Solutions for Engineering Vibration 5th Edition by Inman

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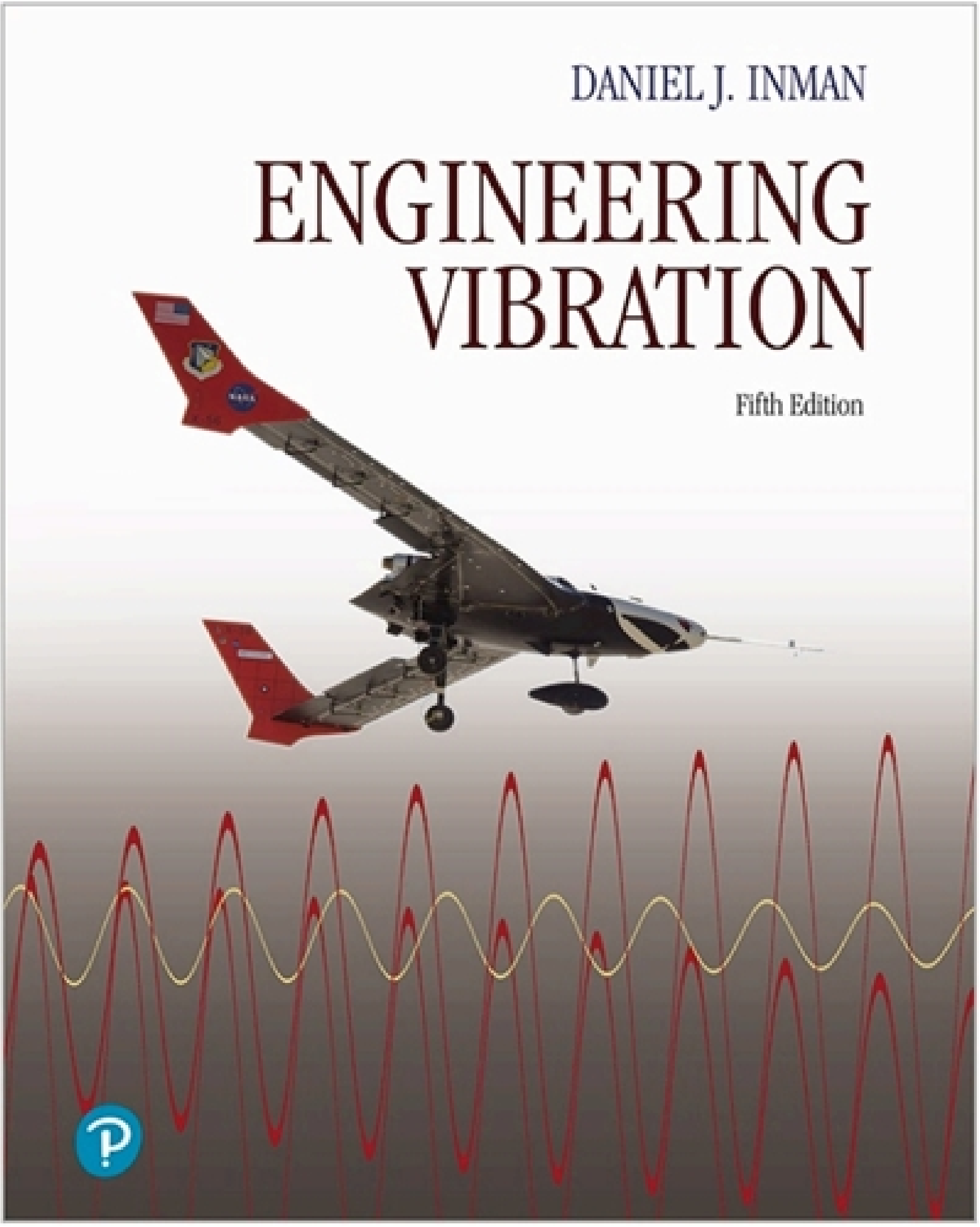


Solutions

Problems and Solutions Section 1.1 (1.1 through 1.27)

1.1 A spring-mass system has a mass of 100 kg and a stiffness of 10,000 N/m. What is its period of oscillation?

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000 \text{ N/m}}{100 \text{ kg}}} = 10 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega_n} = \frac{2\pi}{10} = 0.62 \text{ s}$$

1.2 The mass of a passenger car is about 2500 kg and has a stiffness of 161,255 N/m. Compare the frequency and period of the car empty to that if 181 kg of passengers are added to the car.

Solution: The frequency and period of the empty car are:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{161{,}255 \text{ N/m}}{2500 \text{ kg}}} = 8.031 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega_n} = \frac{2\pi}{8.031} = 0.81 \text{ s}$$

With 181 kg of passengers the frequency and period become

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{161{,}255 \text{ N/m}}{2681 \text{ kg}}} = 7.555 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega_n} = \frac{2\pi}{7.555} = 0.78 \text{ s}$$

Thus in this case the difference in period of oscillation with and without the passengers is an imperceptible 3 hundredths of a second. For lighter cars this difference could become perceptible.

1.3 Consider a simple pendulum (see Example 1.1.1) and compute the magnitude of the restoring force if the mass of the pendulum is 2 kg and the length of the pendulum is 0.5 m. Assume the pendulum is at the surface of the earth at sea level.

Solution: From example 1.1.1, the restoring force of the pendulum is $mgl\sin\theta$, which has maximum value

$$mgl = 2 \cdot 9.81 \cdot 0.5 \frac{\text{kg}\cdot\text{m}\cdot\text{m}}{\text{sec}^2} = \underline{9.81 \text{ N}\cdot\text{m}}$$

1.4 Compute the period of oscillation of a pendulum of length 1 m at the North Pole where the acceleration due to gravity is measured to be 9.832 m/s^2.

Solution: The natural frequency and period can be computed with the following relationships:

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\omega_n = \sqrt{\frac{9.832\,\frac{m}{s^2}}{1m}}$$

$$T = \frac{2\pi}{\omega_n}$$

$$\omega_n = 3.1356\,\frac{rad}{s} \qquad T = 2.004\,s$$

1.5 The spring of Figure 1.2, repeated here as Figure P1.3, is loaded with mass of 15 kg and the corresponding (static) displacement is 0.01 m. Calculate the spring's stiffness.

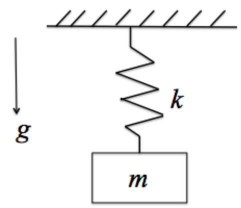
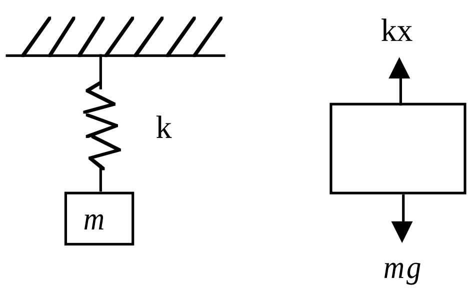


Figure P1.3

Solution:

Free-body diagram:



From the free-body diagram and static equilibrium:

$$kx = mg \quad (g = 9.81 m/s^2)$$

$$k = mg/x$$

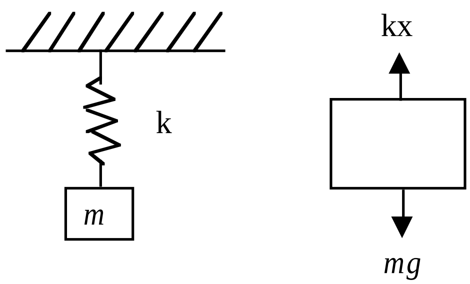
$$= \frac{15 \times 9.81}{0.01}\frac{\text{N}}{\text{m}} = \underline{14715\ \text{N/m}}$$

1.6 The spring of Figure P1.3 is successively loaded with mass and the corresponding (static) displacement is recorded below. Plot the data and calculate the spring's stiffness. Note that the data contain some error. Also calculate the standard deviation.

m(kg)	10	11	12	13	14	15	16
x(m)	1.14	1.25	1.37	1.48	1.59	1.71	1.82

Solution:

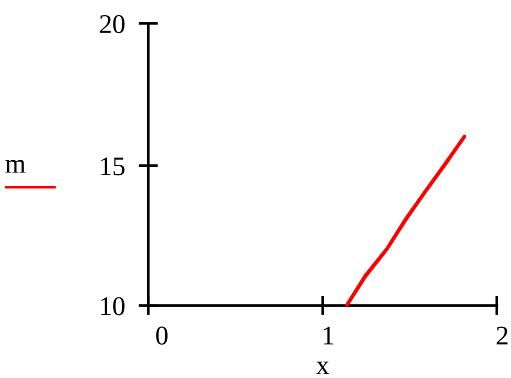
Free-body diagram:



From the free-body diagram and static equilibrium:

$$kx = mg \quad (g = 9.81 m/s^2)$$
$$k = mg / x$$

$$\mu = \frac{\Sigma k_i}{n} = 86.164$$



The sample standard deviation in computed stiffness is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (k_i - \mu)^2}{n-1}} = \mathbf{0.164}$$

Plot of mass in kg versus displacement in m

Computation of slope from *mg*/*x*

m(kg)	*x*(m)	*k*(N/m)
10	1.14	86.05
11	1.25	86.33
12	1.37	85.93
13	1.48	86.17
14	1.59	86.38
15	1.71	86.05
16	1.82	86.24

1.7 Consider the pendulum of Example 1.1.1 and compute the amplitude of the restoring force if the mass of the pendulum is 2 kg and the length of the pendulum is 0.5 m if the pendulum is at the surface of the moon.

Solution: From example 1.1.1, the restoring force of the pendulum is $mgl\sin\theta$, which has maximum value

$$1.5\quad mgl = 2\times\frac{9.81}{6}\times 0.5\frac{\text{kg}\cdot\text{m}\cdot\text{m}}{\text{sec}^2} = \underline{1.635\ \text{N}\cdot\text{m}}$$

1.8 Consider the pendulum of Example 1.1.1 and compute the angular natural frequency (radians per second) of vibration for the linearized system if the mass of the pendulum is 2 kg and the length of the pendulum is 0.5 m if the pendulum is at the surface of the earth. What is the period of oscillation in seconds?

Solution: The natural frequency and period are:

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\omega_n = \sqrt{\frac{9.81\,\frac{m}{s^2}}{0.5m}}$$

$$T = \frac{2\pi}{\omega_n}$$

$$\omega_n = 4.43\frac{rad}{s} \qquad T = 1.42\ s$$

1.9 Derive the solution of $m\ddot{x} + kx = 0$ and plot the result for at least two periods for the case with ω_n = 2 rad/s, x_0 = 1 mm, and v_0 = $\sqrt{5}$ mm/s.

Solution:

Given:

$$m\ddot{x} + kx = 0 \qquad (1)$$

Assume: $x(t) = ae^{rt}$. Then: $\dot{x} = are^{rt}$ and $\ddot{x} = ar^2e^{rt}$. Substitute into equation (1) to get:

$$mar^2e^{rt} + kae^{rt} = 0$$
$$mr^2 + k = 0$$
$$r = \pm\sqrt{\frac{k}{m}}\ i$$

Thus there are two solutions:

$$x_1 = c_1 e^{\left(\sqrt{\frac{k}{m}}i\right)t}, \text{ and } x_2 = c_2 e^{\left(-\sqrt{\frac{k}{m}}i\right)t}$$
$$\text{where } \omega_n = \sqrt{\frac{k}{m}} = 2 \text{ rad/s}$$

The sum of x_1 and x_2 is also a solution so that the total solution is:

$$x = x_1 + x_2 = c_1e^{2it} + c_2e^{-2it}$$

Substitute initial conditions: x_0 = 1 mm, v_0 = $\sqrt{5}$ mm/s

$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow \underline{c_2 = 1 - c_1}$, and $v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = \sqrt{5}$ mm/s

$\Rightarrow \underline{-2c_1 + 2c_2 = \sqrt{5}\,i}$. Combining the two underlined expressions (2 eqs in 2 unkowns):

$-2c_1 + 2 - 2c_1 = \sqrt{5}\,i \Rightarrow \underline{\underline{c_1 = \frac{1}{2} - \frac{\sqrt{5}}{4}i}}$, and $\underline{\underline{c_2 = \frac{1}{2} + \frac{\sqrt{5}}{4}i}}$

Therefore the solution is:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)e^{2it} + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)e^{-2it}$$

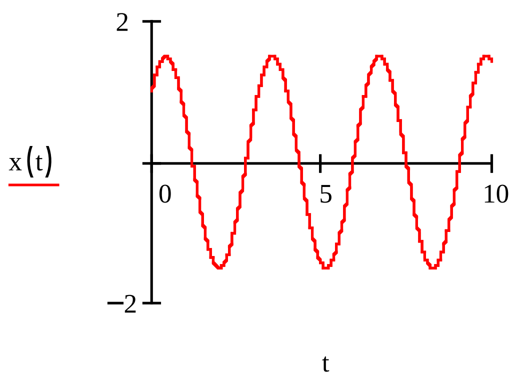
Using the Euler formula to evaluate the exponential terms yields:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)\left(\cos 2t + i\sin 2t\right) + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)\left(\cos 2t - i\sin 2t\right)$$

$$\Rightarrow \underline{x(t) = \cos 2t + \frac{\sqrt{5}}{2}\sin 2t = \frac{3}{2}\sin\left(2t + 0.7297\right)}$$

Using Mathcad the plot is:

$$x(t) := \cos(2\cdot t) + \frac{\sqrt{5}}{2}\cdot\sin(2\cdot t)$$



1.10 Solve $m\ddot{x} + kx = 0$ for k = 4 N/m, m = 1 kg, x_0 = 1 mm, and v_0 = 0. Plot the solution.

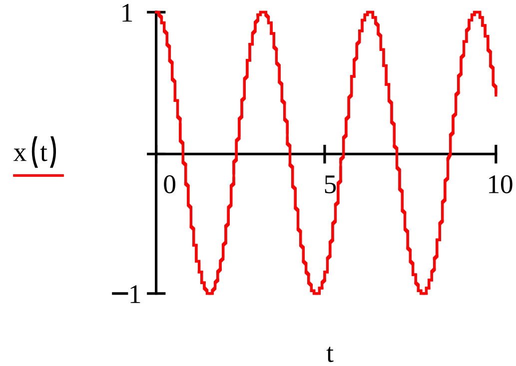
Solution: Here $v_0 = 0$. $\left(w_n = \sqrt{\frac{k}{m}} = 2 \text{ rad/s}\right)$. Calculating the initial conditions:

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow c_2 = 1 - c_1$$
$$v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = 0 \Rightarrow c_2 = c_1$$
$$c_2 = c_1 = 0.5$$
$$x(t) = \frac{1}{2}e^{2it} + \frac{1}{2}e^{-2it} = \frac{1}{2}(\cos 2t + i\sin 2t) + \frac{1}{2}(\cos 2t - i\sin 2t)$$

$$\underline{x(t) = \cos(2t)}$$

The following plot is from Mathcad:

$$x(t) := \cos(2 \cdot t)$$



Alternately students may use equation (1.10) directly to get

$$x(t) = \frac{\sqrt{2^2(1)^2 + 0^2}}{2}\sin(2t + \tan^{-1}[\frac{2 \times 1}{0}])$$

$$= 1\sin(2t + \frac{\pi}{2}) = \cos 2t$$

1.11 The amplitude of vibration of a spring-mass system is measured to be 1 mm. The phase shift from $t = 0$ is measured to be 2 rad and the frequency is found to be 5 rad/s. Calculate the initial conditions that caused this vibration to occur. Assume the response is of the form $x(t) = A\sin(\omega_n t + \phi)$.

Solution:

Given: $A = 1\text{mm}$, $\phi = 2\,\text{rad}$, $\omega = 5\,\text{rad/s}$. For an *undamped* system:

$$x(t) = A\sin(\omega_n t + \phi) = 1\sin(5t + 2) \quad \text{and}$$
$$v(t) = \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi) = 5\cos(5t + 2)$$

Setting $t = 0$ in these expressions yields:

$$x(0) = 1\sin(2) = \underline{0.9093 \text{ mm}}$$

$$v(0) = 5\cos(2) = \underline{-2.081 \text{ mm/s}}$$

1.12 Determine the stiffness of a single-degree-freedom, spring-mass system with a mass of 100 kg such that the natural frequency is 10 Hz.

Solution: First change Hertz to radians and then use the formula for natural frequency:

$$10\ \text{Hz} = 10\frac{\text{cycle}}{\text{sec}}\frac{2\pi\text{rad}}{\text{cycle}} = 20\pi\ \text{rad / sec}$$

$$\omega_n^2 = \frac{k}{m} \Rightarrow k = m\omega_n^2 = 100\text{kg}(20\pi)^2\frac{1}{\text{sec}^2} = \underline{394{,}784\ \text{N/m}}$$

1.13 Find the equation of motion for the system of Figure P1.11, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

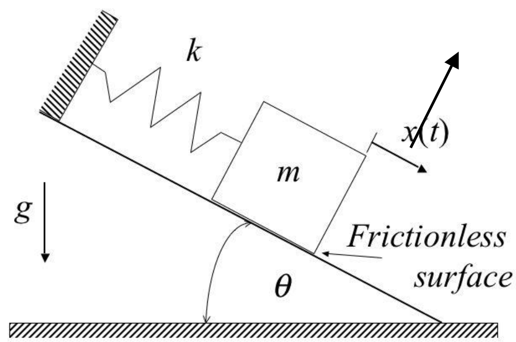
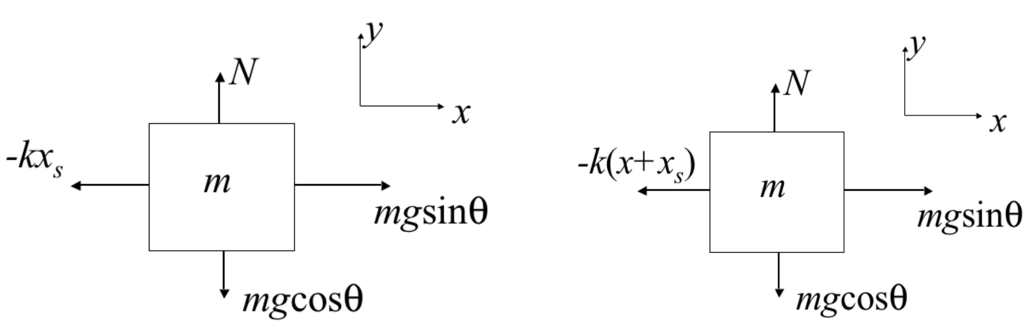


Figure P1.11

Solution:

Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$.

Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$
$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$
$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$
$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}} \text{ rad/s}$$

1.14 An undamped system vibrates with a frequency of 10 Hz and amplitude 1 mm. Calculate the maximum amplitude of the system's velocity and acceleration.

Solution:

Given: First convert Hertz to rad/s: $\omega_n = 2\pi f_n = 2\pi(10) = 20\pi$ rad/s.We also have that A= 1 mm.

For an undamped system:

$$x(t) = A\sin(\omega_n t + \phi)$$

and differentiating yields the velocity: $v(t) = A\omega_n\cos(\omega_n t + \phi)$. Realizing that both the sin and cos functions have maximum values of 1 yields:

$$v_{\max} = A\omega_n = 1(20\pi) = \mathbf{62.8\ mm/s}$$

Likewise for the acceleration: $a(t) = -A\omega_n^2\sin(\omega_n t + \phi)$

$$a_{\max} = A\omega_n^2 = 1(20\pi)^2 = \mathbf{3948\ mm/s^2}$$

1.15 Show by calculation that $A\sin(\omega_n t + \phi)$ can be represented as $A_1\sin\omega_n t + A_2\cos\omega_n t$ and calculate A_1 and A_2 in terms of A and ϕ.

Solution:

This trig identity is useful: $\sin(a + b) = \sin a\cos b + \cos a\sin b$

Given: $A\sin(\omega_n t + \phi) = A\sin(\omega_n t)\cos(\phi) + A\cos(\omega_n t)\sin(\phi)$

$$= A_1\sin\omega_n t + A_2\cos\omega_n t$$

where $A_1 = A\cos\phi$ and $A_2 = A\sin\phi$

1.16 Using the solution of equation (1.2) in the form $x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$ calculate the values of $A_1$and A_2 in terms of the initial conditions x_0 and v_0.

Solution:

Using the solution of equation (1.2) in the form

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

and differentiate to get:

$$\dot{x}(t) = \omega_n A_1 \cos(\omega_n t) - \omega_n A_2 \sin(\omega_n t)$$

Now substitute the initial conditions into these expressions for the position and velocity to get:

$$\begin{aligned} x_0 &= x(0) = A_1 \sin(0) + A_2 \cos(0) = A_2 \\ v_0 &= \dot{x}(0) = \omega_n A_1 \cos(0) - \omega_n A_2 \sin(0) \\ &= \omega_n A_1 (1) - \omega_n A_2 (0) = \omega_n A_1 \end{aligned}$$

Solving for A_1 and A_2 yields:

$$A_1 = \frac{v_0}{\omega_n}, \text{ and } \mathrm{A}_2 = x_0$$

Thus $$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t$$

1.17 Using the drawing in Figure 1.7, verify that equation (1.10) satisfies the initial velocity condition.

Solution: Following the lead given in Example 1.1.2, write down the general expression of the velocity by differentiating equation (1.10):

$$\begin{aligned} & x(t) = A \sin(\omega_n t + \phi) \Rightarrow \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi) \\ & \Rightarrow v(0) = A\omega_n \cos(\omega_n 0 + \phi) = A\omega_n \cos(\phi) \end{aligned}$$

From the figure:

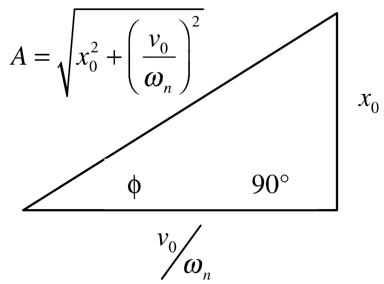


Figure 1.7

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \quad \cos\phi = \frac{v_0/\omega_n}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}}$$

Substitution of these values into the expression for $v(0)$ yields

$$v(0) = A\omega_n \cos\phi = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}\,(\omega_n)\frac{\frac{v_0}{\omega_n}}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}} = v_0$$

verifying the agreement between the figure and the initial velocity condition.

1.18 A 0.5 kg mass is attached to a linear spring of stiffness 0.1 N/m. a) Determine the natural frequency of the system in hertz. b) Repeat this calculation for a mass of 50 kg and a stiffness of 10 N/m. Compare your result to that of part a.

Solution: From the definition of frequency and equation (1.12)

$$(a) \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5}{0.1}} = 0.447 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{2.236}{2\pi} = \underline{0.071 \text{ Hz}}$$

$$(b) \qquad \omega_n = \sqrt{\frac{50}{10}} = 0.447 \text{rad/s}, \; f_n = \frac{\omega_n}{2\pi} = \underline{0.071 \text{ Hz}}$$

Part (b) is the same as part (a) thus very different systems can have same natural frequencies.

