1
0
1
2
3
4
5
6
x
y = g(x)
Symmetric interval 0 < |x − 2| < 1
that guarantees |g(x) − 3| < 2

Figure 2.68 (a)



CLICK HERE TO ACCESS THE COMPLETE Solutions

Figure 2.68 (b)



Steps for proving that $\lim_{x \to a} f(x) = L$

1. **Find δ.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.

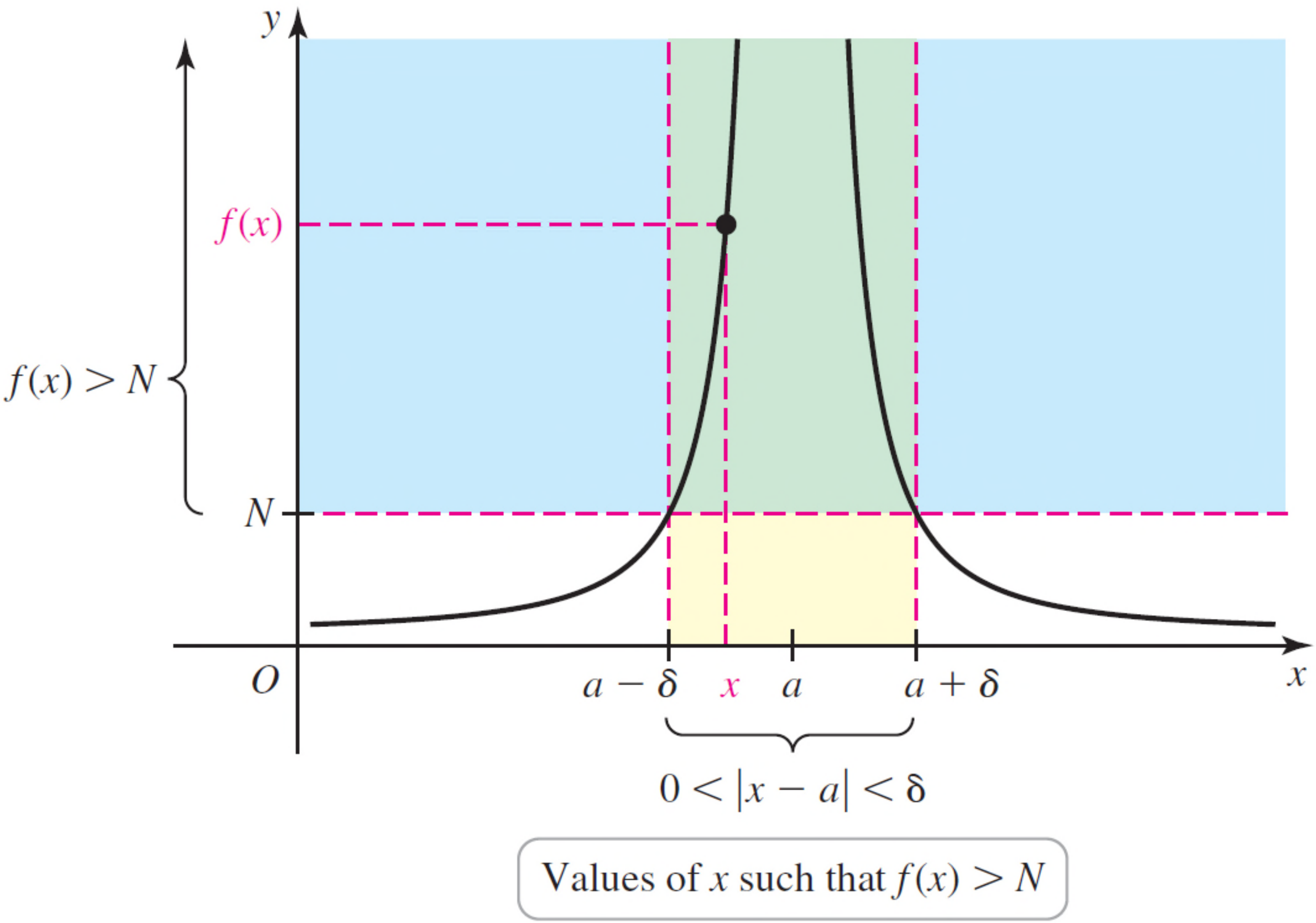
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

DEFINITION Two-Sided Infinite Limit

The **infinite limit** $\lim_{x \to a} f(x) = \infty$ means that for any positive number N there exists a corresponding $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Figure 2.69



Steps for proving that $\lim_{x \to a} f(x) = \infty$

1. **Find δ.** Let N be an arbitrary positive number. Use the statement $f(x) > N$ to find an inequality of the form $|x - a| < \delta$, where δ depends only on N.

2. **Write a proof.** For any $N > 0$, assume $0 < |x - a| < \delta$ and use the relationship between N and δ found in Step 1 to prove that $f(x) > N$.