1.19 Derive the solution of the single degree of freedom system of Figure 1.4 by writing Newton's law, *ma* = –*kx*, in differential form using *adx* = *vdv* and integrating twice.

Solution: Substitute *a* = *vdv/dx* into the equation of motion *ma* = –*kx*, to get *mvdv* = –*kxdx*. Integrating yields:

$$\frac{v^2}{2} = -\omega_n^2\frac{x^2}{2} + c^2, \quad \text{where } c \text{ is a constant}$$

$$\text{or} \quad v^2 = -\omega_n^2 x^2 + c^2$$

$$v = \frac{dx}{dt} = \sqrt{-\omega_n^2 x^2 + c^2}$$

$$dt = \frac{dx}{\sqrt{-\omega_n^2 x^2 + c^2}}, \quad \text{write } u = \omega_n x \text{ to get:}$$

$$t - 0 = \frac{1}{\omega_n}\int \frac{du}{\sqrt{c^2 - u^2}} = \frac{1}{\omega_n}\sin^{-1}\left(\frac{u}{c}\right) + c_2$$

Here c_2 is a second constant of integration that is convenient to write as $c_2 = -\phi/\omega_n$. Rearranging yields

$$\omega_n t + \phi = \sin^{-1}\left(\frac{\omega_n x}{c}\right)$$

$$\frac{\omega_n x}{c} = \sin(\omega_n t + \phi)$$

$$\underline{x(t) = A\sin(\omega_n t + \phi)}, \quad A = \frac{c}{\omega_n}$$

in agreement with equation (1.19).

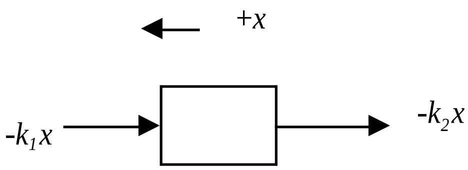
1.20 Determine the natural frequency of the two systems illustrated.

(a) (b)

Figure P1.18

Solution:

(a) Summing forces from the free-body diagram in the *x* direction yields:



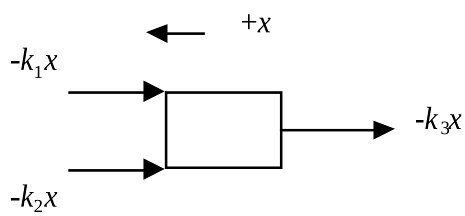
Free-body diagram for part a

$$m\ddot{x} = -k_1x - k_2x \Rightarrow$$
$$m\ddot{x} + k_1x + k_2x = 0$$
$m\ddot{x} + x(k_1 + k_2) = 0$, dividing by m yields:
$$\ddot{x} + \left(\frac{k_1 + k_2}{m}\right)x = 0$$

Examining the coefficient of x yields:

$$w_n = \sqrt{\frac{k_1 + k_2}{m}}$$

(b) Summing forces from the free-body diagram in the x direction yields:



Free-body diagram for part b

$$m\ddot{x} = -k_1x - k_2x - k_3x, \Rightarrow$$
$$m\ddot{x} + k_1x + k_2x + k_3x = 0 \Rightarrow$$
$$m\ddot{x} + (k_1 + k_2 + k_3)x = 0 \Rightarrow \ddot{x} + \frac{(k_1 + k_2 + k_3)}{m}x = 0$$
$$\Rightarrow w_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}$$

1.21* Plot the solution given by equation (1.11) for the case k = 1000 N/m and m = 10 kg for two complete periods for each of the following sets of initial conditions: a) x_0 = 0 m, v_0 = 1 m/s, b) x_0 = 0.01 m, v_0 = 0 m/s, and c) x_0 = 0.01 m, v_0 = 1 m/s.

Solution: Here we use Mathcad:

a) all units in m, kg, s

$m := 10 \qquad k := 1000$

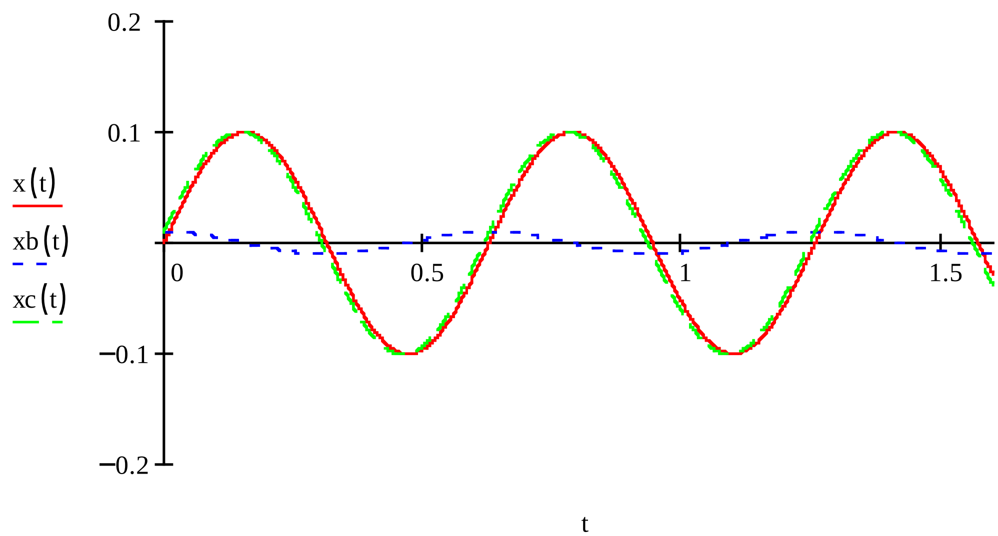
$x0 := 0.0 \qquad v0 := 1$

$fn := \frac{\omega n}{2 \cdot \pi} \qquad T := \frac{2 \cdot \pi}{\omega n}$

$\phi := \operatorname{atan}\left(\frac{\omega n \cdot x0}{v0}\right) \qquad x(t) := A \cdot \sin(\omega n \cdot t + \phi)$

parts b and c are plotted in the above by simply changing the initial conditions as appropriate

$$A := \frac{1}{\omega n} \cdot \sqrt{x0^2 \cdot \omega n^2 + v0^2}$$



1.22 A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.21. Ignore the mass of pendulum's rod and derive the equation of motion. Then following the procedure used in Example 1.1.1, linearize the equation of motion and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

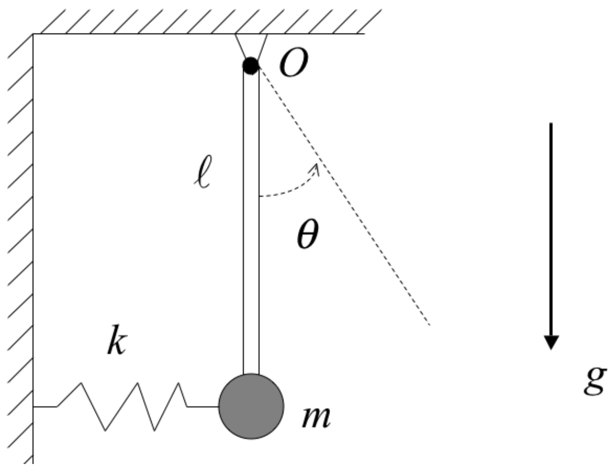
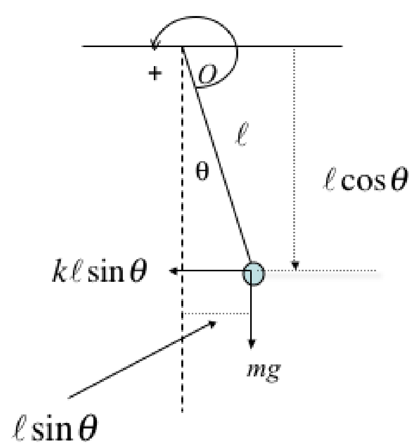


Figure P1.21

Solution: Consider the free body diagram of the mass displaced from equilibrium:



There are two forces acting on the system to consider, if we take moments about point O (then we can ignore any forces at O). This yields

$$\sum M_O = J_O\alpha \Rightarrow m\ell^2\ddot{\theta} = -mg\ell\sin\theta - k\ell\sin\theta\cdot\ell\cos\theta$$

$$\Rightarrow \underline{m\ell^2\ddot{\theta} + mg\ell\sin\theta + k\ell^2\sin\theta\cos\theta = 0}$$

Next consider the small θ approximations to that $\sin\theta \sim \theta$ and $\cos\theta = 1$. Then the linearized equation of motion becomes:

$$\ddot{q}(t) + \left(\frac{mg + k\ell}{m\ell}\right) q(t) = 0$$

Thus the natural frequency is

$$w_n = \sqrt{\frac{mg + k\ell}{m\ell}} \text{ rad/s}$$

1.23 A pendulum has length of 250 mm. What is the system's natural frequency in Hertz?

Solution:

Given: l =250 mm

Assumptions: small angle approximation of sin

From Window 1.1, the equation of motion for the pendulum is as follows: $I_O\ddot{q} + mgq = 0$, where $I_O = ml^2$ Þ $\ddot{q} + \frac{g}{l}q = 0$

The coefficient of θ yields the natural frequency as:

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.25 \text{ m}}} = 6.26 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 0.996 \text{ Hz}$$

1.24 The pendulum in Example 1.1.1 is required to oscillate once every second. What length should it be?

Solution:

Given: $f = 1$ Hz (one cycle per second)

$$\omega_n = 2\pi f = \sqrt{\frac{g}{l}}$$

$$\therefore\ l = \frac{g}{(2\pi f)^2} = \frac{9.81}{4\pi^2} = 0.248\ m$$

1.25 The approximation of sin θ = θ, is reasonable for θ less than 10°. If a pendulum of length 0.5 m, has an initial position of θ 0) = 0, what is the maximum value of the initial angular velocity that can be given to the pendulum with out violating this small angle approximation? (be sure to work in radians)

Solution: From Window 1.1, the linear equation of the pendulum is

$$\ddot{\theta}(t) = + \frac{g}{\ell}\theta(t) = 0$$

For zero initial position, the solution is given in equation (1.10) by

$$q(t) = \frac{v_0\sqrt{\ell}}{\sqrt{g}}\sin(\sqrt{\frac{g}{\ell}}t) \text{ Þ } |q| \text{ £ } \frac{v_0\sqrt{\ell}}{\sqrt{g}}$$

since sin is always less then one. Thus if we need θ< 10°= 0.175 rad, then we need to solve:

$$\frac{v_0\sqrt{0.5}}{\sqrt{9.81}} = 0.175$$

for v_0 which yields:

$$v_0 \leq \mathbf{0.773 \text{ rad/s.}}$$

1.26 A machine, modeled as a simple spring-mass system, oscillates in simple harmonic motion. Its acceleration is measured to have an amplitude of 10,000 mm/s^2 with a frequency of 8 Hz. Compute the maximum displacement the machine undergoes during this oscillation.

Solution: the equations of motion for position and acceleration are:

$$x = A\sin(w_n t + f) \quad \text{and} \quad \ddot{x} = - Aw_n^2 \sin(w_n t + f)$$

Since the sin is max at 1, the maximum acceleration is

$$Aw_n^2 = 10{,}000 \text{ mm/s}^2$$

$$w_n = 2pf = 2p(8) = 16p \text{ rad/s}$$

Solving for A yields:

$$A = \frac{10{,}000}{\omega_n^2} = \frac{10{,}000}{(16\pi)^2} = \underline{3.96 \text{ mm}}$$

1.27 Derive the relationships given in Window 1.4 for the constants a_1 and a_2 used in the exponential form of the solution in terms of the constants A_1 and A_2 used in sum of sine and cosine form of the solution. Use the Euler relationships for sine and cosine in terms of exponentials as given following equation (1.18).

Solution: Let θ = ω*t* for ease of notation. Then:

$$2\sin\theta j = e^{\theta j} - e^{\theta j} \text{ and } 2\cos\theta = e^{\theta j} + e^{\theta j}$$

$$\Rightarrow A_1 \sin\theta = A_1\frac{e^{\theta j} - e^{\theta j}}{2j} \text{and } A_2 \cos = A_2\frac{e^{\theta j} + e^{\theta j}}{2}$$

Adding these to in order to form $x(t)$ yields:

$$x(t) = A_1\frac{e^{\theta j}}{2j} - A_1\frac{e^{-\theta j}}{2j} + A_2\frac{e^{\theta j}}{2} + A_2\frac{e^{-\theta j}}{2}$$

$$\Rightarrow x(t) = -A_1\frac{e^{\theta j}}{2}j + A_1\frac{e^{-\theta j}}{2}j + A_2\frac{e^{\theta j}}{2} + A_2\frac{e^{-\theta j}}{2}$$

$$\Rightarrow x(t) = (A_2 - A_1 j)\frac{e^{\theta j}}{2} + (A_1 j + A_2)\frac{e^{-\theta j}}{2}$$

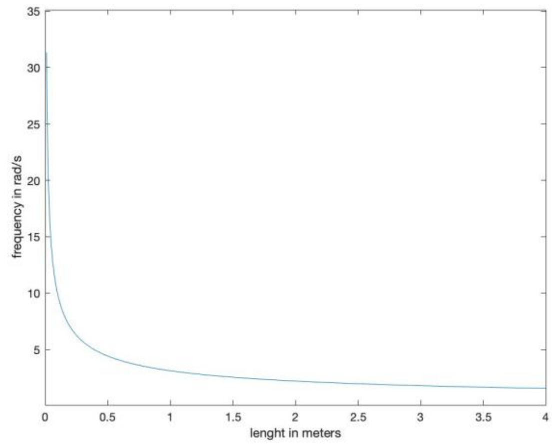
Comparing this last expression to $x(t) = a_1 e^{\theta j} + a_2 e^{-\theta j}$ yields:

$$a_1 = \frac{A_2 - A_1 j}{2} \text{ and } a_2 = \frac{A_2 + A_1 j}{2}$$

1.28 For a pendulum at the earth's surface, plot the frequency versus the length of the pendulum for values of the length between 0 and 4 meters.

Solution: Using the formula for frequency of a pendulum, typing in MATLAB

```
>> L=0:1/100:4;
>> x=sqrt((1./L)*9.81); % use the dot to perform element by element division
>> plot(L,x)
>> xlabel('lenght in meters')
>> ylabel('frequency in rad/s')
```

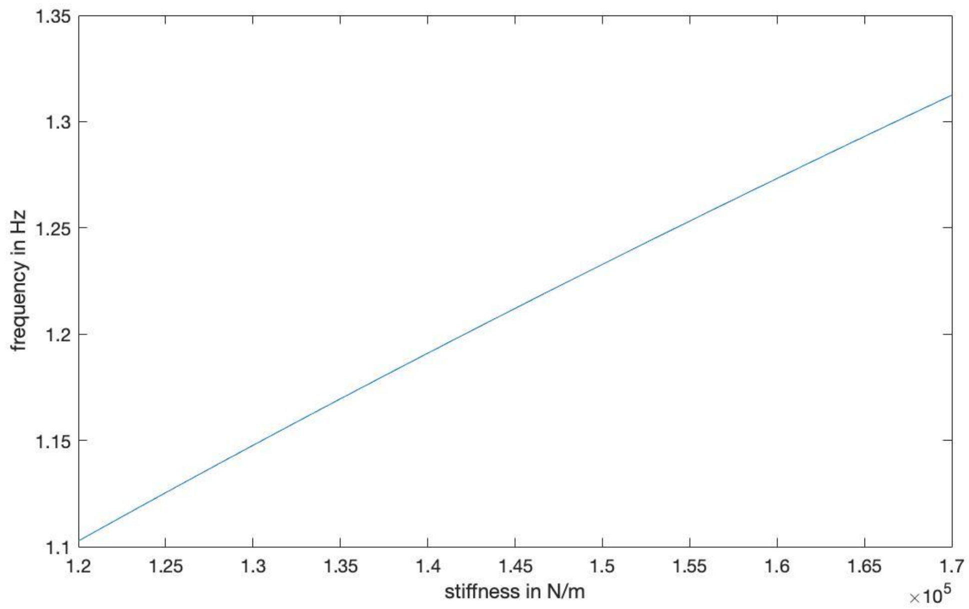


Note how little the frequency changes once past a quarter of a meter

1.29 Plot how the frequency changes in a 2500 kg car as the possible stiffness values range from 120,000 N/m to 170,000 N/m. Express your answer in Hz.

Solution: Typing in MATLAB's command window

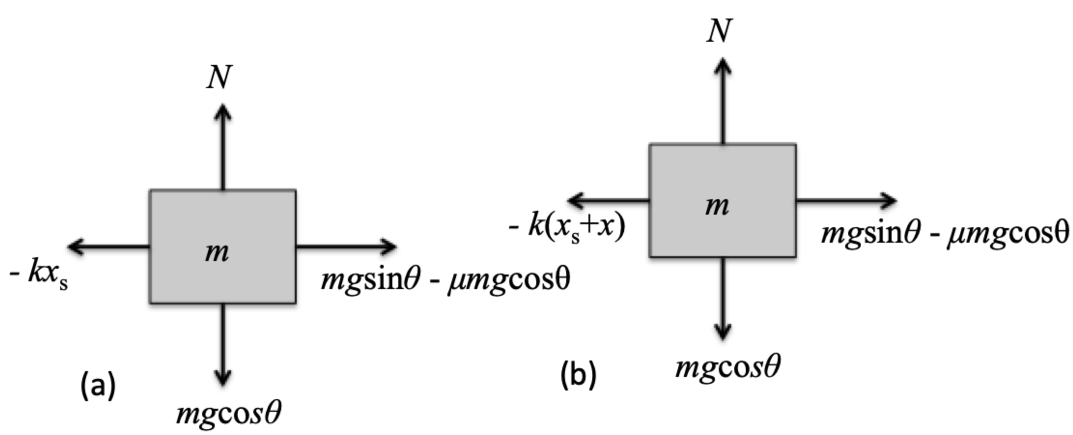
```
>> m=2500;
>> K=120000:50000/500:170000;
>> omega=sqrt((1/m).*K);
>> w=omega/(2*pi);
>> plot(K,w)
>> xlabel('stiffness in N/m')
>> ylabel('frequency in Hz')
```



So the frequency changes from 1.1 Hz to about 1.32 Hz.

1.30 Consider the system of Problem 1.13 and Figure P1.13. Suppose the surface of the plane provides Coulomb friction and determine the equation for vibration.

Solution: Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, *N* is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin q = 0$.

Summing forces in (b) yields:

$$\sum F_x = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta - \mu_k \operatorname{sgn}(\dot{x})mg\cos\theta$$

$$\Rightarrow m\ddot{x}(t) + kx + \mu_k \operatorname{sgn}(\dot{x})mg\cos\theta = -kx_s + mg\sin\theta = 0$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx + \mu_k \operatorname{sgn}(\dot{x})mg\cos\theta = 0}$$

Problems and Solutions for Section 1.2 and Section 1.3 (1.31 to 1.72)

Problems and Solutions Section 1.2 (Numbers 1.31 through 1.48)

1.31 The acceleration of a machine part modeled as a spring mass system is measured and recorded in Figure P 1.31. Compute the amplitude of the displacement of the mass.

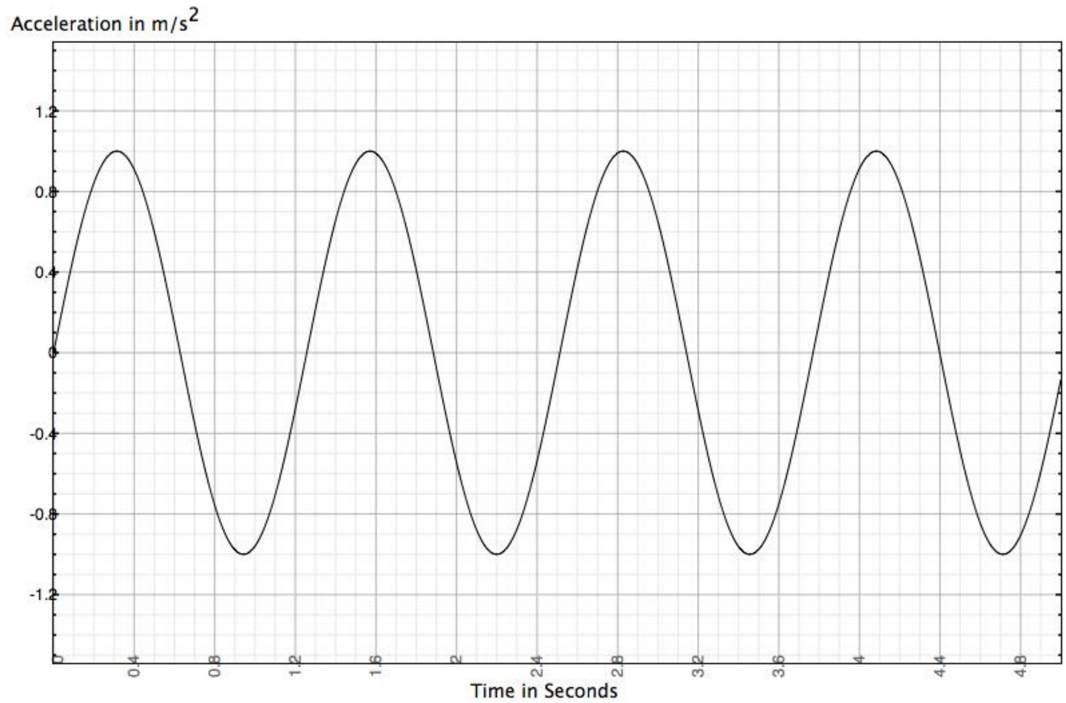


Figure P1.31

Solution: From Window 1.3 the maximum amplitude of the acceleration versus time plot is just $\omega_n^2 A$ where A is the maximum amplitude of the displacement and the quantity to be determined here. Looking at P1.31, not that the plot repeats itself twice after 2.5 s so that $T = 2.5/2 = 1.25$ s. Also the plot has 1 m/s^2 as its maximum value. Thus $\omega_n^2 A = 1$ and

$$A = \frac{1}{\omega_n^2}\text{ m/s}^2 = \frac{1\text{ m/s}^2}{\left(\frac{2\pi}{T}\right)^2 \frac{1}{\text{s}^2}} = \left(\frac{T}{2\pi}\right)^2\text{ m} = \left(\frac{1.25}{2\pi}\right)^2\text{ m} = \underline{0.0396\text{ m}}$$

1.32 Resolve Example 1.2.1 using English Engineering Units.

Solution: First change the mass given in kg into slugs using 1 lb-sec^2/ft = 14.5939 kg. So

$$m = 49.2 \times 10^{-3}\text{ kg} \times \frac{1\text{ lb}\cdot\text{sec}^2/\text{ft}}{14.5939\text{ kg}} = 3.37127 \times 10^{-3}\text{lb}\cdot\text{sec}^2/\text{ft}$$

Using the conversions: 1 N = 0.224808 lb and 1 m = 3.280839 ft, the stiffness becomes:

$$k = \frac{857.8\text{ N}}{\text{m}} \times \frac{0.224808\text{ lb}}{\text{N}} \times \frac{1\text{ m}}{3.280839\text{ ft}} = 58.78\frac{\text{lb}}{\text{ft}}$$

Thus the frequency becomes

$$\omega_n = \sqrt{\frac{58.78\text{ lb/ft}}{3.37127\text{ (lb/ft)}\cdot\text{sec}^2}} = 132.05\text{ rad/s}$$

which agrees with the example as it should.

The period is of course the same. To compute the maximum amplitude change mm to in using 1 millimeter = 0.039 370 078 74 inch. So x_0 = 10 mm = 0.39 in and the max amplitude becomes A = 0.39 in and the max acceleration becomes

$$|\ddot{x}(t)| = \omega_n^2 A = (132)^2 \cdot 0.39\text{ in} = 6.8 \times 10^3\text{ in/sec}^2$$

Since the velocity is zero the phase is 90° and the solution is

$$x(t) = 0.39\cos(132t)\text{ in}$$

Which is identical to the solution in mm.

1.33 Referring to Example 1.2.2, determine the length in feet by using the formula for the period as done in the Example.

Solution: From the example the formula for the length of a pendulum is

$$l = \frac{gT^2}{4\pi^2} = \frac{(32.174\text{ ft/s}^2)(3\text{ s})^2}{4\pi^2} = 7.335\text{ ft}$$

As a check 2.237 m = 2.237 m x (3.280 ft/m) = 7.339 ft a little difference in round off.

1.34 Calculate the moon's acceleration due to gravity in ft/s^2.

Solution:

$$g_m = g/6 = (32.174\text{ ft/s}^2)/6 = 5.362\text{ ft/s}^2$$

1.35 A vibrating spring and mass system has a measured acceleration amplitude of 8 mm/s^2 and measured displacement amplitude of 2 mm. Calculate the systems natural frequency.

Solution: The amplitude of displacement is A = 2mm, and that of acceleration is

$$\omega_n^2 A = 8 \Rightarrow \omega_n^2 = 4 \Rightarrow \underline{\omega_n = 2 \text{ rad/s}}$$

1.36 A spring-mass system has measured period of 5 seconds and a known mass of 20 kg. Calculate the spring stiffness.

Solution: Using the basic formulas for period and frequency:

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 5 \text{ s} \Rightarrow k = \left(\frac{2\pi}{5}\right)^2 \times m = \left(\frac{2\pi}{5}\right)^2 20\frac{\text{kg}}{\text{s}^2} = \underline{31.583 \text{ N/m}}$$

1.37* Plot the solution of a linear, spring and mass system with frequency ω_n =1 rad/s, x_0 = 2 mm and v_0 = 2 mm/s, for at least two periods.

Solution: From the formula in Window 1.2, the plot can be formed by computing:

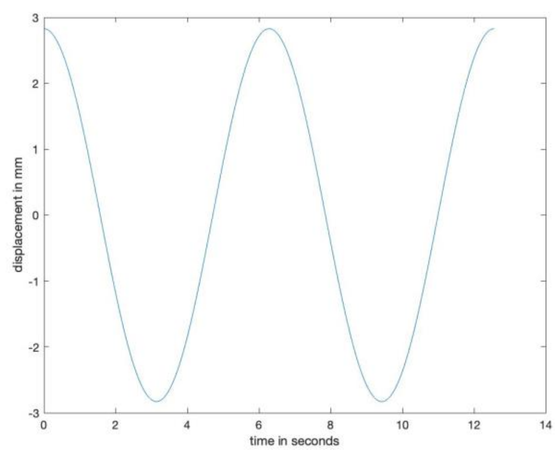
$$A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2} = 2.8284 \text{ mm}, \quad \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = 0.7854 \text{ rad/s}$$

$$x(t) = A\sin(\omega_n t + \phi) = 2.2\sin(t + 0.7854)$$

The period is $2\pi/\omega_n = 2\pi$, so the plot needs to run to 4π.

The solution in Matlab is

```
>> wn=1;x0=2;v0=2;
>> A=sqrt(wn^2*x0^2+v0^2)/wn;
>> p=atan2(wn*x0,v0);
>> figure
>> t=(0:0.01:4*pi);
>> x=A*sin(wn*t+pi/2);
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel('displacement in mm'
>> A
>>A = 2.8284
```

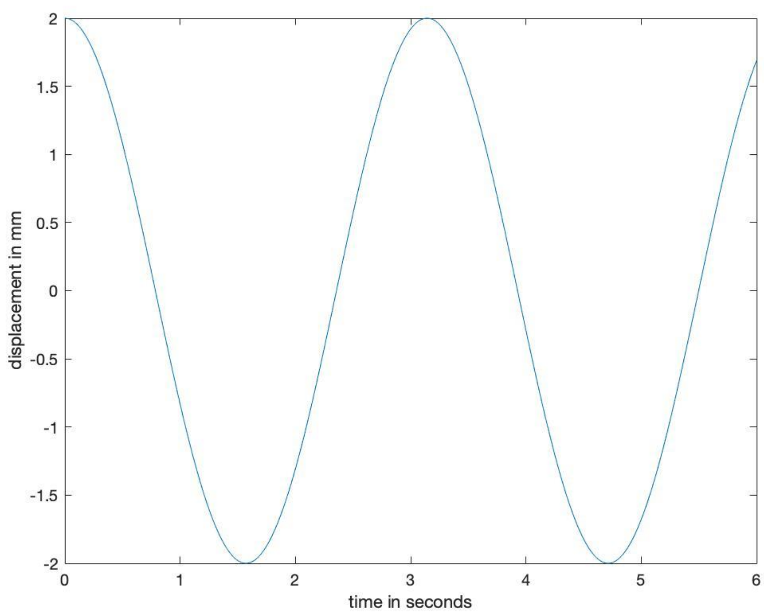


```
>> p
p = 0.7854
```

1.38* Compute the natural frequency and plot the solution of a spring-mass system with mass of 1 kg and stiffness of 4 N/m, and initial conditions of x_0 = 2 mm and v_0 = 0 mm/s, for at least two periods.

Solution: Working entirely in the command window of Matlab, and using the units of mm yields:

```
>> m=1;k=4;x0=2;v0=0;
>> wn=sqrt(k/m)
wn = 2
>> A=(1/wn)*sqrt(wn^2*x0^2+v0^2)
A = 2
>> figure
>> t=[0:0.01:6];
>> x=A*sin(wn*t+pi/2);
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel('displacement in mm')
```



1.39 When designing a linear spring-mass system it is often a matter of choosing a spring constant such that the resulting natural frequency has a specified value. Suppose that the mass of a system is 4 kg and the stiffness is 100 N/m. How much must the spring stiffness be changed in order to increase the natural frequency by 10%?

Solution: Given m = 4 kg and k = 100 N/m the natural frequency is

$$\omega_n = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}$$

Increasing this value by 10% requires the new frequency to be 5 x 1.1 = 5.5 rad/s. Solving for k given m and ω_n yields:

$$5.5 = \sqrt{\frac{k}{4}} \Rightarrow k = (5.5)^2(4) = 121 \text{ N/m}$$

Thus the stiffness k must be increased by about 20%.

1.40 The pendulum in the Chicago Museum of Science and Industry has a length of 20 m and the acceleration due to gravity at that location is known to be 9.803 m/s^2. Calculate the period of this pendulum.

Solution: Following along through Example 1.2.2:

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{g/l}} = \frac{2\pi}{\sqrt{9.803/20}} = 9.975 \text{ s}$$

1.41 Calculate the RMS values of displacement, velocity and acceleration for the undamped single degree of freedom system of equation (1.19) with zero phase.

Solution: Calculate RMS values

Let

$$x(t) = A\sin\omega_n t$$
$$\dot{x}(t) = A\omega_n \cos\omega_n t$$
$$\ddot{x}(t) = -A\omega_n^2 \sin\omega_n t$$

Mean Square Value: $\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\sin^2\omega_n t\,dt = \lim_{T\to\infty} \frac{A^2}{T}\int_0^T (1 - \cos 2\omega_n t)\,dt = \frac{A^2}{2}$$

$$\overline{\dot{x}^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2 w_n^2 \cos^2 w_n t \, dt = \lim_{T\to\infty} \frac{A^2 w_n^{\ 2}}{T}\int_0^T \frac{1}{2}(1 + \cos 2w_n t)\, dt = \frac{A^2 w_n^{\ 2}}{2}$$

$$\overline{\ddot{x}^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2 w_n^4 \sin^2 w_n t \, dt = \lim_{T\to\infty} \frac{A^2 w_n^{\ 4}}{T}\int_0^T \frac{1}{2}(1 + \cos 2w_n t)\, dt = \frac{A^2 w_n^{\ 4}}{2}$$

Therefore,

$$x_{rms} = \sqrt{\overline{x}^2} = \frac{\sqrt{2}}{2} A$$

$$\dot{x}_{rms} = \sqrt{\overline{\dot{x}^2}} = \frac{\sqrt{2}}{2} A w_n$$

$$\ddot{x}_{rms} = \sqrt{\overline{\ddot{x}^2}} = \frac{\sqrt{2}}{2} A w_n^2$$

1.42 Calculate the coefficients A_1 and A_2 of the solution given in Window 1.4 and write down the solution in that form for the case x_0 =1 mm, v_o = 1mm/s and the natural frequency is 10 rad/s.

Solution:

$$x_0 = x(0) = A_1 \sin(0) + A_2 \cos(0) = \underline{A_2 = 1 \text{ mm}}$$

$$v_0 = v(0) = w_n A_1 \cos(0) - A_2 w_n \sin(0) = 0.1 \text{ mm/s}$$

$$\Rightarrow A_1 = (0.1 \text{ mm/s}) / 10 \text{ rad/s} \Rightarrow \underline{A_1 = 0.01 \text{ mm}}$$

$$\Rightarrow \underline{x(t) = \left[0.01\sin(10t) + 1\cos(10t)\right] \text{ mm}}$$

1.43 A foot pedal mechanism for a machine is crudely modeled as a pendulum connected to a spring as illustrated in Figure P1.43. The purpose of the spring is to keep the pedal roughly vertical. Compute the spring stiffness needed to keep the pendulum at 1° from the horizontal and then compute the corresponding natural frequency. Assume that the angular deflections are small, such that the spring deflection can be approximated by the arc length, that the pedal may be treated as a point mass and that pendulum rod has negligible mass. The values in the figure are m = 0.5 kg, g = 9.8 m/s^2, $\ell_1 = 0.2$ m and $\ell_2 = 0.3$ m.

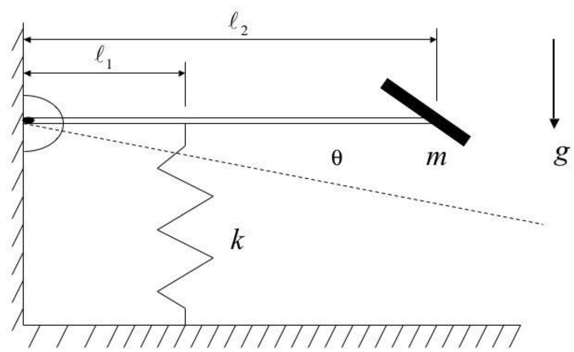
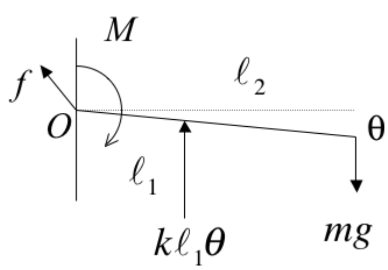


Figure P1.43

Solution: You may want to note to your students, that many systems with springs are often designed based on static deflections, to hold parts in specific positions as in this case, and yet allow some motion. The free-body diagram for the system is given in the figure.



For static equilibrium the sum of moments about point O yields (θ_1 is the static deflection):

$$\sum M_0 = -\ell_1\theta_1(\ell_1)k + mg\ell_2 = 0$$

$$\Rightarrow \ell_1^2\theta_1 k = mg\ell_2$$

$$\Rightarrow k = \frac{mg\ell_2}{\ell_1^2\theta_1} = \frac{0.5\cdot 9.8\cdot 0.3}{(0.2)^2\frac{\pi}{2}} = \underline{2106\text{ N/m}} \qquad (1)$$

Again take moments about point O to get the dynamic equation of motion:

$$\sum M_O = J\ddot{\theta} = m\ell_2^2\ddot{\theta} = -\ell_1^2 k(\theta+\theta_1) + mg\ell_2 = -\ell_1^2 k\theta + \ell_1^2 k\theta_1 - mg\ell_2\theta$$

Next using equation (1) above for the static deflection yields:

$$m\ell_2^2\ddot{q} + \ell_1^2 kq = 0$$

$$\Rightarrow \ddot{q} + \left(\frac{\ell_1^2 k}{m\ell_2^2}\right)q = 0$$

$$\Rightarrow \omega_n = \frac{\ell_1}{\ell_2}\sqrt{\frac{k}{m}} = \frac{0.2}{0.3}\sqrt{\frac{2106}{0.5}} = \underline{43.27 \text{ rad/s}}$$

1.44 An automobile is modeled as a 1000-kg mass supported by a spring of stiffness k = 400,000 N/m. When it oscillates it does so with a maximum deflection of 10 cm. When loaded with passengers, the mass increases to as much as 1300 kg. Calculate the change in frequency, velocity amplitude, and acceleration amplitude if the maximum deflection remains 10 cm.

Solution:

Given: $m_1 = 1000\ kg$

$m_2 = 1300\ kg$

$k = 400{,}000\ N/m$

$x_{max} = A = 10$ cm

$$\omega_{n1} = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{400{,}000}{1000}} = 20\ rad/s$$

$$\omega_{n2} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{400{,}000}{1300}} = 17.54\ rad/s$$

$$\Delta\omega = 17.54 - 20 = -2.46\ rad/s$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \left|\frac{-2.46}{2\pi}\right| = 0.392\ Hz$$

$v_1 = A\omega_{n1} = 10\ cm \times 20\ rad/s = 200\ cm/s$

$v_2 = A\omega_{n2} = 10\ cm \times 17.54$ rad/s $= 175.4\ cm/s$

$\Delta v = 175.4 - 200 = \mathbf{-24.6\ cm/s}$

$a_1 = A\omega_{n1}^2 = 10\ cm \times (20\ rad/s)^2 = 4000\ cm/s^2$

$a_2 = A\omega_{n2}^2 = 10\ cm \times (17.54$ rad/s$)^2 = 3077\ cm/s^2$

$\Delta a = 3077 - 4000 = \mathbf{-923\ cm/s^2}$

1.45 The front suspension of some cars contains a torsion rod as illustrated in Figure P1.45 to improve the car's handling. (a) Compute the frequency of vibration of the wheel assembly given that the torsional stiffness is 2000 N m/rad and the wheel assembly has a mass of 38 kg. Take the distance $x = 0.26$ m.

(b) Sometimes owners put different wheels and tires on a car to enhance the appearance or performance. Suppose a thinner tire is put on with a larger wheel raising the mass to 45 kg. What effect does this have on the frequency?

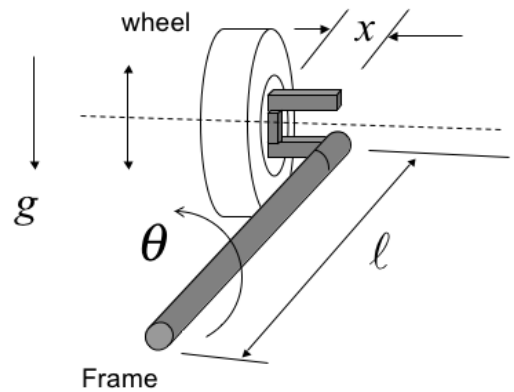


Figure P1.45

Solution: (a) Ignoring the moment of inertial of the rod, and computing the moment of inertia of the wheel as mx^2, the frequency of the shaft mass system is

$$w_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N}\times\text{m}}{38 \times\text{kg}\,(0.26\text{ m})^2}} = \underline{27.9\text{ rad/s}}$$

(b) The same calculation with 45 kg will *reduce* the frequency to

$$w_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N}\times\text{m}}{45 \times\text{kg}\,(0.26\text{ m})^2}} = \underline{25.6\text{ rad/s}}$$

This corresponds to about an 8% change in unsprung frequency and could influence wheel hop etc. You could also ask students to examine the effect of increasing *x*, as commonly done on some trucks to extend the wheels out for appearance sake.

1.46 A machine oscillates in simple harmonic motion and appears to be well modeled by an undamped single-degree-of-freedom oscillation. Its acceleration is measured to have an amplitude of 10,000 mm/s^2 at 7 Hz. What is the machine's maximum displacement?

Solution:

Given: a_{max} = 10,000 mm/s^2 @ 7 *Hz*

The equations of motion for position and acceleration are:

$$x = A\sin(\omega_n t + \phi) \qquad (1.3)$$

$$\ddot{x} = -A\omega_n^2 \sin(\omega_n t + \phi) \qquad (1.5)$$

The amplitude of acceleration is $A\omega_n^2 = 10{,}000$ mm/s^2 and ω_n = 2πf = 2π(7) = 14π rad/s, from equation (1.12).

The machine's displacement is $A = \dfrac{10{,}000}{\omega_n^2} = \dfrac{10{,}000}{(14\pi)^2} = \underline{5.169 \text{ mm}}$

1.47 A simple undamped spring-mass system is set into motion from rest by giving it an initial velocity of 100 mm/s. It oscillates with a maximum amplitude of 15 mm. What is its natural frequency?

Solution:

Given: x_0 = 0, v_0 = 100 mm/s, A = 15 mm

From equation (1.9), $A = \dfrac{v_0}{\omega_n}$ or $\omega_n = \dfrac{100}{15} = 6.667$, so that: **$\omega_n$= 6.667 rad/s.**

1.48 An automobile exhibits a vertical oscillating displacement of maximum amplitude 1 cm and a measured maximum acceleration of 2000 cm/s^2. Assuming that the automobile can be modeled as a single-degree-of-freedom system in the vertical direction, calculate the natural frequency of the automobile.

Solution:

Given: A = 1 cm. From equation (1.15)

$$|\ddot{x}| = A\omega_n^2 = 2000 \text{ cm/s}$$

Solving for ω_n yields:

$$\omega_n = \sqrt{\frac{2000}{1}} = \underline{44.72 \text{ rad/s}}$$

Problems Section 1.3 (Numbers 1.49 through 1.72)

1.49 Consider a spring mass damper system, like the one in Figure 1.10, with the following values: m =10 kg, c = 3 N/s and k = 1000 N/m. a) Is the system overdamped, underdamped or critically damped? b) Compute the solution if the system is given initial conditions x_0 = 0.01 m and v_0 = 0.

Solution: a) Using equation 1.30 the damping ratio is

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{3}{2\sqrt{10 \times 1000}} = 0.015 < 1$$

Thus the system is underdamped.

b) Using equations (1.38) the amplitude and phase can be calculated from the initial conditions:

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}} =$$

$$\frac{1}{9.999}\sqrt{(0.015 \times 10 \times 0.01)^2 + (0.01 \times 9.999)^2} = 0.01 \text{ m}$$

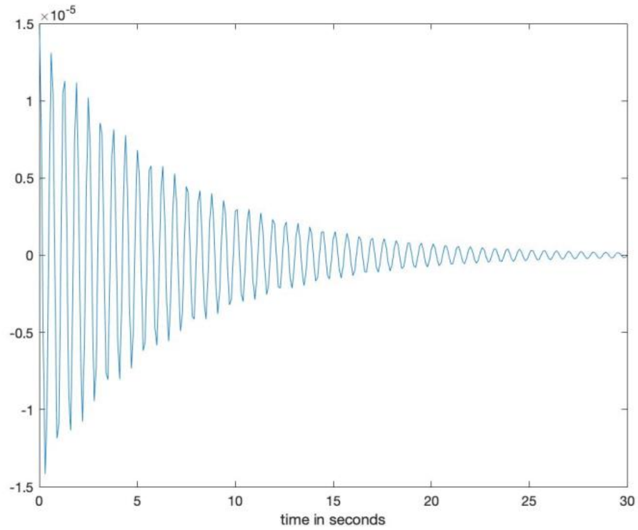
$$\phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 1.556 \text{ rad}$$

So the solution is $Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = \underline{0.01e^{-0.15t}\sin(9.999t + 1.556)}$ m.

Note that for any system with v_0 = 0 the phase is strictly a function of the damping ratio. Also note that the code given below can be used to generate many problems for homework or quizzes by just rearranging the numbers, always making sure to keep the damping low enough to be underdamped. Typing the following in the command window in Matlab:

```
>> m=10;c=3;k=1000;x0=0.01;v0=0.0;
>> wn=sqrt(k/m);
>> z=c/(2*sqrt(k*m));
>> wd=wn*sqrt(1-z^2);
>> A=sqrt(((v0+z*wn*x0)^2*(x0*wd)^2)/wd^2);p=atan2(wd*x0,
(v0+z*wn*x0));
>> t=[0:0.1:30];
>> x=A*exp(-z*wn*t).*sin(wn*t+p);
>> figure
>> plot(t,x)
```

```
>> xlabel('time in seconds')
>> ylabel('displacement in m')
```



1.50 Consider a spring-mass-damper system with equation of motion given by $\ddot{x}(t)+0.2\dot{x}(t)+2x(t)=0$. Compute the damping ratio and determine if the system is overdamped, underdamped or critically damped.

Solution: The parameter values are m = 1, k = 2 and c = 0.2. From equation (1.30) the damping ratio is

$$\zeta=\frac{c}{2\sqrt{km}}=\frac{0.2}{2\sqrt{2\times1}}=0.0707<1$$

Hence the system is underdamped.

1.51 Consider the system $\ddot{x}+4\dot{x}+x=0$ for x_0 = 1 mm, v_0 = 0 mm/s. Is this system overdamped, underdamped or critically damped? Compute the solution and determine which root dominates as time goes on (that is, one root will die out quickly and the other will persist.

Solution: From equation (1.30) the damping ratio is

$$\zeta=\frac{c}{2\sqrt{km}}=\frac{4}{2\sqrt{1\times1}}=2>1$$

Hence the system is overdamped.

Given $\ddot{x}+4\dot{x}+x=0$ where $x_0=1\,\text{mm}, v_0=0$

$$x=ae^{rt}\Rightarrow\dot{x}=are^{rt}\Rightarrow\ddot{x}=ar^2e^{rt}$$

Substitute these into the equation of motion to get:

$$ar^2e^{rt} + 4are^{rt} + ae^{rt} = 0$$
$$\Rightarrow\ r^2 + 4r + 1 = 0 \Rightarrow\ r_{1,2} = -2 \pm \sqrt{3}$$

So

$$x = a_1 e^{\left(-2+\sqrt{3}\right)t} + a_2 e^{\left(-2-\sqrt{3}\right)t}$$
$$\dot{x} = \left(-2+\sqrt{3}\right)a_1 e^{\left(-2+\sqrt{3}\right)t} + \left(-2-\sqrt{3}\right)a_2 e^{\left(-2-\sqrt{3}\right)t}$$

Applying initial conditions yields,

$$x_0 = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$
$$v_0 = \left(-2+\sqrt{3}\right)a_1 + \left(-2-\sqrt{3}\right)a_2 \qquad (2)$$

Substitute equation (1) into (2)

$$v_0 = \left(-2+\sqrt{3}\right)(x_0 - a_2) + \left(-2-\sqrt{3}\right)a_2$$
$$v_0 = \left(-2+\sqrt{3}\right)x_0 - 2\sqrt{3}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

Substituting the value of a_2 into equation (1), and solving for a_1 yields,

$$a_1 = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

$$\therefore\quad x(t) = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}e^{\left(-2+\sqrt{3}\right)t} + \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}e^{\left(-2-\sqrt{3}\right)t}$$

The response is dominated by the root: $\underline{-2+\sqrt{3}}$ as the other root dies off very fast.

1.52 Compute the solution to $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and write down the closed form expression for the response.

Solution:

The parameter values are $m = 1$, $k = 2$ and $c = 2$. From equation (1.30) the damping ratio is

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2}{2\sqrt{2\cdot 1}} = 0.707 < 1$$

Hence the system is underdamped. The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{1}} = \sqrt{2}$$

Thus equations (1.36) and (1.38) can be used directly or one can follow the last expression in Example 1.3.3:

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t + x_0\cos\omega_d t\right)$$

$$= e^{-\frac{1}{\sqrt{2}}\sqrt{2}t}\left(\frac{v_0}{\omega_d}\sin\omega_d t\right)$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{2}\times\sqrt{1-\frac{1}{2}} = 1$$

Thus the solution is

$$\underline{x(t) = e^{-t}\sin t \text{ mm}}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.12.

1.53 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the definition of the damping ratio.

Solution:

Equation (1.28): $\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$

Rewrite, $\lambda_{1,2} = -\left(\frac{c}{2\sqrt{m}\sqrt{m}}\right)\left(\frac{\sqrt{k}}{\sqrt{k}}\right) \pm \frac{1}{2\sqrt{m}\sqrt{m}}\left(\frac{\sqrt{k}}{\sqrt{k}}\right)\left(\frac{c}{c}\right)\sqrt{c^2 - \left(2\sqrt{km}\right)^2\left(\frac{c}{c}\right)^2}$

Rearrange, $\lambda_{1,2} = -\left(\frac{c}{2\sqrt{km}}\right)\left(\frac{\sqrt{k}}{\sqrt{m}}\right) \pm \frac{c}{2\sqrt{km}}\left(\frac{\sqrt{k}}{\sqrt{m}}\right)\left(\frac{1}{c}\right)\sqrt{c^2\left[1 - \left(\frac{2\sqrt{km}}{c}\right)^2\right]}$

Substitute:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{c}{2\sqrt{km}} \Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \zeta\omega_n\left(\frac{1}{c}\right)c\sqrt{1-\left(\frac{1}{\zeta^2}\right)}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2\left[1-\left(\frac{1}{\zeta^2}\right)\right]}$$

$$\Rightarrow \underline{\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

1.54 Use the Euler formulas to derive equation (1.36) from equation (1.35) and to determine the relationships listed in Window 1.4.

Solution:

Equation (1.35): $x(t) = e^{-zw_n t}\left(a_1 e\right)^{jw_n\sqrt{1-z^2}t} - a_2 e^{-jw_n\sqrt{1-z^2}t}$

From Euler,

$$x(t) = e^{-zw_n t}(a_1\cos\left(w_n\sqrt{1-z^2}t\right) + a_1 j\sin\left(w_n\sqrt{1-z^2}t\right)$$
$$+ a_2\cos\left(w_n\sqrt{1-z^2}t\right) - a_2 j\sin\left(w_n\sqrt{1-z^2}t\right))$$
$$= e^{-zw_n t}\left(a_1 + a_2\right)\cos w_d t + j\left(a_1 - a_2\right)\sin w_d t$$

Let: $A_1 = \left(a_1 + a_2\right)$, $A_2 = \left(a_1 - a_2\right)$, then this last expression becomes

$$x(t) = e^{-zw_n t} A_1\cos w_d t + A_2\sin w_d t$$

Next use the trig identity:

$$A = \sqrt{A_1 + A_2}, \quad f = \tan^{-1}\frac{A_1}{A_2}$$

to get: $\underline{x(t) = e^{-zw_n t} A\sin(w_d t + f)}$

1.55 Using equation (1.35) as the form of the solution of the underdamped system, calculate the values for the constants a_1 and a_2 in terms of the initial conditions x_0 and v_0.

Solution:

Equation (1.35):

$$x(t) = e^{-zw_n t}\left(a_1 e^{jw_n\sqrt{1-z^2}t} + a_2 e^{-jw_n\sqrt{1-z^2}t}\right)$$

$$\dot{x}(t) = (-zw_n + jw_n\sqrt{1-z^2})a_1 e^{\left(-zw_n + jw_n\sqrt{1-z^2}\right)t} + (-zw_n - jw_n\sqrt{1-z^2})a_2 e^{\left(-zw_n - jw_n\sqrt{1-z^2}\right)t}$$

Initial conditions

$$x_0 = x(0) = a_1 + a_2 \ \text{Þ}\ a_1 = x_0 - a_2 \qquad (1)$$

$$v_0 = \dot{x}(0) = (-zw_n + jw_n\sqrt{1-z^2})a_1 + (-zw_n - jw_n\sqrt{1-z^2})a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-zw_n + jw_n\sqrt{1-z^2}\right)(x_0 - a_2) + \left(-zw_n - jw_n\sqrt{1-z^2}\right)a_2$$

$$v_0 = \left(-zw_n + jw_n\sqrt{1-z^2}\right)x_0 - 2jw_n\sqrt{1-z^2}\, a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

1.56 Calculate the constants A and ϕ in terms of the initial conditions and thus verify equation (1.38) for the underdamped case.

Solution:

From Equation (1.36),

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$$

Applying initial conditions ($t = 0$) yields,

$$x_0 = A\sin\phi \qquad (1)$$

$$v_0 = \dot{x}_0 = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi \qquad (2)$$

Next solve these two simultaneous equations for the two unknowns A and ϕ.

From (1),

$$A = \frac{x_0}{\sin\phi} \qquad (3)$$

Substituting (3) into (1) yields

$$v_0 = -\zeta\omega_n x_0 + \frac{\omega_d x_0}{\tan\phi} \quad \Rightarrow \quad \tan\phi = \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}.$$

Hence,

$$\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] \qquad (4)$$

From (3), $$\sin\phi = \frac{x_0}{A} \qquad (5)$$

and From (4), $$\cos\phi = \frac{v_0 + \zeta\omega_n x_0}{\sqrt{(x_0\omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2}} \qquad (6)$$

Substituting (5) and (6) into (2) yields,

$$A = \sqrt{\frac{(v_0 + zw_n x_0)^2 + (x_0 w_d)^2}{w_d^2}}$$

which are the same as equation (1.38)

1.57 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equations (1.42) and (1.43) for the overdamped case.

Solution: From Equation (1.41)

$$x(t) = e^{-zw_n t}\left(a_1 e^{w_n \sqrt{z^2-1}\, t} + a_2 e^{-w_n \sqrt{z^2-1}\, t}\right)$$

taking the time derivative yields:

$$\dot{x}(t) = (-zw_n + w_n\sqrt{z^2-1})a_1 e^{\left(-zw_n + w_n\sqrt{z^2-1}\right)t} + (-zw_n - w_n\sqrt{z^2-1})a_2\, e^{\left(-zw_n - w_n\sqrt{z^2-1}\right)t}$$

Applying initial conditions yields,

$$x_0 = x(0) = a_1 + a_2 \quad Þ \quad x_0 - a_2 = a_1 \qquad (1)$$

$$v_0 = \dot{x}(0) = \left(-zw_n + w_n\sqrt{z^2-1}\right)a_1 + \left(-zw_n - w_n\sqrt{z^2-1}\right)a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-zw_n + w_n\sqrt{z^2-1}\right)(x_0 - a_2) + \left(-zw_n - w_n\sqrt{z^2-1}\right)a_2$$

$$v_0 = \left(-zw_n + w_n\sqrt{z^2-1}\right)x_0 - 2w_n\sqrt{z^2-1}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - zw_n x_0 + w_n\sqrt{z^2-1}\, x_0}{2w_n\sqrt{z^2-1}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + zw_n x_0 + w_n\sqrt{z^2-1}\, x_0}{2w_n\sqrt{z^2-1}}$$

1.58 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equation (1.46) for the critically damped case.

Solution:

From Equation (1.45),

$$x(t) = (a_1 + a_2 t)e^{-w_n t}$$

$$Þ\ \dot{x}_0 = -w_n a_1 e^{-w_n t} - w_n a_2 t e^{-w_n t} + a_2 e^{-w_n t}$$

Applying the initial conditions yields:

$$x_0 = a_1 \qquad (1)$$

and

$$v_0 = \dot{x}(0) = a_2 - \omega_n a_1 \qquad (2)$$

solving these two simultaneous equations for the two unknowns a_1 and a_2. Substituting (1) into (2) yields,

$$a_1 = x_0$$

$$a_2 = v_0 + \omega_n x_0$$

which are the same as equation (1.46).

1.59 Using the definition of the damping ratio and the undamped natural frequency, derive equitation (1.48) from (1.47).

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{thus,} \quad \frac{k}{m} = \omega_n^{\,2}$$

$$\zeta = \frac{c}{2\sqrt{km}} \quad \text{thus,} \quad \frac{c}{m} = \frac{2\zeta\sqrt{km}}{m} = 2\zeta\omega_n$$

Therefore, $\ddot{x} + \dfrac{c}{m}\dot{x} + \dfrac{k}{m}x = 0$

becomes,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^{\,2}x(t) = 0$$

1.60 For a damped system, *m*, *c*, and *k* are known to be *m* = 1 kg, *c* = 2 kg/s, *k* = 10 N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

Solution:

Given: *m* = 1 kg, *c* = 2 kg/s, *k* = 10 N/m

Natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10}{1}} = 3.16\ rad/s$

Damping ratio: $\zeta = \dfrac{c}{2\omega_n m} = \dfrac{2}{2(3.16)(1)} = 0.316$

Damped natural frequency: $\omega_d = \sqrt{10}\sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 3.0 \text{ rad/s}$

Since $0 < \zeta < 1$, the system is **underdamped**.

1.61 Plot $x(t)$ for a damped system of natural frequency $\omega_n = 2$ rad/s and initial conditions $x_0 = 1$ mm, $v_0 = 1$ mm, for the following values of the damping ratio: $\zeta = 0.01$, $\zeta = 0.2$, $\zeta = 0.1$, $\zeta = 0.4$, and $\zeta = 0.8$.

Solution:

Given: $\omega_n = 2$ rad/s, $x_0 = 1$ mm, $v_0 = 1$ mm, $\zeta_i = [0.01;\ 0.2;\ 0.1;\ 0.4;\ 0.8]$

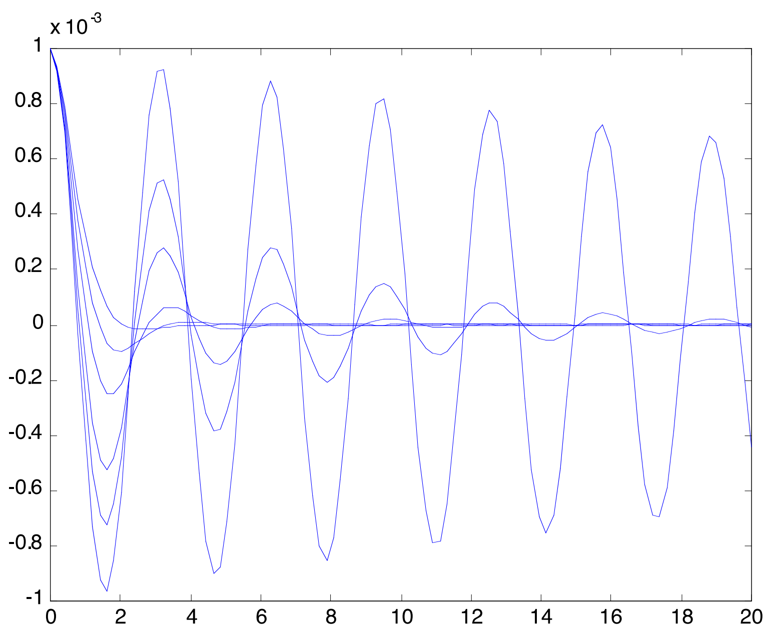
Underdamped cases:

$$\therefore \ \omega_{di} = \omega_n\sqrt{1 - \zeta_i^2}$$

From equation 1.38,

$$A_i = \sqrt{\frac{\left(v_0 + \zeta_i\omega_n x_0\right)^2 + \left(x_0\omega_{di}\right)^2}{\omega_{di}^2}} \qquad \phi_i = \tan^{-1}\frac{x_0\omega_{di}}{v_0 + \zeta_i\omega_n x_0}$$

The response is plotted for each value of the damping ratio in the following using Matlab:



1.62 Plot the response $x(t)$ of an underdamped system with $\omega_n = 2$ rad/s, $\zeta = 0.1$, and $v_0 = 0$ for the following initial displacements: $x_0 = 10$ mm and $x_0 = 100$ mm.

Solution:

Given: $\omega_n = 2$ rad/s, $\zeta = 0.1$, $v_0 = 0$, $x_0 = 10$ mm and $x_0 = 100$ mm.

Underdamped case:

$$\therefore \ \omega_d = \omega_n\sqrt{1-\zeta_i^2} = 2\sqrt{1-0.1^2} = 1.99 \text{ rad/s}$$

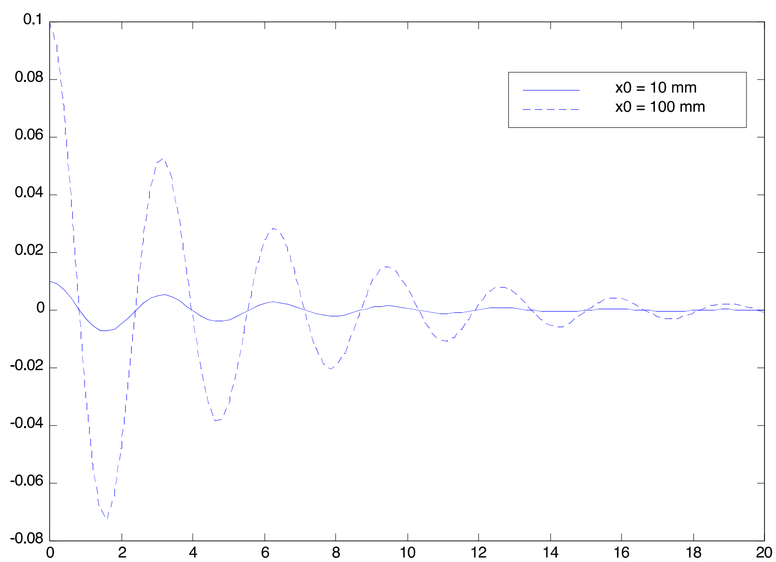
$$A = \sqrt{\frac{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}{\omega_d^{\,2}}} = 1.01\,x_0$$

$$\phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = 1.47 \text{ rad}$$

where

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$$

The following is a plot from Matlab.



1.63 Calculate the solution to $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \quad \text{where } \; j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow \; A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t) =$$

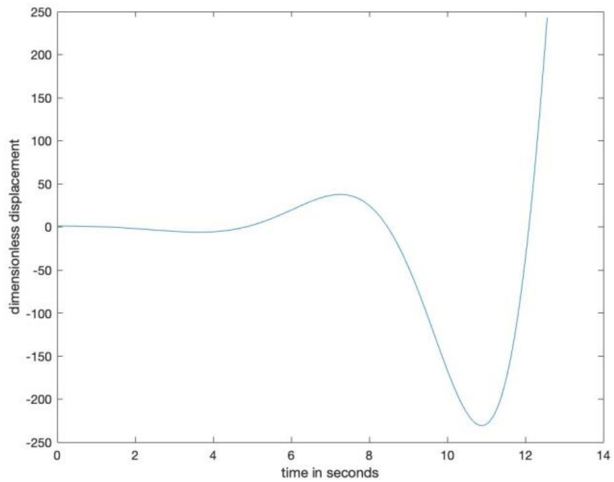
$$\frac{1}{2}e^0(A_1 \cos\frac{\sqrt{3}}{2}0 + A_2 \sin\frac{\sqrt{3}}{2}0) + e^0\frac{\sqrt{3}}{2}(-A_1 \sin\frac{\sqrt{3}}{2}0 + A_2 \cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow \; A_1 + \sqrt{3}(A_2) = 0 \Rightarrow \; A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow \; x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. Working in the command window of Matlab:

```
>> t=(0:0.01:4*pi);
>> x=(exp(0.5*t)).*(cos(t*sqrt(3)/2)–(1/sqrt(3))*sin(t*sqrt(3)/2));
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel('dimensionless displacement')
```



1.64 A spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000 \text{ N/m}}{100 \text{ kg}}} = 5.477 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{300}{2\sqrt{km}} = \frac{300}{2\sqrt{(3000)(100)}} = 0.274$$

Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 5.27 \text{ rad/s}$.

1.65 A sketch of a valve and rocker arm system for an internal combustion engine is give in Figure P1.65. Model the system as a pendulum attached to a spring and a mass and assume the oil provides viscous damping in the range of $\zeta = 0.01$. Determine the equations of motion and calculate an expression for the natural frequency and the damped natural frequency. Here *J* is the rotational inertia of the rocker arm about its pivot point, *k* is the stiffness of the valve spring and *m* is the mass of the valve and stem. Ignore the mass of the spring.

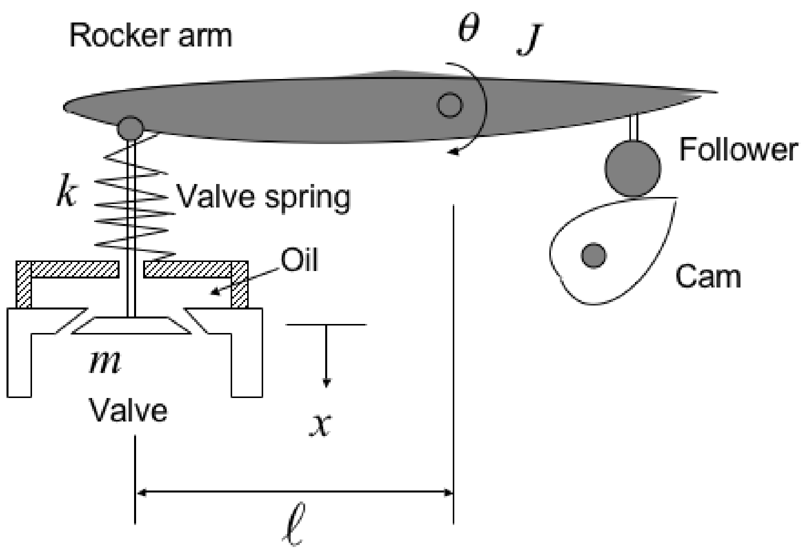
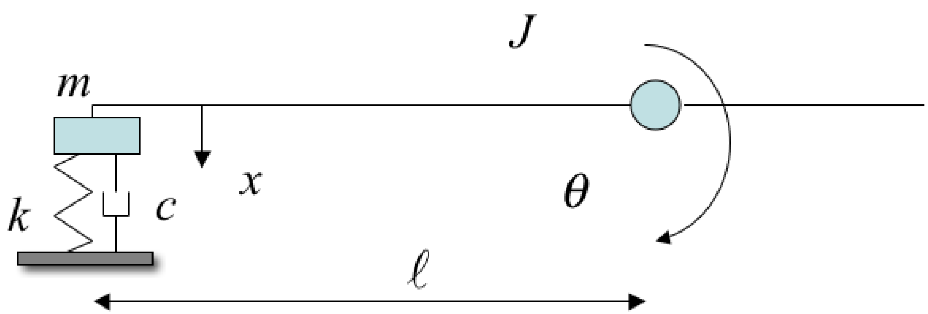


Figure P1.65

Solution: The model is of the form given in the figure. You may wish to give this figure as a hint as it may not be obvious to all students.



Taking moments about the pivot point yields:

$$(J + m\ell^2)\ddot{\theta}(t) = -kx\ell - c\dot{x}\ell = -k\ell^2\theta - c\ell^2\dot{\theta}$$

$$\Rightarrow \underline{(J + m\ell^2)\ddot{\theta}(t) + c\ell^2\dot{\theta} + k\ell^2\theta = 0}$$

Next divide by the leading coefficient to get;

$$\ddot{\theta}(t) + \left(\frac{c\ell^2}{J + m\ell^2}\right)\dot{\theta}(t) + \frac{k\ell^2}{J + m\ell^2}\theta(t) = 0$$

From the coefficient of θ, the undamped natural frequency is

$$\underline{\omega_n = \sqrt{\frac{k\ell^2}{J + m\ell^2}} \text{ rad/s}}$$

From equation (1.37), the damped natural frequency becomes

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \underline{0.99995\sqrt{\frac{k\ell^2}{J+m\ell^2}}} \sim \sqrt{\frac{k\ell^2}{J+m\ell^2}}$$

This is effectively the same as the undamped frequency for any reasonable accuracy. However, it is important to point out that the resulting response will still decay, even though the frequency of oscillation is unchanged. So even though the numerical value seems to have a negligible effect on the frequency of oscillation, the small value of damping still makes a substantial difference in the response.

1.66 A spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{150 \text{ kg}}} = 3.162 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{200}{2\sqrt{km}} = \frac{200}{2\sqrt{(1500)(150)}} = 0.211$$

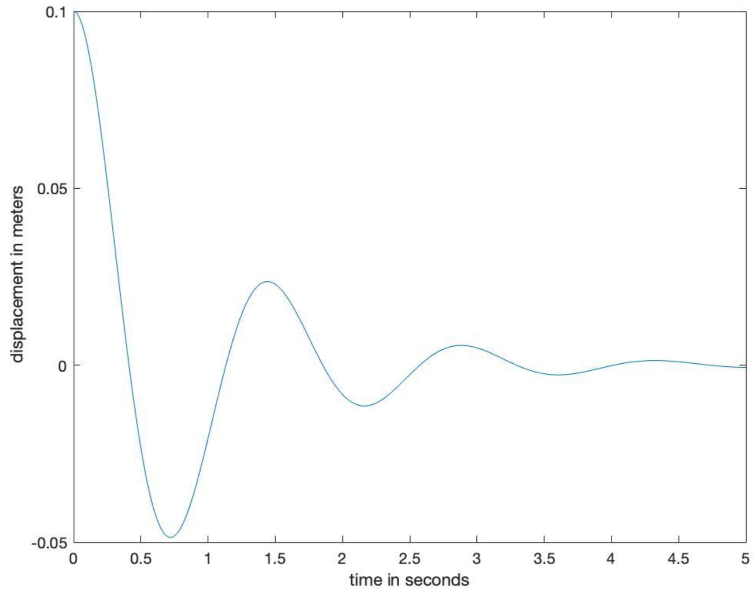
This last expression follows from the equation following equation (1.29). Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.091$ rad/s, which follows from equation (1.37).

1.67* The spring mass system of 150 kg mass, stiffness of 3000 N/m and damping coefficient of 300 Ns/m is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the form of the response and plot it for as long as it takes to die out.

Solution: Working from equation (1.38) and working in Matlab's command window:

```
>>m=150;c=300;k=3000;wn=sqrt(k/m);z=(1/2)*c/sqrt(m*k);
>> wd=wn*sqrt(1-z^2);v0=0;x0=0.1;
>> A=(1/wd)*sqrt((v0+z*wn*x0)^2+(x0*wd)^2);
```

```
>> t=(0:0.01:5);
>> p=atan2(wd*x0,v0+z*wn*x0);
>> x=A*sin(wd*t+p).*exp(-z*wn*t);
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel(' displacement in meters')
```



1.68* The spring mass system of 100 kg mass, stiffness of 1500 N/m and damping coefficient of 200 Ns/m is given an initial velocity of 10 mm/s and an initial displacement of –5 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Referring to Example 1.3.5, the time it takes to die out is defined by the settling time

$$T_s = \frac{4}{\zeta\omega_m} \ .$$

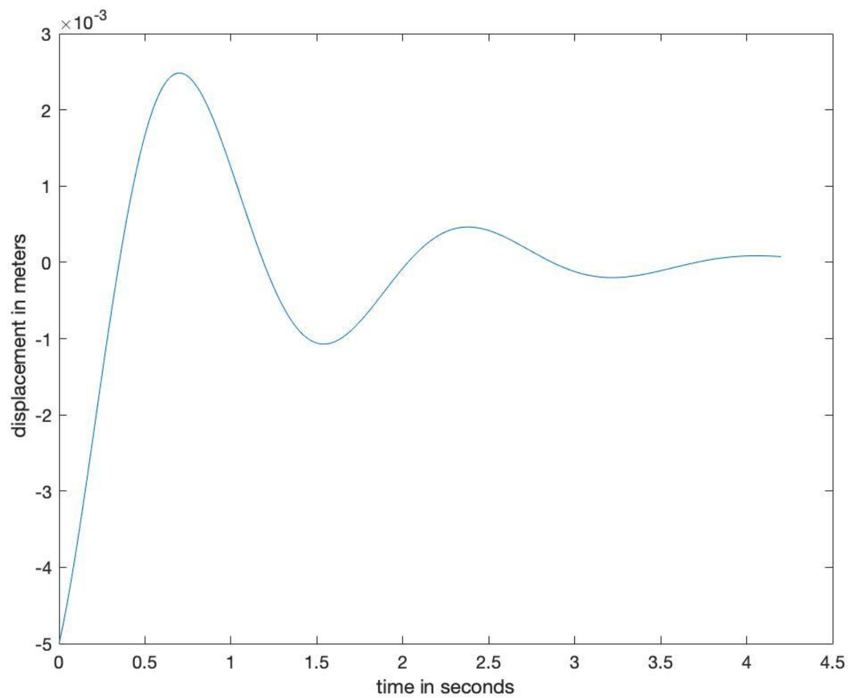
First compute ω_n and ζ:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{100 \text{ kg}}} = 3.873 \text{ rad/s}, \ \zeta = \frac{c}{2\sqrt{mk}} = 0.258$$

$$T_s = \frac{4}{\zeta\omega_d} = \frac{4}{0.258 \times 3.873} = 4.003 \text{ sec}$$

Thus in plotting the response time must run past 4.003 sec, say 4.2 sec. Now working from equation (1.38), the form of the response is programmed in the command window of Matlab:

```
>> m=100;k=1500;c=200;wn=sqrt(k/m);z=(1/2)*c/sqrt(m*k);
>> wd=wn*sqrt(1-z^2);v0=0.01;x0=-0.005;
>> A=(1/wd)*sqrt((v0+z*wn*x0)^2+(x0*wd)^2);
>> p=atan2(wd*x0,v0+z*wn*x0);
>> t=(0:0.01:4.2);
>> x=A*sin(wd*t+p).*exp(-z*wn*t);
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel(' displacement in meters')
```



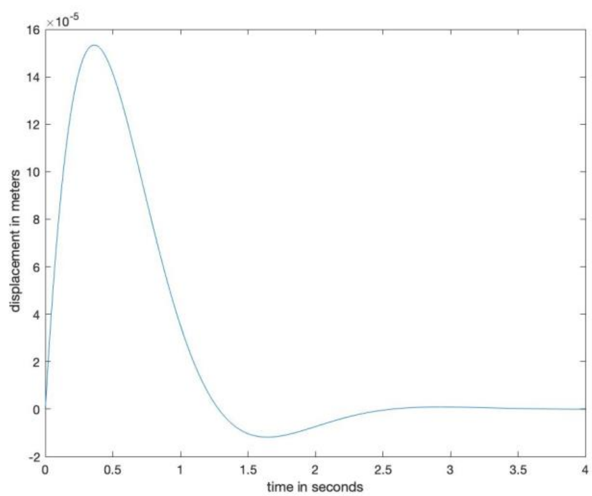
1.69 Choose the damping coefficient of a spring-mass-damper system with mass of 200 kg and stiffness of 2000 N/m such that it's response will die out after 2 s, given a zero initial position and an initial velocity of 0.001 m/s. Plot the solution to see if our answer is reasonable.

Solution: Working with the settling time as given in Example 1.3.5 the settling time is given as

$$T_s = 2 = \frac{4}{\zeta\omega_n} \text{ so that } \zeta = \frac{2}{\omega_n} = \frac{c}{2m\omega_n} \Rightarrow c = 4m = 4(200) = 800 \text{ Ns/m}$$

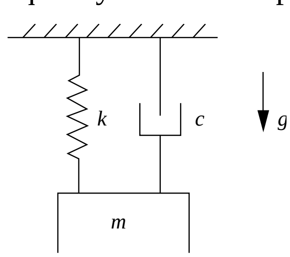
where equation (1.30) is used to remove ζ in favor of *c*. As a check plot this in Matlab

```
>> wd=wn*sqrt(1-z^2);v0=0.001;x0=0;
>> A=(1/wd)*sqrt((v0+z*wn*x0)^2+(x0*wd)^2);
>> p=atan2(wd*x0,v0+z*wn*x0);
>> t=(0:0.01:4);
>> x=A*sin(wd*t+p).*exp(-z*wn*t);
>> plot(t, x)
>> xlabel('time in seconds')
>> ylabel(' displacement in meters')
```

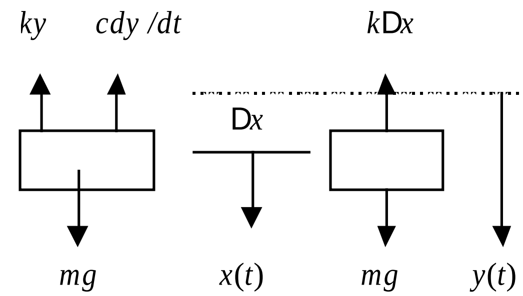


The plot clearly gets within 2% of zero.

1.70 Derive the equation of motion of the system in Figure P1.70 and discuss the effect of gravity on the natural frequency and the damping ratio.



Solution: This requires two free body diagrams. One for the dynamic case and one to show static equilibrium.



(a) (b)

From the free-body diagram of static equilibrium (b) we have that $mg = k\Delta x$, where Δx represents the static deflection. From the free-body diagram of the dynamic case given in (a) the equation of motion is:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) - mg = 0$$

From the diagram, $y(t) = x(t) + \Delta x$. Since Δx is a constant, differentiating and substitution into the equation of motion yields:

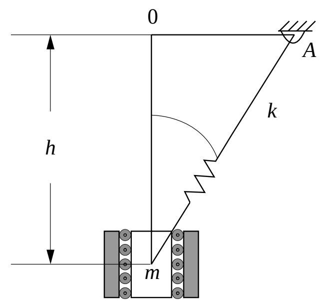
$$\dot{y}(t) = \dot{x}(t) \text{ and } \ddot{y}(t) = \ddot{x}(t) \Rightarrow$$
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \underbrace{(k\Delta x - mg)}_{=0} = 0$$

where the last term is zero from the relation resulting from static equilibrium. Dividing by the mass yields the standard form

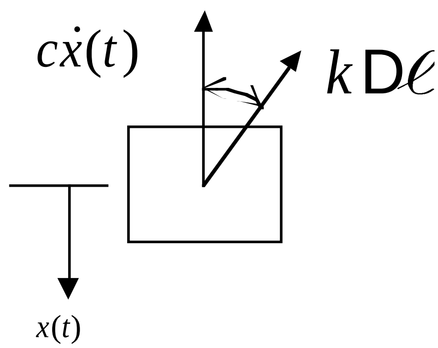
$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

It is clear that gravity has no effect on the damping ratio ζ or the natural frequency ω_n. Not that the damping force is not present in the static case because the velocity is zero.

1.71 Derive the equation of motion of the system in Figure P1.71 and discuss the effect of gravity on the natural frequency and the damping ratio. You may have to make some approximations of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. (From the A. Diaz-Jimenez, *South African Mechanical Engineer*, Vol. 26, pp. 65-69, 1976)



Solution: First consider a free-body diagram of the system:



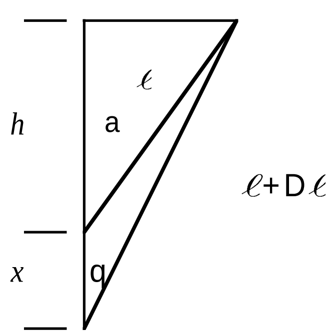
Let α be the angel between the damping and stiffness force. The equation of motion becomes

$$m\ddot{x}(t) = -c\dot{x}(t) - k(\Delta\ell + \delta_s)\cos\alpha$$

From static equilibrium, the free-body diagram (above with $c = 0$ and stiffness force $k\delta_s$) yields: $\sum F_x = 0 = mg - k\delta_s\cos\alpha$. Thus the equation of motion becomes

$$m\ddot{x} + c\dot{x} + k\Delta\ell\cos\alpha = 0 \qquad (1)$$

Next consider the geometry of the dynamic state:



From the definition of cosine applied to the two different triangles:

$$\cos\alpha = \frac{h}{\ell} \quad \text{and} \quad \cos\theta = \frac{h+x}{\ell+\Delta\ell}$$

Next assume small deflections so that the angles are nearly the same cos α = cos θ, so that

$$\frac{h}{\ell} \approx \frac{h+x}{\ell+\Delta\ell} \Rightarrow \Delta\ell \approx x\frac{\ell}{h} \Rightarrow \Delta\ell \approx \frac{x}{\cos\alpha}$$

For small motion, then this last expression can be substituted into the equation of motion (1) above to yield:

$$m\ddot{x} + c\dot{x} + kx = 0, \ \alpha \text{ and } x \text{ small}$$

Thus the frequency and damping ratio have the standard values and are not effected by gravity. If the small angle assumption is not made, the frequency can be approximated as

$$\omega_n = \sqrt{\frac{k}{m}\cos^2\alpha + \frac{g}{h}\sin^2\alpha}, \quad \zeta = \frac{c}{2m\omega_n}$$

as detailed in the reference above. For a small angle these reduce to the normal values of

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

as derived here.

1.72 An Embraer ERJ-145 has a mass of 12,007 kg when empty. The three landing gear suspension systems share the load evenly. When loaded with 4000 kg the suspension system pictured deflects 0.2 m. What value of viscous damping in the suspension system would cause the system to be critically damped?

Solution: First calculate the undamped natural frequency from the static deflection by realizing that the slop of the curve in Figure 1.4 can be used to determine the ratio of $k = mg/\delta$ where δ is the static deflection. Then

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{\delta}\frac{1}{m}} = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81 \text{ m/s}^2}{0.2 \text{ m}}} = 7.004 \text{ rad/s}$$

From equation (1.30) with ζ = 1,

$$c = 2\zeta m\omega_n = 2(1)(4000 \text{ kg})(7.004 \text{ rad/s}) = 56{,}032 \text{ kg/s (N}\cdot\text{s/m)}$$

Problems and Solutions Section 1.4 (problems 1.73 through)

1.73 Calculate the frequency of the compound pendulum of Figure P1.73 if a mass m_T is added to the tip, by using the energy method. Assume the mass of the pendulum is evenly distributed so that its center of gravity is in the middle of the pendulum of length *l*.

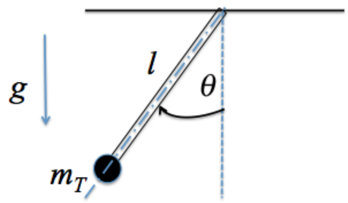


Figure P1.73 A compound pendulum with a tip mass.

Solution Adding a tip mass adds both kinetic and potential energy to the system.

If the mass of the pendulum bar is *m*, and it is lumped at the center of mass the energies become:

Potential Energy:

$$U = \frac{1}{2}(\ell - \ell\cos\theta)mg + (\ell - \ell\cos\theta)m_t g$$
$$= \frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g)$$

Kinetic Energy:

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_t\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}m_t\ell^2\dot{\theta}^2$$
$$= (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2$$

Conservation of energy (Equation 1.51) requires $T + U =$ constant:

$$\frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g) + (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2 = C$$

Differentiating with respect to time yields:

$$\frac{\ell}{2}(\sin\theta)(mg + 2m_t g)\dot{\theta} + (\frac{1}{3}m + m_t)\ell^2\dot{\theta}\ddot{\theta} = 0$$
$$\Rightarrow \ (\frac{1}{3}m + m_t)\ell\ddot{\theta} + \frac{1}{2}(mg + 2m_t g)\sin\theta = 0$$

Rearranging and approximating using the small angle formula sin $\theta \sim \theta$, yields:

$$\ddot{\theta}(t) + \left(\frac{\frac{m}{2} + m_t}{\frac{1}{3}m + m_t} \frac{g}{\ell} \right) \theta(t) = 0 \Rightarrow \underline{\omega_n = \sqrt{\frac{3m + 6m_t}{2m + 6m_t}} \sqrt{\frac{g}{\ell}} \text{ rad/s}}$$

Note that this solution makes sense because if $m_t = 0$ it reduces to the frequency of the pendulum equation for a bar, and if $m = 0$ it reduces to the frequency of a massless pendulum with only a tip mass.

1.74 Calculate the total energy in a damped system with frequency 2 rad/s and damping ratio $\zeta = 0.01$ with mass 10 kg for the case $x_0 = 0.1$ m and $v_0 = 0$. Plot the total energy versus time.

Solution: Given: $\omega_n = 2$ rad/s, $\zeta = 0.01$, $m = 10$ kg, $x_0 = 0.1$ m, $v_0 = 0$.

Calculate the stiffness and damped natural frequency:

$$k = m\omega_n^2 = 10(2)^2 = 40 \text{ N/m}$$
$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 2\sqrt{1 - 0.01^2} = 2 \text{ rad/s}$$

The total energy of the damped system is

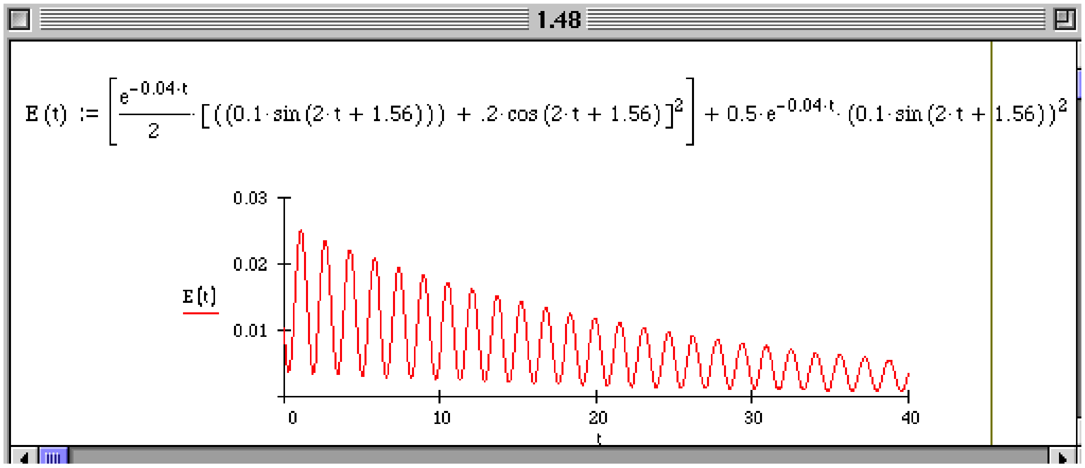
$$E(t) = \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx(t)$$

where
$$x(t) = Ae^{-0.02t}\sin(2t + \phi)$$
$$\dot{x}(t) = -0.02Ae^{-0.02t}\sin(2t + \phi) + 2Ae^{-0.02t}\cos(2t + \phi)$$

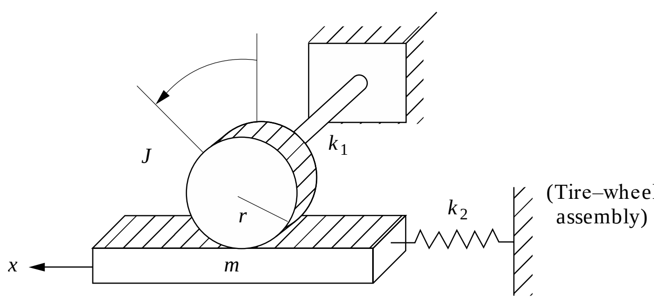
Applying the initial conditions to evaluate the constants of integration yields:

$$x(0) = 0.1 = A\sin\phi$$
$$\dot{x}(0) = 0 = -0.02A\sin\phi + 2A\cos\phi$$
$$\Rightarrow \phi = 1.57 \text{ rad}, \quad A = 0.1 \text{ m}$$

Substitution of these values into $E(t)$ yields:



1.75 Use the energy method to calculate the equation of motion and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in Figure P1.75.



The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_2. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution: From kinematics: $x = r\theta, \Rightarrow \dot{x} = r\dot{\theta}$

Kinetic energy: $$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

Potential energy: $U = \frac{1}{2}k_2 x^2 + \frac{1}{2}k_1 q^2$

Substitute $q = \frac{x}{r}$: $T + U = \frac{1}{2}\frac{J}{r^2}\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_2 x^2 + \frac{1}{2}\frac{k_1}{r^2}x^2$

Derivative: $\frac{d(T+U)}{dt} = 0$

$$\frac{J}{r^2}\ddot{x}\dot{x} + m\ddot{x}\dot{x} + k_2 x\dot{x} + \frac{k_1}{r^2}x\dot{x} = 0$$

$$\left[\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x\right]\dot{x} = 0$$

Equation of motion: $\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x = 0$

Natural frequency: $w_n = \sqrt{\frac{k_2 + \frac{k_1}{r^2}}{\frac{J}{r^2} + m}} = \sqrt{\frac{k_1 + r^2 k_2}{J + mr^2}}$

1.76 Consider the pendulum and spring system of Figure P1.76. Here the mass of the pendulum rod is negligible. Derive the equation of motion using the energy method. Then linearize the system for small angles and determine the natural frequency. The length of the pendulum is *l*, the tip mass is *m*, and the spring stiffness is *k*.

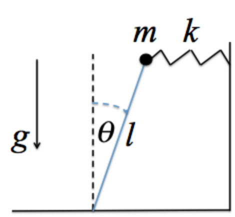


Figure P1.76 A simple pendulum connected to a spring

Solution: Writing down the kinetic and potential energy yields:

$$T = \frac{1}{2}ml^2\dot{q}^2, \quad U = \frac{1}{2}kx^2 + mgh$$

$$U = \frac{1}{2}kl^2\sin^2 q + mgl(1 - \cos q)$$

Here the soring deflects a distance $l\sin\theta$, and the mass drops a distance $l(1 - \cos\theta)$. Adding up the total energy and taking its time derivative yields:

$$\frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}kl^2\sin^2\theta + mgl\cos\theta\right)$$
$$= (ml^2\ddot{\theta})\dot{\theta} + (kl^2\sin\theta\cos\theta)\dot{\theta} - mgl\sin\theta\dot{\theta} = 0$$
$$\Rightarrow ml^2\ddot{\theta} + kl^2\sin\theta\cos\theta - mgl\sin\theta = 0$$

For small θ, this becomes

$$ml^2\ddot{\theta} + kl^2\theta - mgl\theta = 0$$
$$\Rightarrow \underline{\ddot{\theta} + \frac{kl - mg}{ml}\theta = 0}$$
$$\Rightarrow \underline{\omega_n = \sqrt{\frac{kl - mg}{ml}}\ \text{rad/s}}$$

1.77 Consider the pendulum of Figure 1.22 in Example 1.4.8. Repeat the solution given there only this time linearize the energy by assuming small θ before writing down the Lagrange equation and calculate the frequency. Compare your answer to that in the example.

Solution: From the example the potential energy is

$$U = \frac{1}{2}l^2k\sin\theta + mgl(1 - \cos\theta)$$

For small angles this becomes

$$U = \frac{1}{2}l^2k\theta + mgl(1 - 1) = \frac{1}{2}l^2k\theta$$

Thus the Lagrangian is

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}l^2k\theta$$

Substitution into the Lagrange formulation yields

$$ml^2\ddot{\theta} + kl^2\theta = 0 \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Which is very different from the frequency found in Example 1.4.8 and misses the effect of the pendulums inertia and that of gravity. Thus it is important not to linearize too early when using the Lagrangian method.

1.78 A control pedal of an aircraft can be modeled as the single-degree-of-freedom system of Figure P1.78. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at θ = 0.

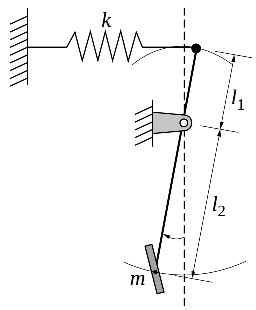


Figure P1.78

Solution: In the figure let the mass at θ = 0 be the lowest point for potential energy. Then, the height of the mass *m* is (1-cos θ)ℓ_2.

Kinematic relation: $x = \ x = \ell_1\theta$

Kinetic Energy: $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\ell_2^2\dot{q}^2$

Potential Energy: $U = \frac{1}{2}k(\ell_1 q)^2 + mg\ell_2(1 - \cos q)$

Taking the derivative of the total energy yields:

$$\frac{d}{dt}(T+U) = m\ell_2^2\dot{q}\ddot{q} + k(\ell_1^2 q)\dot{q} + mg\ell_2(\sin q)\dot{q} = 0$$

Rearranging, dividing by $d\theta/dt$ and approximating sinθ with θ yields: $m\ell_2^2\ddot{\theta} + (k\ell_1^2 + mg\ell_2)\theta = 0$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell_2}{m\ell_2^2}}$$

1.79 To save space, two large pipes are shipped one stacked inside the other as indicated in Figure P1.79. Calculate the natural frequency of vibration of the smaller pipe (of radius R_1) rolling back and forth inside the larger pipe (of radius R). Use the energy method and assume that the inside pipe rolls without slipping and has a mass m.

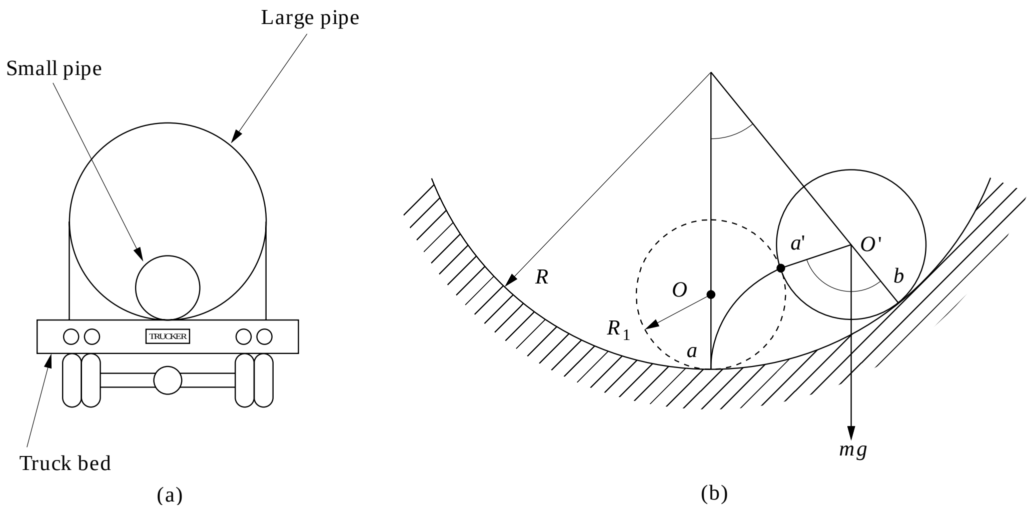
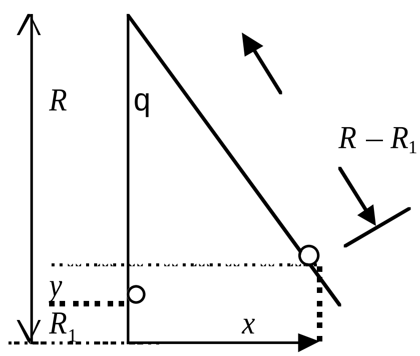


Figure P1.79

Solution: Let θ be the angle that the line between the centers of the large pipe and the small pipe make with the vertical and let θ be the angle that the small pipe rotates through. Let R be the radius of the large pipe and R_1 the radius of the smaller pipe. Then the kinetic energy of the system is the translational plus rotational of the small pipe. The potential energy is that of the rise in height of the center of mass of the small pipe.



From the drawing:

$$y + (R - R_1)\cos\theta + R_1 = R$$

$$\Rightarrow \; y = (R - R_1)(1 - \cos\theta)$$

$$\Rightarrow \; \dot{y} = (R - R_1)\sin(\theta)\dot{\theta}$$

Likewise examination of the value of x yields:

$$x = (R - R_1)\sin\theta$$

$$\Rightarrow \; \dot{x} = (R - R_1)\cos\theta\dot{\theta}$$

Let β denote the angle of rotation that the small pipe experiences as viewed in the inertial frame of reference (taken to be the truck bed in this case). Then the total kinetic energy can be written as:

$$T = T_{trans} + T_{rot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$= \frac{1}{2}m(R - R_1)^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$\Rightarrow \; T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

The total potential energy becomes just:

$$V = mgy = mg(R - R_1)(1 - \cos\theta)$$

Now it remains to evaluate the angle β Let α be the angle that the small pipe rotates in the frame of the big pipe as it rolls (say) up the inside of the larger pipe. Then

$\beta = \theta - \alpha$

were α is the angle "rolled" out as the small pipe rolls from *a* to *b* in figure P1.56. The rolling with out slipping condition implies that arc length *a'b* must equal arc length *ab*. Using the arc length relation this yields that $R\theta = R_1\alpha$. Substitution of the expression $\beta = \theta - \alpha$ yields:

$$R\theta = R_1(\theta - \beta) = R_1\theta - R_1\beta \Rightarrow (R - R_1)\theta = -R_1\beta$$

$$\Rightarrow \; \beta = \frac{1}{R_1}(R_1 - R)\theta \text{ and } \dot{\beta} = \frac{1}{R_1}(R_1 - R)\dot{\theta}$$

which is the relationship between angular motion of the small pipe relative to the ground (β) and the position of the pipe (θ). Substitution of this last expression into the kinetic energy term yields:

$$T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\left(\frac{1}{R_1}(R_1 - R)\dot{\theta}\right)^2$$

$$\Rightarrow \quad T = m(R - R_1)^2\dot{\theta}^2$$

Taking the derivative of $T + V$ yields

$$\frac{d}{d\theta}(T + V) = 2m(R - R_1)^2\dot{\theta}\ddot{\theta} + mg(R - R_1)\sin\theta\dot{\theta} = 0$$

$$\Rightarrow \; 2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\sin\theta = 0$$

Using the small angle approximation for sine this becomes

$$2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\theta = 0$$

$$\Rightarrow \; \ddot{\theta} + \frac{g}{2(R - R_1)}\theta = 0$$

$$\Rightarrow \; \omega_n = \sqrt{\frac{g}{2(R - R_1)}}$$

1.80 Consider the example of a simple pendulum given in Example 1.4.2. The pendulum motion is observed to decay with a damping ratio of $\zeta = 0.001$. Determine a damping coefficient and add a viscous damping term to the pendulum equation.

Solution: From example 1.4.2, the equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

So $\omega_n = \sqrt{\frac{g}{\ell}}$. With viscous damping the equation of motion in normalized form becomes:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \text{ or with } \zeta \text{ as given :}$$

$$\Rightarrow \; \ddot{\theta} + 2(.001)\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

The coefficient of the velocity term is

$$\frac{c}{J} = \frac{c}{m\ell^2} = (.002)\sqrt{\frac{g}{\ell}}$$

$$c = (0.002)m\sqrt{g\ell^3}$$

1.81 Determine a damping coefficient for the disk-rod system of Example 1.4.3. Assuming that the damping is due to the material properties of the rod, determine c for the rod if it is observed to have a damping ratio of $\zeta = 0.01$.

Solution: The equation of motion for a disc/rod in torsional vibration is

$$J\ddot{\theta} + k\theta = 0$$

or
$$\ddot{\theta} + \omega_n^2\theta = 0 \quad \text{where } \omega_n = \sqrt{\frac{k}{J}}$$

Add viscous damping:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$
$$\ddot{\theta} + 2(.01)\sqrt{\frac{k}{J}}\dot{\theta} + \omega_n^2\theta = 0$$

From the velocity term, the damping coefficient must be

$$\frac{c}{J} = (0.02)\sqrt{\frac{k}{J}}$$
$$\Rightarrow \ c = 0.02\sqrt{kJ}$$

1.82 The rod and disk of Window 1.1 are in torsional vibration. Calculate the damped natural frequency if $J = 1000\ \text{m}^2 \cdot \text{kg}$, $c = 20\ \text{N}\cdot\text{m}\cdot\text{s/rad}$, and $k = 400\ \text{N}\cdot\text{m/rad}$.

Solution: The equation of motion is

$$J\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

where
$$\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{400}{1000}} = 0.632 \text{ rad/s}$$

and
$$\zeta = \frac{c}{2\sqrt{kJ}} = \frac{20}{2\sqrt{400 \times 1000}} = 0.0158$$

Thus the damped natural frequency is $\omega_d = 0.632$ rad/s

1.83 Consider the system of P1.83, which represents a simple model of an aircraft landing system. Assume, $x = r\theta$. What is the damped natural frequency?

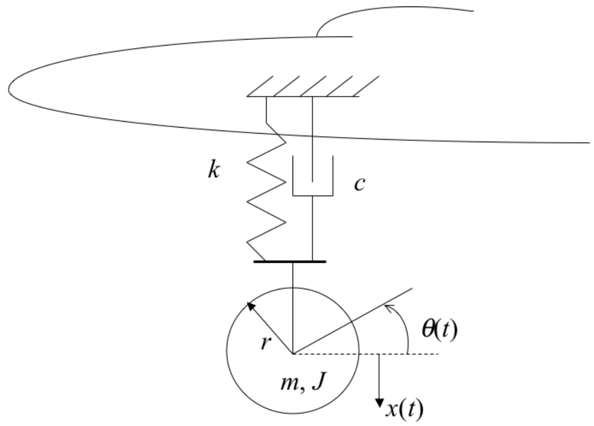


Figure P1.83

Solution: Ignoring the damping and using the energy method the equation of motion is

$$T = \frac{1}{2}J\dot{q}^2 + \frac{1}{2}m\dot{x}^2, \quad U = \frac{1}{2}kx^2, \quad q = \frac{x}{r}$$

$$\frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2}J\frac{\dot{x}^2}{r^2} + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right)$$

$$\square \; \frac{J}{r^2}\dot{x}\ddot{x} + m\dot{x}\ddot{x} + kx\dot{x}$$

Thus the undamped equation of motion is:

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + kx = 0$$

From examining the equation of motion the natural frequency is:

$$w_n = \sqrt{\frac{k}{m_{eq}}} = \sqrt{\frac{k}{m + \frac{J}{r^2}}}$$

An add hoc way do to this is to add the damping force to get the damped equation of motion:

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + c\dot{x} + kx = 0$$

The value of ζ is determined by examining the velocity term:

$$\frac{c}{m+\frac{J}{r^2}} = 2\zeta\omega_n \Rightarrow \zeta = \frac{c}{m+\frac{J}{r^2}}\frac{1}{2\sqrt{\frac{k}{m+\frac{J}{r^2}}}}$$

$$\Rightarrow \zeta = \frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}$$

Thus the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m+\frac{J}{r^2}}}\sqrt{1-\left(\frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}\right)^2}$$

$$\Rightarrow \omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}} = \frac{r}{2(mr^2+J)}\sqrt{4(kmr^2+kJ)-c^2r^2}$$

1.84 Consider Problem 1.83 with k = 400,000 N/m, m = 1500 kg, J = 100 m^2·kg/rad, r = 25 cm, and c = 8000 kg/s. Calculate the damping ratio and the damped natural frequency. How much effect does the rotational inertia have on the undamped natural frequency?

Solution: From problem 1.74:

$$\zeta = \frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}} \text{ and } \omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}$$

Given:

$$k = 4\times 10^5 \text{ N/m}$$
$$m = 1.5\times 10^3 \text{ kg}$$
$$J = 100 \text{ m}^2\text{kg/rad}$$
$$r = 0.25 \text{ m and}$$
$$c = 8\times 10^3 \text{ N}\cdot\text{s/m}$$

Inserting the given values yields

$$\underline{\zeta = 0.114} \text{ and } \underline{\omega_d = 11.29 \text{ rad/s}}$$

For the undamped natural frequency, $\omega_n = \sqrt{\dfrac{k}{m + J/r^2}}$

With the rotational inertia, $\omega_n = 11.36$ rad/s

Without rotational inertia, $\omega_n = 16.33$ rad/s

The effect of the rotational inertia is that it lowers the natural frequency by almost 19%.

1.85 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.85. Model each of the brackets as a spring of stiffness *k*, and assume the inertia of the pulleys is negligible.

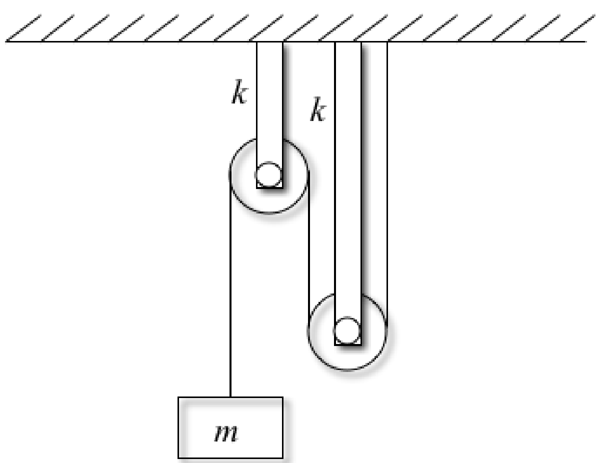


Figure P1.85

Solution: Let *x* denote the distance mass *m* moves, then each spring will deflects a distance *x*/4. Thus the potential energy of the springs is

$$U = 2 \times \frac{1}{2} k \left(\frac{x}{4} \right)^2 = \frac{k}{16} x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2} m \dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right)+\frac{\partial}{\partial x}\left(\frac{kx^2}{16}\right)=0 \Rightarrow \frac{d}{dt}\left(m\dot{x}\right)+\frac{k}{8}x=0$$

$$\Rightarrow \underline{m\ddot{x}+\frac{k}{8}x=0} \Rightarrow \underline{\omega_n=\frac{1}{2}\sqrt{\frac{k}{2m}}} \text{ rad/s}$$

1.86 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.86. This figure represents a simplified model of a jet engine mounted to a wing through a mechanism which acts as a spring of stiffness k and mass m_s. Assume the engine has inertia J and mass m and that the rotation of the engine is related to the vertical displacement of the engine, $x(t)$ by the "radius" r_0 (i.e. $x = r_0\theta$).

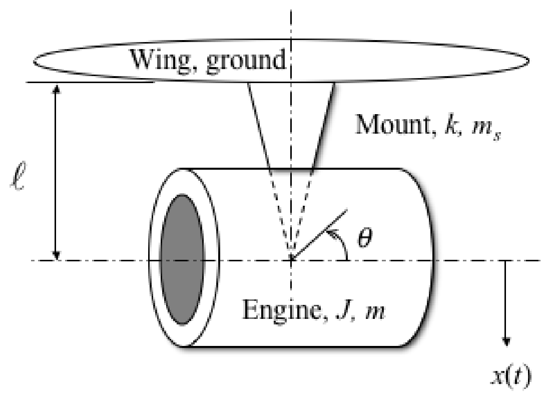


Figure P1.86

Solution: This combines Examples 1.4.1 and 1.4.4. The kinetic energy is

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}J\dot{\theta}^2+T_{\text{spring}}=\frac{1}{2}\left(m+\frac{J}{r_0^2}\right)\dot{x}^2+T_{\text{spring}}$$

The kinetic energy in the spring (see example 1.4.4) is

$$T_{\text{spring}}=\frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

Thus the total kinetic energy is

$$T=\frac{1}{2}\left(m+\frac{J}{r_0^2}+\frac{m_s}{3}\right)\dot{x}^2$$

The potential energy is just

$$U = \frac{1}{2}kx^2$$

Using the Lagrange formulation of Equation (1.64) the equation of motion results from:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow \underline{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\ddot{x} + kx = 0}$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)}}} \text{ rad/s}$$

1.87 Consider the inverted simple pendulum connected to a spring of Figure P1.68. Use Lagrange's formulation to derive the equation of motion.

Solution: The energies are (see the solution to 1.68):

$$T = \frac{1}{2}ml^2\dot{\theta}^2, \quad U = \frac{1}{2}kx^2 + mgh$$

Choosing θ as the generalized coordinate, the spring compresses a distance $x = l \sin \theta$ and the mass moves a distance $h = l \cos \theta$ from the reference position. So the Lagrangian becomes:

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}kl^2\sin^2\theta - mgl\cos\theta$$

The terms in Lagrange's equation are

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -kl^2\sin\theta\cos\theta + mgl\sin\theta$$

Thus from the Lagrangian the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \underline{ml^2\ddot{q} + kl^2 \sin q \cos q - mgl \sin q = 0}$$

$$\square \quad \ddot{q} + \left(\frac{lk - mg}{ml}\right) q = 0$$

Where the last expression is the linearized version for small θ.

1.88 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.63) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i = \frac{1}{2} c \dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension driven by a dynamometer illustrated in Figure P1.88. Assume here that the dynamometer drives the system such that $x = r\theta$.

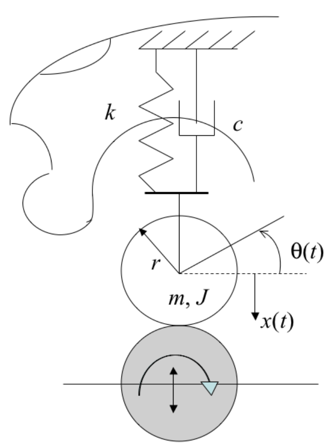


Figure P1.88

Solution: The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{q}^2 = \frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U = \frac{1}{2}kx^2$$

The Rayleigh dissipation function is

$$R = \frac{1}{2}c\dot{x}^2$$

The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

$$\square \quad \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right) = 0$$

$$\square \quad \underline{(m + \frac{J}{r^2})\ddot{x} + c\dot{x} + kx = 0}$$

1.89 Consider the disk of Figure P1.89 connected to two springs. Use the energy method to calculate the system's natural frequency of oscillation for small angles θ(t).

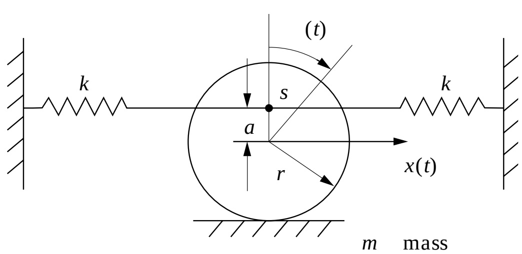


Figure P1.89

Solution:

Known: $x = rq, \dot{x} = r\dot{q}$ and $J_o = \frac{1}{2}mr^2$

Kinetic energy:

$$T_{rot} = \frac{1}{2}J_o\dot{q}^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right)\dot{q}^2 = \frac{1}{4}mr^2\dot{q}^2$$

$$T_{trans} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{q}^2$$

$$T = T_{rot} + T_{trans} = \frac{1}{4}mr^2\dot{q}^2 + \frac{1}{2}mr^2\dot{q}^2 = \frac{3}{4}mr^2\dot{q}^2$$

Potential energy: $U = 2\left(\frac{1}{2}k\left[(a+r)q\right]^2\right) = k(a+r)^2q^2$

Conservation of energy:

$$T + U = \text{ Constant}$$

$$\frac{d}{dt}(T+U) = 0$$

$$\frac{d}{dt}\left(\frac{3}{4}mr^2\dot{q}^2 + k(a+r)^2q^2\right) = 0$$

$$\frac{3}{4}mr^2(2\dot{q}\ddot{q}) + k(a+r)^2(2\dot{q}q) = 0$$

$$\frac{3}{2}mr^2\ddot{q} + 2k(a+r)^2q = 0$$

Natural frequency:

$$w_n = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{2k(a+r)^2}{\frac{3}{2}mr^2}}$$

$$w_n = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}} \text{ rad/s}$$

1.90 A pendulum of negligible mass is connected to a spring of stiffness k at halfway along its length, l, as illustrated in Figure P1.90. The pendulum has two masses fixed to it, one at the connection point with the spring and one at the top. Derive the equation of motion using the Lagrange formulation, linearize the equation and compute the systems natural frequency. Assume that the angle remains small enough so that the spring only stretches significantly in the horizontal direction.

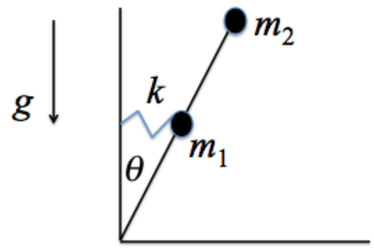


Figure P1.90

Solution: Using the Lagrange formulation the relevant energies are:

$$T = \frac{1}{2}m_1\left(\frac{l}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2$$

$$U = \frac{1}{2}kx^2 + m_1 g h_1 + m_2 g h_2$$

From the trigonometry of the drawing:

$$x = \frac{l}{2}\sin\theta,\ \ h_1 = \frac{l}{2}\cos\theta,\ \ h_2 = l\cos\theta$$

So the potential energy writing in terms of θ is:

$$U = \frac{1}{2}k\left(\frac{l}{2}\sin\theta\right)^2 + m_1 g\frac{l}{2}\cos\theta + m_2 gl\cos\theta$$

Setting $L = T\text{-}U$ and taking the derivatives required for the Lagrangian yields:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(\frac{1}{2}m_1\left(\frac{l}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2\right)$$

$$= \frac{d}{dt}\left(\frac{1}{4}m_1 l^2\dot{\theta} + m_2 l^2\dot{\theta}\right) = \frac{m_1 l^2 + 4m_2 l^2}{4}\ddot{\theta}$$

$$-\frac{\partial L}{\partial\theta} = \frac{kl}{2}\sin\theta\cos\theta - \frac{l}{2}m_1 g\sin\theta - m_2 gl\sin\theta$$

Thus the equation of motion becomes

$$\frac{m_1 l^2 + 4m_2 l^2}{4}\ddot{\theta} + \frac{kl}{2}\sin\theta\cos\theta - \frac{l}{2}m_1 g\sin\theta - m_2 gl\sin\theta = 0$$

Linearizing for small θ this becomes

$$\frac{m_1 l^2 + 4m_2 l^2}{4}\ddot{\theta} + \left(\frac{kl}{2} - \frac{l}{2}m_1 g - m_2 gl\right)\theta = 0$$

So the natural frequency is

$$\omega_n = \sqrt{\frac{2k - 2m_1 g - 4m_2 g}{m_1 l + 4m_2 l}}$$

Problems and Solutions Section 1.5 (1.91 through 1.105)

1.91 A bar of negligible mass fixed with a tip mass forms part of a machine used to punch holes in a sheet of metal as it passes past the fixture as illustrated in Figure P1.91. The impact to the mass and bar fixture causes the bar to vibrate and the speed of the process demands that frequency of vibration not interfere with the process. The static design yields a mass of 50 kg and that the bar be made of steel of length 0.25 m with a cross sectional area of 0.01 m^2. Compute the system's natural frequency.

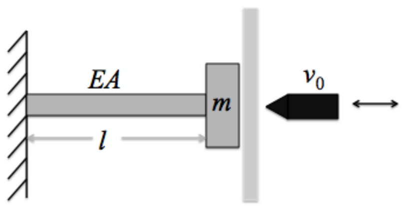


Figure P1.91 A bar model of a punch fixture.

Solution: From equation (1.63)

$$\omega_n = \sqrt{\frac{EA}{lm}} = \sqrt{\frac{(2.0\times 10^{11})(0.01)}{50(0.25)}\frac{(\text{N/m}^2)\text{m}^2}{\text{kg}\cdot\text{m}}} = \underline{1.26\times 10^4\,\text{rad/s}}$$

This is about 2000 Hz, which is likely too high to be a problem but could cause some undesirable noise.

1.92 Consider the punch fixture of Figure P1.91. If the system is giving an initial velocity of 10 m/s, what is the maximum displacement of the mass at the tip if the mass is 1000 kg and the bar is made of steel of length 0.25 m with a cross sectional area of 0.01 m^2?

Solution: First compute the frequency:

$$\omega_n = \sqrt{\frac{EA}{lm}} = \sqrt{\frac{(2.0\times 10^{11})(0.01)}{1000(0.25)}\frac{(\text{N/m}^2)\text{m}^2}{\text{kg}\cdot\text{m}}} = \underline{2.828\times 10^3\,\text{rad/s}}$$

From equation (1.9) the maximum amplitude is

$$A_{\max} = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{v_0}{\omega_n} = \frac{10}{2828}\ \frac{\text{m/s}}{1/\text{s}} = \underline{0.0035\ \text{m}},$$

or about 0.35 mm, not much.

1.93 Consider the punch fixture of Figure P1.91. If the punch strikes the mass off center it is possible that the steel bar may vibrate in torsion. The mass is 1000 kg and the bar 0.25 m-long, with a square cross section of 0.1 m on a side. The mass polar moment of inertia of the tip mass is 10 kg/m^2. The polar moment of inertia for a square bar is $b^4/6$, where b is the length of the side of the square. Compute both the torsion and longitudinal frequencies. Which is larger?

Solution: First compute the longitudinal frequency of the bar:

$$\omega_n = \sqrt{\frac{EA}{lm}} = \sqrt{\frac{\left(2.0\times 10^{11}\right)(0.01)}{1000(0.25)}\ \frac{(\text{N/m}^2)\text{m}^2}{\text{kg}\cdot\text{m}}} = \underline{2.828\times 10^3\,\text{rad/s}}$$

Next compute the torsional frequency of the bar (square cross section):

$$\omega_n = \sqrt{\frac{GJ_p}{lJ}} = \sqrt{\frac{8\times 10^8(0.1^4/6)}{0.25\times 10}} = \underline{73.03\ \text{rad/s}}$$

In this case the torsional frequency is lower and should be considered in any design.

1.94 A 3,000 kg/m spring is compressed 5 cm (at the surface of the earth). How much force was used to compress it?

Solution: From the definition of static deflection:

$$\Delta = \frac{mg}{k} = \frac{F}{k} \Rightarrow F = \Delta k = (0.05\ \text{m})(3{,}000\ \text{N/m}) = 150\ \text{N}$$

1.95 The Mars' exploration rover squats (deflects) 5 cm when it is set down on the test track at JPL (See Figure P1.95). How much will it squat when it is set down on the surface of Mars?

Solution: Using the static deflection formula from Example 1.5.7:

$$\Delta_E = \frac{mg_E}{k},\ \Delta_M = \frac{mg_M}{k}$$

Here the subscript *E* refers to values at earth and the subscript *M* refers to values on Mars. Dividing the expression for deflection on Mars by that on Earth yields

$$\frac{\Delta_M}{\Delta_E} = \frac{mg_M/k}{mg_E/k} = \frac{g_M}{g_E} \Rightarrow \Delta_M = \frac{g_M}{g_E}\Delta_E = \frac{0.38 g_E}{g_E}\Delta_E = 0.38(0.05) = \underline{0.019 \text{ m}}$$

Students can find the *g* on Mars from Google.

1.96 A helicopter landing gear consists of a metal framework rather than the coil spring based suspension system used in a fixed-wing aircraft. The vibration of the frame in the vertical direction can be modeled by a spring made of a slender bar as illustrated in Figure 1.24, where the helicopter is modeled as ground. Here $l = 0.4$ m, $E = 20 \times 10^{10}$ N/m^2, and $m = 100$ kg. Calculate the cross-sectional area that should be used if the natural frequency is to be $f_n = 500$ Hz.

Solution: From equation (1.63)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{EA}{lm}} \qquad (1)$$

and

$$\omega_n = 500 \text{ Hz}\left(\frac{2\pi \text{ rad}}{1 \text{ cycle}}\right) = 3142 \text{ rad/s}$$

Solving (1) for *A* yields:

$$A = \frac{\omega_n^2 lm}{E} = \frac{(3142)^2(.4)(100)}{20 \times 10^{10}} = 0.001974$$
$$A \approx 0.0020 \text{ m}^2 = 20 \text{cm}^2$$

1.97 The frequency of oscillation of a person on a diving board can be modeled as the transverse vibration of a beam as indicated in Figure 1.27. Let *m* be the mass of the diver ($m = 100$ kg) and $l = 1.5$ m. If the diver wishes to oscillate at 3 Hz, what value of *EI* should the diving board material have?

Solution: From equation (1.67),

$$\omega_n^2 = \frac{3EI}{ml^3}$$

and

$$\omega_n = 3Hz\left(\frac{2\pi \text{ rad}}{1 \text{ cycle}}\right) = 6\pi \text{ rad/s}$$

Solving for *EI*

$$EI = \frac{\omega_n^2 ml^3}{3} = \frac{(6\pi)^2(100)(1.5)^3}{3} = \underline{3.997\times 10^4 \text{ Nm}^2}$$

1.98 Consider four springs, labeled k_1, k_2, k_3 and k_4 connected in series. Compute the formula for the equivalent stiffness.

Solution: Basically add one more step to Example 1.5.8 to get

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} = \frac{k_2k_3k_4 + k_1k_3k_4 + k_1k_2k_4 + k_1k_2k_3}{k_1k_2k_3k_4}$$

$$\Rightarrow k_{eq} = \frac{k_1k_2k_3k_4}{k_2k_3k_4 + k_1k_3k_4 + k_1k_2k_4 + k_1k_2k_3}$$

1.99 Consider the spring system of Figure 1.33. Let $k_1 = k_5 = k_2 = 100$ N/m, $k_3 = 50$ N/m, and $k_4 = 1$ N/m. What is the equivalent stiffness?

Solution: Given: $k_1 = k_2 = k_5 = 100$ N/m, $k_3 = 50$ N/m, $k_4 = 1$ N/m

From Example 1.5.4

$$k_{eq} = k_1 + k_2 + k_5 + \frac{k_3k_4}{k_3 + k_4}$$

$$\Rightarrow \underline{k_{eq} = 300.98 \text{ N/m}}$$

1.100 Springs are available in stiffness values of 10, 100, and 1000 N/m. Design a spring system using these values only, so that a 100-kg mass is connected to ground with frequency of about 1.5 rad/s.

Solution: Using the definition of natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

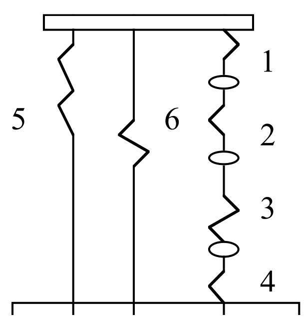
With m = 100 kg and ω_n = 1.5 rad/s the equivalent stiffness must be:

$$k_{eq} = m\omega_n^2 = (100)(1.5)^2 = 225 \text{ N/m}$$

There are many configurations of the springs given and no clear way to determine one configuration over another. Here is one possible solution. Choose two 100 N/m springs in parallel to get 200 N/m, then use four 100 N/m springs in series to get an equivalent spring of 25 N/m to put in parallel with the other 3 springs since

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}} = \frac{1}{4/100} = 25$$

Thus using six 100 N/m springs in the following arrangement will produce an equivalent stiffness of 225 N/m



1.101 Calculate the natural frequency of the system in Figure 1.33(a) if $k_1 = k_2 = 0$. Choose m and nonzero values of k_3, k_4, and k_5 so that the natural frequency is 100 Hz.

Solution: Given: $k_1 = k_2 = 0$ and $\omega_n = 2\pi(100) = 628.3$ rad/s

From Figure 1.29, the natural frequency is

$$\omega_n = \sqrt{\frac{k_5k_3 + k_5k_4 + k_3k_4}{m(k_3 + k_4)}} \quad \text{and} \quad k_{eq} = \left(k_5 + \frac{k_3k_4}{k_3 + k_4}\right)$$

Equating the given value of frequency to the analytical value yields:

$$\omega_n^2 = (628.3)^2 = \frac{k_5k_3 + k_5k_4 + k_3k_4}{m(k_3 + k_4)}$$

Any values of k_3, k_4, k_5, and m that satisfy the above equation will do. Again, the answer is *not unique.* One solution is

$$k_3 = 1\text{ N/m}, k_4 = 1\text{ N/m}, k_5 = 50{,}000\text{ N/m, and } m = 0.127\text{ kg}$$

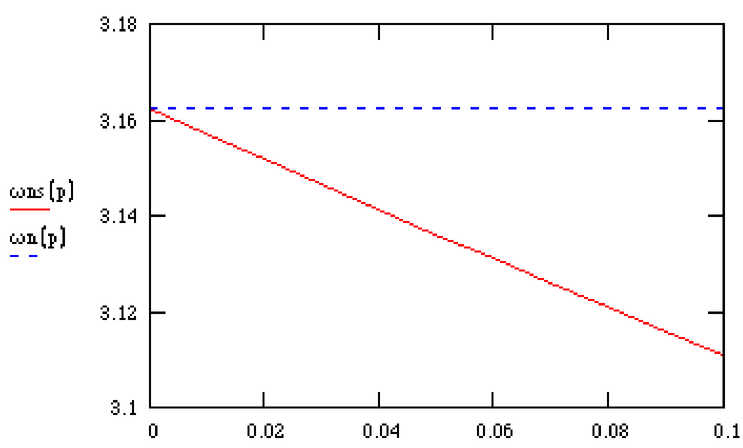
1.102* Example 1.4.4 examines the effect of the mass of a spring on the natural frequency of a simple spring-mass system. Use the relationship derived there and plot the natural frequency (normalized by the natural frequency, ω_n, for a massless spring) versus the percent that the spring mass is of the oscillating mass. Determine from the plot (or by algebra) the percentage where the natural frequency changes by 1% and therefore the situation when the mass of the spring should not be neglected.

Solution: The solution here depends on the value of the stiffness and mass ratio and hence the frequency. Almost any logical discussion is acceptable as long as the solution indicates that for smaller values of m_s, the approximation produces a reasonable frequency. Here is one possible answer. For

$k := 1000 \qquad m := 100$

$p := 0, 0.01 .. 0.1$

$$\omega ns(p) := \sqrt{\frac{k}{m + \frac{p \cdot m}{3}}} \qquad \omega n(p) := \sqrt{\frac{k}{m}}$$



From this plot, for these values of m and k, ***a 10 % spring mass causes less then a 1 % error in the frequency.***

1.103 Calculate the natural frequency and damping ratio for the system in Figure P1.103 given the values m = 10 kg, c = 100 kg/s, k_1 = 4000 N/m, k_2 = 200 N/m and k_3 = 1000 N/m. Assume that no friction acts on the rollers. Is the system overdamped, critically damped or underdamped?

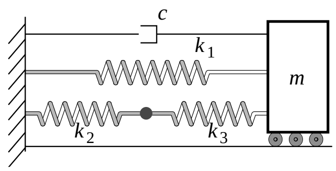


Figure P1.103

Solution: Following the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + \frac{k_2 k_3}{k_2 + k_3} = 4000 + \frac{(200)(1000)}{1200} = 4167 \text{ N/m}$$

Then using the standard formulas for frequency and damping ratio:

$$w_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4167}{10}} = 20.412 \text{ rad/s}$$

$$z = \frac{c}{2mw_n} = \frac{100}{2(10)(20.412)} = 0.245$$

Thus the system is underdamped.

1.104 Calculate the natural frequency and damping ratio for the system in Figure P1.104. Assume that no friction acts on the rollers. Is the system overdamped, critically damped or underdamped?.

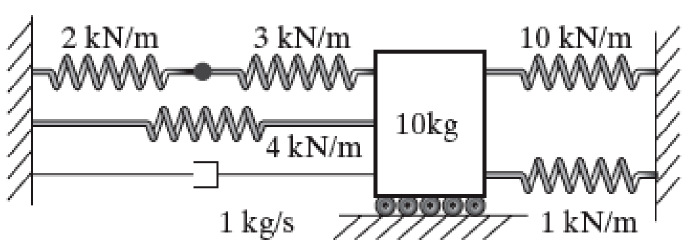


Figure P1.104

Solution: Again using the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + k_2 + k_3 + \frac{k_4 k_5}{k_4 + k_5} = (10 + 1 + 4 + \frac{2 \times 3}{2 + 3})\text{kN/m} = 16.2 \text{ kN/m}$$

Then using the standard formulas for frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{16.2 \times 10^3}{10}} = 40.25 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{1}{2(10)(40.25)} = 0.001242 \approx \underline{0.001}$$

Thus the system is underdamped, in fact very lightly damped.

1.105 A manufacturer makes a cantilevered leaf spring from steel ($E = 2 \times 10^{11}$ N/m^2) and sizes the spring so that the device has a specific frequency. Later, to save weight, the spring is made of aluminum ($E = 7.1 \times 10^{10}$ N/m^2). Assuming that the mass of the spring is much smaller than that of the device the spring is attached to, determine if the frequency increases or decreases and by how much.

Solution: Use equation (1.67) to write the expression for the frequency twice:

$$\omega_{\text{al}} = \sqrt{\frac{3E_{\text{al}}}{m\ell^3}} \quad \text{and} \quad \omega_{\text{steel}} = \sqrt{\frac{3E_{\text{steel}}}{m\ell^3}} \text{ rad/s}$$

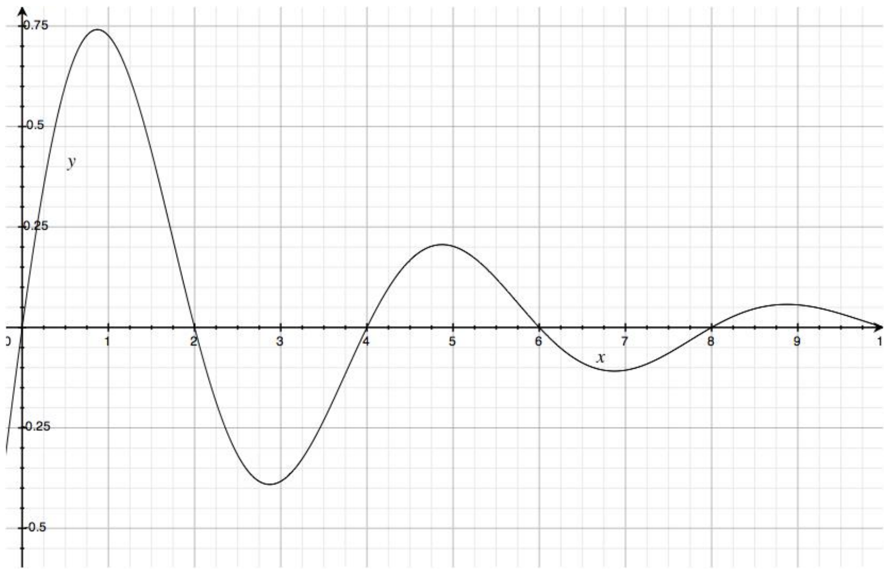
Dividing yields:

$$\frac{\omega_{\text{al}}}{\omega_{\text{steel}}} = \frac{\sqrt{\dfrac{3E_{\text{al}}}{m\ell^3}}}{\sqrt{\dfrac{3E_{\text{steel}}}{m\ell^3}}} = \sqrt{\frac{7.1 \times 10^{10}}{2 \times 10^{11}}} = 0.596$$

Thus the ***frequency is decreased by about 40% by using aluminum.***

Problems and Solutions Section 1.6 (1.106 through 1.113)

1.106 The displacement of a vibrating spring-mass-damper system is recorded on an *x* – *y* plotter and reproduced in Figure P1.106. The *y* coordinate is the displacement in cm and the *x* coordinate is time in seconds. From the plot determine the natural frequency, the damping ratio and the damped natural frequency.



P1.106 A plot of displacement versus time for a vibrating system.

Solution: From the plot, the period is *T* =4 seconds. Thus the damped natural frequency is

$$\omega_d = \frac{2\pi}{T} = \frac{2\pi}{4\text{ s}} = \underline{1.5708\text{ rad/s}}$$

Using the formula for log decrement and noting from the plot that the first peak is about $y(t_1) = 0.74$ cm and the second peak is about $y(t_2) = 0.2$ cm yields

$$\delta = \ln\left(\frac{0.74}{0.2}\right) = 1.3083$$

Using equation (1.83) the damping ration is then:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.2039$$

The undamped natural frequency is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.5709}{\sqrt{1-(0.2039)^2}} \approx \underline{1.6046 \text{ rad/s}}$$

1.107 Show that the logarithmic decrement is equal to

$$\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$$

where x_n is the amplitude of vibration after n cycles have elapsed.

Solution:

$$\ln\left[\frac{x(t)}{x(t+nT)}\right] = \ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t+nt)}\sin(\omega_d t+\omega_d nT+\phi)}\right] \qquad (1)$$

Since $n\omega_d T = n(2\pi)$, $\sin(\omega_d t + n\omega_d T + \phi) = \sin(\omega_d t + \phi)$

Hence, Eq. (1) becomes

$$\ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t+nT)}e^{-\zeta\omega_n nt}\sin(\omega_d t+\omega_d nt+\phi)}\right] = \ln\left(e^{\zeta\omega_n nT}\right) = n\zeta\omega_n T$$

Since $$\ln\left[\frac{x(t)}{x(t+T)}\right] = \zeta\omega_n T = \delta,$$

Then $$\ln\left[\frac{x(t)}{x(t+nT)}\right] = n\delta$$

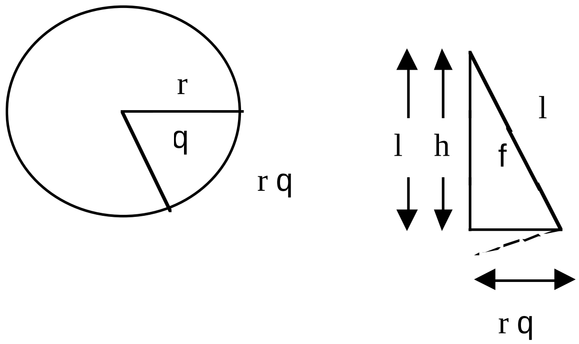
Therefore,

$$\delta = \frac{1}{n}\ln\frac{x_o}{x_n} \qquad \begin{array}{l}\leftarrow \text{original amplitude} \\ \leftarrow \text{amplitude } n \text{ cycles later}\end{array}$$

Here $x_0 = x(0)$.

1.108 Derive the equation (1.78) for the trifalar suspension system.

Solution: Using the notation given for Figure 1.33, and the following geometry:



Write the kinetic and potential energy to obtain the frequency:

Kinetic energy: $$T_{\max} = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$$

From geometry, $x = r\theta$ and $\dot{x} = r\dot{\theta}$

$$T_{\max} = \frac{1}{2}(I_o + I)\frac{\dot{x}^2}{r^2}$$

Potential Energy:

$$U_{\max} = (m_o + m)g(l - l\cos\phi)$$

Two term Taylor Series Expansion of $\cos\phi \approx 1 - \frac{\phi^2}{2}$:

$$U_{\max} = (m_o + m)gl\left(\frac{\phi^2}{2}\right)$$

For geometry, $\sin\phi = \frac{r\theta}{l}$, and for small ϕ, $\sin\phi = \phi$ so that $\phi = \frac{r\theta}{l}$

$$U_{\max} = (m_o + m)gl\left(\frac{r^2\theta^2}{2l^2}\right)$$

$$U_{\max} = (m_o + m)g\left(\frac{r^2\theta^2}{2l}\right) \quad \text{where } r\theta = x$$

$$U_{\max} = \frac{(m_o + m)g}{2l}x^2$$

Conservation of energy requires that:

$$T_{\max} = U_{\max} \quad \Rightarrow$$

$$\frac{1}{2}\frac{(I_o + I)}{r^2}\dot{x}^2 = \frac{(m_o + m)g}{2l}x^2$$

At maximum energy, x = A and $\dot{x} = \omega_n A$

$$\frac{1}{2}\frac{(I_o + I)}{r^2}\omega_n^2 A^2 = \frac{(m_o + m)g}{2l}A^2$$

$$\Rightarrow \ (I_o + I) = \frac{gr^2(m_o + m)}{\omega_n^2 l}$$

Substitute $\omega_n = 2\pi f_n = \frac{2\pi}{T}$

$$(I_o + I) = \frac{gr^2(m_o + m)}{(2\pi / T)^2 l}$$

$$I = \frac{gT^2 r^2 (m_o + m)}{4\pi^2 l} - I_o$$

were T is the period of oscillation of the suspension.

1.109 A prototype composite material is formed and hence has an unknown modulus. An experiment is performed consisting of forming it into a cantilevered beam of length 1 m and $I = 10^{-9}$ m^4 with a 10-kg mass attached at its end. The system is given an initial displacement and found to oscillate with a period of 0.5 s. Calculate the modulus E.

Solution: Using equation (1.65) for a cantilevered beam,

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{ml^3}{3EI}}$$

Solving for E and substituting the given values yields

$$E = \frac{4\pi^2 ml^3}{3T^2 I} = \frac{4\pi^2(10)(1)^3}{3(.5)^2(10^{-9})}$$

$$\Rightarrow \ \underline{E = 5.75 \times 10^{11} \text{ N/m}^2}$$

1.110 The free response of a 1000-kg car with stiffness of k = 400,000 N/m is observed to be of the form given in Figure 1.36. Modeling the car as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution: Given: x_1 = 2 cm and x_2 = 0.22 cm where $t_2 = T + t_1$

Logarithmic Decrement: $$\delta = \ln\frac{x_1}{x_2} = \ln\frac{2}{0.22} = 2.207$$

Damping Ratio: $$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.207}{\sqrt{4\pi^2 + (2.207)^2}} = 0.331$$

Damping Coefficient: $$c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{(400,000)(1000)} = 13,240 \text{ kg/s}$$

1.111 A pendulum decays from 10 cm to 1 cm over one period. Determine its damping ratio.

Solution: Using Figure 1.36: $x_1 = 10$ cm and $x_2 = 1$ cm

Logarithmic Decrement: $$\delta = \ln\frac{x_1}{x_2} = \ln\frac{10}{1} = 2.303$$

Damping Ratio: $$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.303}{\sqrt{4\pi^2 + (2.303)^2}} = 0.344$$

1.112 The relationship between the log decrement δ and the damping ratio ζ is often approximated as $\delta = 2\pi\zeta$. For what values of ζ would you consider this a good approximation to equation (1.82)?

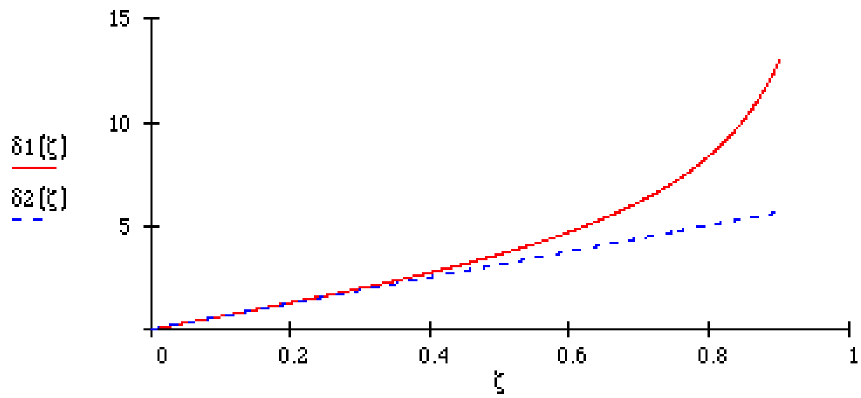
Solution: From equation (1.82), $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

For small ζ, $$\delta = 2\pi\zeta$$

A plot of these two equations is shown:

$\zeta := 0, 0.001 .. 0.9$

$$\delta 1(\zeta) := \frac{2\cdot\pi\cdot\zeta}{\sqrt{1-\zeta^2}} \qquad \delta 2(\zeta) := 2\cdot\pi\cdot\zeta$$



The lower curve represents the approximation for small ζ, while the upper curve is equation (1.82). The approximation appears to be valid to about $\zeta = 0.3$.

1.113 A damped system is modeled as illustrated in Figure 1.10. The mass of the system is measured to be 5 kg and its spring constant is measured to be 5000 N/m. It is observed that during free vibration the amplitude decays to 0.25 of its initial value after five cycles. Calculate the viscous damping coefficient, *c*.

Solution:

Note that for any two consecutive peak amplitudes,

$$\frac{x_o}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = e^{d} \text{ by definition}$$

$$\backslash \;\; \frac{x_o}{x_5} = \frac{1}{0.25} = \frac{x_0}{x_1} \times \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} = e^{5d}$$

So,

$$d = \frac{1}{5}\ln(4) = 0.277$$

and

$$z = \frac{d}{\sqrt{4p^2 + d^2}} = 0.044$$

Solving for c,

$$c = 2z\sqrt{km} = 2(0.044)\sqrt{5000(5)}$$

$$Þ \;\; c = \underline{13.914\ \text{N-s/m}}$$

Problems and Solutions Section 1.7 (1.114 through 1.122)

1.114 Consider the system of Example 1.7.2 consisting of a helical spring of stiffness 10^3 N/m attached to a 10-kg mass. Place a dashpot parallel to the spring and choose its viscous damping value so that the resulting damped natural frequency is reduced to 9 rad/s.

Solution:

The frequency of oscillation is $\omega_d = \omega_n\sqrt{1-\zeta^2}$

From example 1.7.2: $m = 10$ kg, $k = 10^3$ N/m, and $\omega_n = \sqrt{\dfrac{1000}{10}} = 10$ rad/s

So,
$$9 = 10\sqrt{1-\zeta^2}$$
$$\Rightarrow\ 0.9 = \sqrt{1-\zeta^2} \Rightarrow\ (0.9)^2 = 1-\zeta^2$$
$$\zeta = \sqrt{1-(0.9)^2} = 0.436$$

Then

$$c = 2m\omega_n\zeta = 2(10)(10)(0.436) = \underline{87.2\ \text{kg/s}}$$

1.115 For an underdamped system, x_0 = 0 mm and v_0 = 10 mm/s. Determine *m*, *c*, and *k* such that the amplitude is less than 1 mm.

Solution: Note there are multiple correct solutions. The expression for the amplitude is:

$$A^2 = x_0^2 + \frac{(v_o + \zeta\omega_n x_o)^2}{\omega_d^2}$$

$$\text{for } x_o = 0 \Rightarrow\ A = \frac{v_o}{\omega_d} < 0.001\ \text{m} \Rightarrow\ \omega_d > \frac{v_o}{0.001} = \frac{0.01}{0.001} = 10$$

So

$$w_d = \sqrt{\frac{k}{m}\left(1 - z^2\right)} > 10$$

$$Þ \ \frac{k}{m}\left(1 - z^2\right) > 100, Þ \ k = m\frac{100}{1 - z^2}$$

(1) Choose $z = 0.01 \, Þ \ \frac{k}{m} > 100.01$

(2) Choose $\underline{m = 1\,\text{kg}}$ $Þ \ k > 100.01$

(3) Choose $k = 144$ N/m > 100.01

$$Þ \ w_n = \sqrt{144}\,\frac{\text{rad}}{\text{s}} = 12\,\frac{\text{rad}}{\text{s}}$$

$$Þ \ w_d = 11.99\,\frac{\text{rad}}{\text{s}}$$

$$Þ \ c = 2mzw_n = \underline{\underline{2.4 \ \frac{\text{kg}}{\text{s}}}}$$

1.116 Repeat problem 1.115 if the mass is restricted to be between 10 kg < *m* < 15 kg.

Solution: Referring to the above problem, the relationship between *m* and *k* is

$$k > 1.01\text{x}10^{-4}\, m$$

after converting to meters from mm. Choose m =10 kg and repeat the calculation at the end of Problem 1.115 to get ω_n (again taking $\zeta = 0.01$). Then k = 1000 N/m and:

$$Þ \ w_n = \sqrt{\frac{1.0 ´ \ 10^3}{10}}\,\frac{\text{rad}}{\text{s}} = 10 \ \frac{\text{rad}}{\text{s}}$$

$$Þ \ w_d = 9.998 \ \frac{\text{rad}}{\text{s}}$$

$$Þ \ c = 2mzw_n = \underline{\underline{2.000 \ \frac{\text{kg}}{\text{s}}}}$$

1.117 Use the formula for the torsional stiffness of a shaft from Table 1.1 to design a 1-m shaft with torsional stiffness of 10^5 N·m/rad.

Solution: Referring to equation (1.64) the torsional stiffness is

$$k_t = \frac{GJ_p}{\ell}$$

Assuming a solid shaft, the value of the shaft polar moment is given by

$$J_p = \frac{\pi d^4}{32}$$

Substituting this last expression into the stiffness yields:

$$k_t = \frac{G\pi d^4}{32\ell}$$

Solving for the diameter *d* yields

$$d = \left(\frac{k_t(32)\ell}{G\pi}\right)^{1/4}$$

Thus we are left with the design variable of the material modulus (G). Choose steel, then solve for *d*. For steel $G = 8 \times 10^{10}$ N/m^2. From the last expression the numerical answer is

$$d = \left[\frac{10^5 \frac{\text{Nm}}{\text{rad}}(32)(1\text{m})}{\left(8 \times 10^{10} \frac{\text{N}}{\text{m}^2}\right)(\pi)}\right]^{1/4} = 0.0597 \text{ m}$$

1.118 Consider designing a helical spring made of aluminum, such that when it is attached to a 10-kg mass the resulting spring-mass system has a natural frequency of 10 rad/s. Thus repeat Example 1.7.2 which uses steel for the spring and note any difference.

Solution:

For aluminum $G = 25 \times 10^9$ N/m^2

From example 1.7.2, the stiffness is $k = 10^3 = \dfrac{Gd^4}{64nR^3}$ and $d = .01$ m

So,

$$10^3 = \frac{(25 \times 10^9)(.01)^4}{64nR^3}$$

Solving for nR^3 yields: $nR^3 = 3.906 \times 10^{-3}\text{m}^3$

Choose R = 10 cm = 0.1 m, so that

$$n = \frac{3.906 \times 10^{-3}}{(0.1)^3} = 4\,\text{turns}$$

Thus, aluminum requires 1/3 fewer turns than steel.

1.119 Try to design a bar that has the same stiffness as the helical spring of Example 1.7.2 (i.e., $k = 10^3$ N/m). This amounts to computing the length of the bar with its' cross sectional area taking up about the same space at the helical spring (R = 10 cm). Note that the bar must remain at least 10 times as long as it is wide in order to be modeled by the stiffness formula given for the bar in Figure 1.24.

Solution:

From Figure 1.21, $k = \dfrac{EA}{l}$

For steel, $E = 210 \times 10^9$ N/m^2

From Example 1.7.2, $k = 10^3$ N/m

So, $$10^3 = \frac{(210 \times 10^9)A}{l}$$

$$l = (2.1 \times 10^8)A$$

If $A = 0.01$ m^2 (10 cm^2), then

$$l = (2.1 \times 10^8)(10^{-2}) = 2.1 \times 10^6 \text{ m}$$

Not very practical at all. Sometimes in the course of design, the requirements cannot be met.

1.120 Repeat Problem 1.119 using plastic ($E = 1.40 \times 10^9$ N/m^2) and rubber ($E = 7 \times 10^6$ N/m^2). Are any of these feasible?

Solution:

From problem 1.53, $k = 10^3 \text{ N/m} = \dfrac{EA}{l}$

For plastic, $E = 1.40 \times 10^9 \text{ N/m}^2$

So, $l = 140 \text{ m}$

For rubber, $E = 7 \times 10^6 \text{ N/m}^2$

So, $l = 0.7 \text{ m}$

Rubber may be feasible, plastic would not.

1.121 Consider the diving board of Figure P1.121. For divers, a certain level of static deflection is desirable, denoted by Δ. Compute a design formula for the dimensions of the board (b, h and ℓ) in terms of the static deflection, the average diver's mass, m, and the modulus of the board.

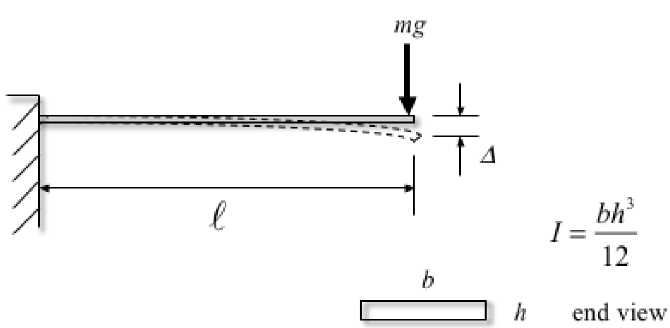


Figure P1.121

Solution: From Figure 1.16 (b), $\Delta k = mg$ holds for the static deflection. The period is:

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/\Delta}} = 2\pi\sqrt{\frac{\Delta}{g}} \qquad (1)$$

From Figure 1.27, we also have that

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m\ell^3}{3EI}} \qquad (2)$$

Equating (1) and (2) and replacing I with the value from the figure yields:

$$2\pi\sqrt{\frac{m\ell^3}{3EI}} = 2\pi\sqrt{\frac{12m\ell^3}{3Ebh^3}} = 2\pi\sqrt{\frac{\Delta}{g}} \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

Alternately just use the static deflection expression and the expression for the stiffness of the beam from Figure 1.28 to get

$$\Delta k = mg \Rightarrow \Delta\frac{3EI}{\ell^3} = mg \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

1.122 In designing a vehicle suspension system using a “quarter car model” consisting of a spring, mass and damper system, studies show the a desirable damping ratio is $\zeta = 0.25$. If the model has a mass of 750 kg and a frequency of 15 Hz, what should the damping coefficient be?

Solution: First convert the 15 Hz to rad/sec:

$$\omega_n = 15\frac{\text{cycle}}{\text{sec}}\frac{2\pi\ \text{rad}}{\text{cycle}} = 94.25\frac{\text{rad}}{\text{sec}}$$

Next use equation for the damping ratio

$$\zeta = \frac{c}{2m\omega_n} \Rightarrow c = 2\zeta m\omega_n$$
$$\Rightarrow c = 2 \cdot 0.25 \cdot 750 \cdot 94.25 = \underline{35{,}343.75\ \text{kg/s}}$$

Problems and Solutions Section 1.8 (1.123 through 1.127)

1.123 Consider the system of Figure P1.123. (a) Write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. (b) Discuss the stability of the linear system's solutions in terms of the physical constants, *m*, *k*, and ℓ. Assume the mass of the rod acts at the center as indicated in the figure.

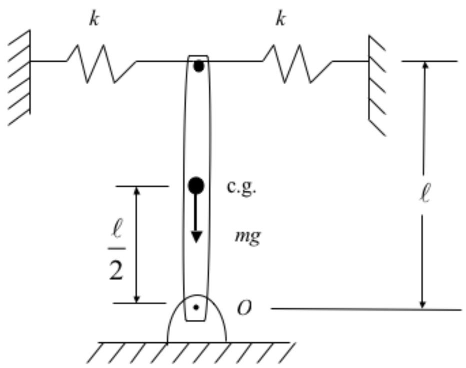


Figure P1.123

Solution: Note that from the geometry, the springs deflect a distance $kx = k(\ell \sin\theta)$ and the cg is a distance $\frac{\ell}{2}\cos\theta$ up from the horizontal line through point 0 taken as zero gravitational potential. Thus the total potential energy is

$$U = 2 \times \frac{1}{2}k(\ell \sin\theta)^2 + \frac{mg\ell}{2}\cos\theta$$

Using the inertia for a thin rod of length l rotating about its end, the total kinetic energy is

$$T = \frac{1}{2}J_O\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + \frac{\partial U}{\partial \theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot{\theta}\right) + 2k\ell \sin\theta\cos\theta - \frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin q » q$ and $\cos q » 1$ yields

$$\text{a)}\quad \underline{\frac{m\ell^2}{3}\ddot{q} + \left(2k\ell^2 - \frac{mg\ell}{2}\right)q = 0}$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

b)
if $4k\ell - mg > 0$ Þ $4k\ell > mg$ Þ the system is stable
if $4k = mg$ Þ $q(t) = at + b$ Þ the system is unstable
if $4k\ell - mg < 0$ Þ $4k\ell < mg$ Þ the system is unstable

Note that physically these results state that the system's response is stable as long as the spring stiffness is large enough to overcome the force of gravity.

1.124 Consider the inverted pendulum of Figure 1.42 as discussed in Example 1.8.1 and repeated here as Figure P1.112. Assume that a dashpot (of damping rate *c*) also acts on the pendulum parallel to the two springs. How does this affect the stability properties of the pendulum?

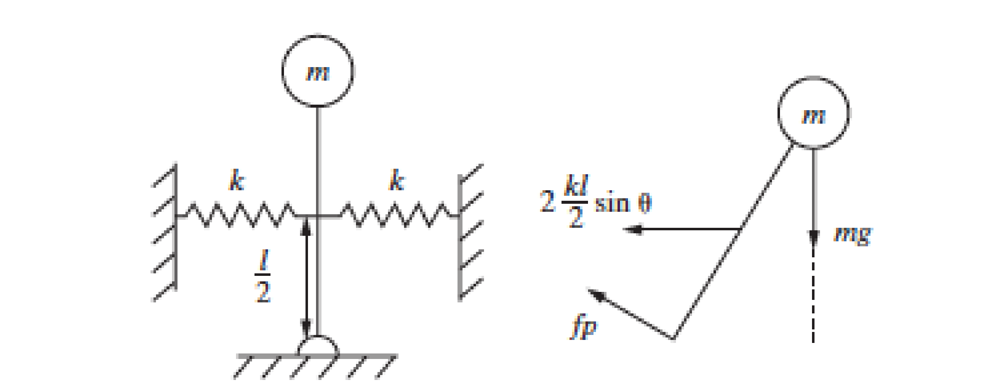
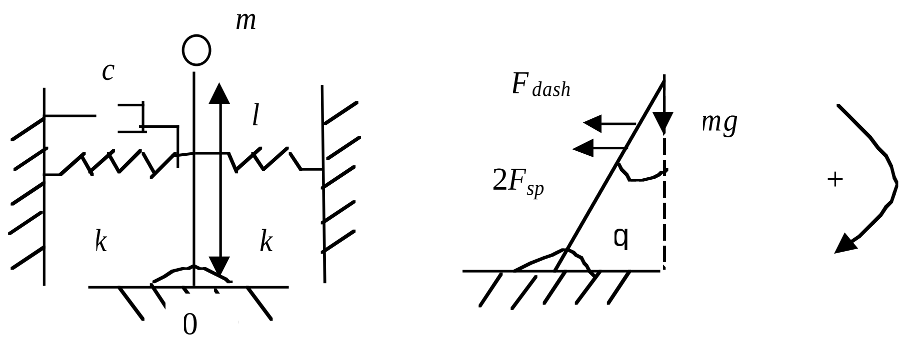


Figure P1.124 The inverted pendulum of Example 1.8.1

Solution: The equation of motion is found from the following FBD:



Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) - c\left(\frac{l}{2}\dot{\theta}\right)\left(\frac{l}{2}\cos\theta\right)$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1

$$ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + \left(\frac{kl^2}{2} - mgl\right)\theta = 0$$

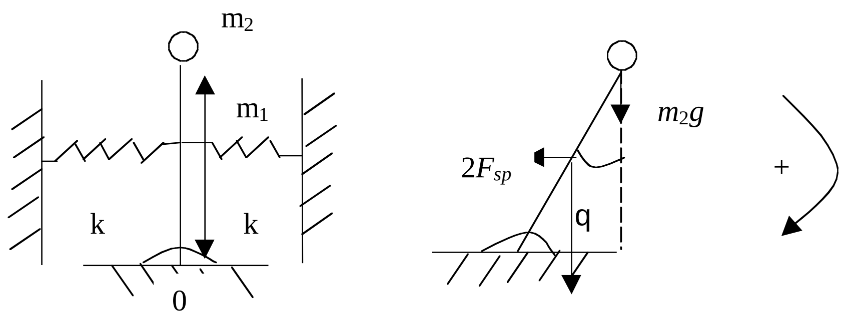
$$ml\ddot{\theta} + \frac{cl}{4}\dot{\theta} + \left(\frac{kl}{2} - mg\right)\theta = 0$$

For stability, $\frac{kl}{2} > mg$ and $c \geq 0$.

The result of adding a dashpot is to make the system asymptotically stable.

1.125 Replace the massless rod of the inverted pendulum of Figure 1.42 with a solid object compound pendulum of Figure 1.21(b). Calculate the equations of vibration and discuss values of the parameter relations for which the system is stable.

Solution:



Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$m_1 g\frac{l}{2}\sin\theta + m_2 g l\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) = \left(\frac{1}{3}m_1 l^2 + m_2 l^2\right)\ddot{\theta}$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1.

$$\left(\frac{m_1}{3} + m_2\right)l^2\ddot{\theta} + \left(\frac{kl^2}{2} - \frac{m_1}{2}gl - m_2 gl\right)\theta = 0$$

$$\left(\frac{m_1}{3} + m_2\right)l\ddot{\theta} + \left[\frac{kl}{2} - \left(\frac{m_1}{2} + m_2\right)g\right]\theta = 0$$

For stability, $\frac{kl}{2} > \left(\frac{m_1}{2} + m_2\right)g.$

1.126 A simple model of a control tab for an airplane is sketched in Figure P1.125. The equation of motion for the tab about the hinge point is written in terms of the angle θ from the centerline to be

$$J\ddot{\theta} + (c - f_d)\dot{\theta} + k\theta = 0.$$

Here J is the moment of inertia of the tab, k is the rotational stiffness of the hinge, c is the rotational damping in the hinge and $f_d\dot{\theta}$ is the negative damping provided by the aerodynamic forces (indicated by arrows in the figure). Discuss the stability of the solution in terms of the parameters c and f_d .

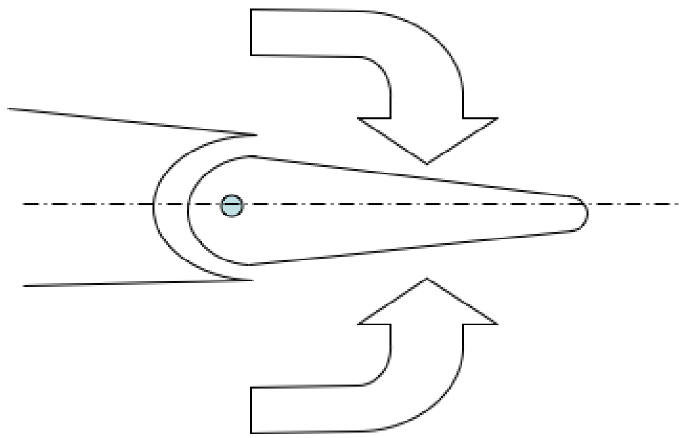


Figure P1.126 A simple model of an airplane control tab

Solution: The stability of the system is determined by the coefficient of $\dot{q}$ since the inertia and stiffness terms are both positive. There are three cases

Case 1 $c - f_d > 0$ and the system's solution is of the form $q(t) = e^{-at}\sin(w_n t + f)$ and the solution is asymptotically stable.

Case 2 $c - f_d < 0$ and the system's solution is of the form $q(t) = e^{at}\sin(w_n t + f)$ and the solution oscillates and grows without bound, and exhibits flutter instability as illustrated in Figure 1.38.

Case 3 $c = f_d$ and the system's solution is of the form $q(t) = A\sin(w_n t + f)$ and the solution is stable as illustrated in Figure 1.39.

1.127* In order to understand the effect of damping in design, develop some sense of how the response changes with the damping ratio by plotting the response of a single degree of freedom system for a fixed amplitude, frequency and phase as ζ changes through the following set of values ζ = 0.01, 0.05, 0.1, 0.2, 0.3 and 0.4. That is plot the response $x(t) = e^{-10zt}\sin(10\sqrt{1 - z^2}\,t)$ for each value of ζ.

Solution: Any plotting program can be used (Grapher, Excel, Matlab, etc.). Here is a Matcad version. If assigned it's good to discuss the differences as the damping increases.

$\zeta 1 := 0.01$ $\quad x1(t) := e^{-10\cdot\zeta 1\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 1^2}\right)$

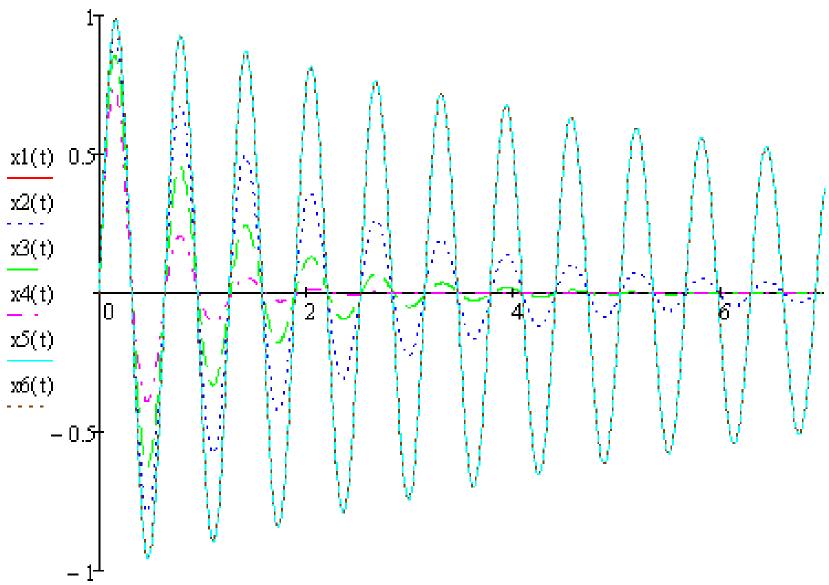
$\zeta 2 := 0.05$ $\quad x2(t) := e^{-10\cdot\zeta 2\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 2^2}\right)$

$\zeta 3 := 0.1$ $\quad x3(t) := e^{-10\cdot\zeta 3\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 3^2}\right)$

$\zeta 4 := 0.2$ $\quad x4(t) := e^{-10\cdot\zeta 4\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 4^2}\right)$

$\zeta 5 := 0.01$ $\quad x5(t) := e^{-10\cdot\zeta 5\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 5^2}\right)$

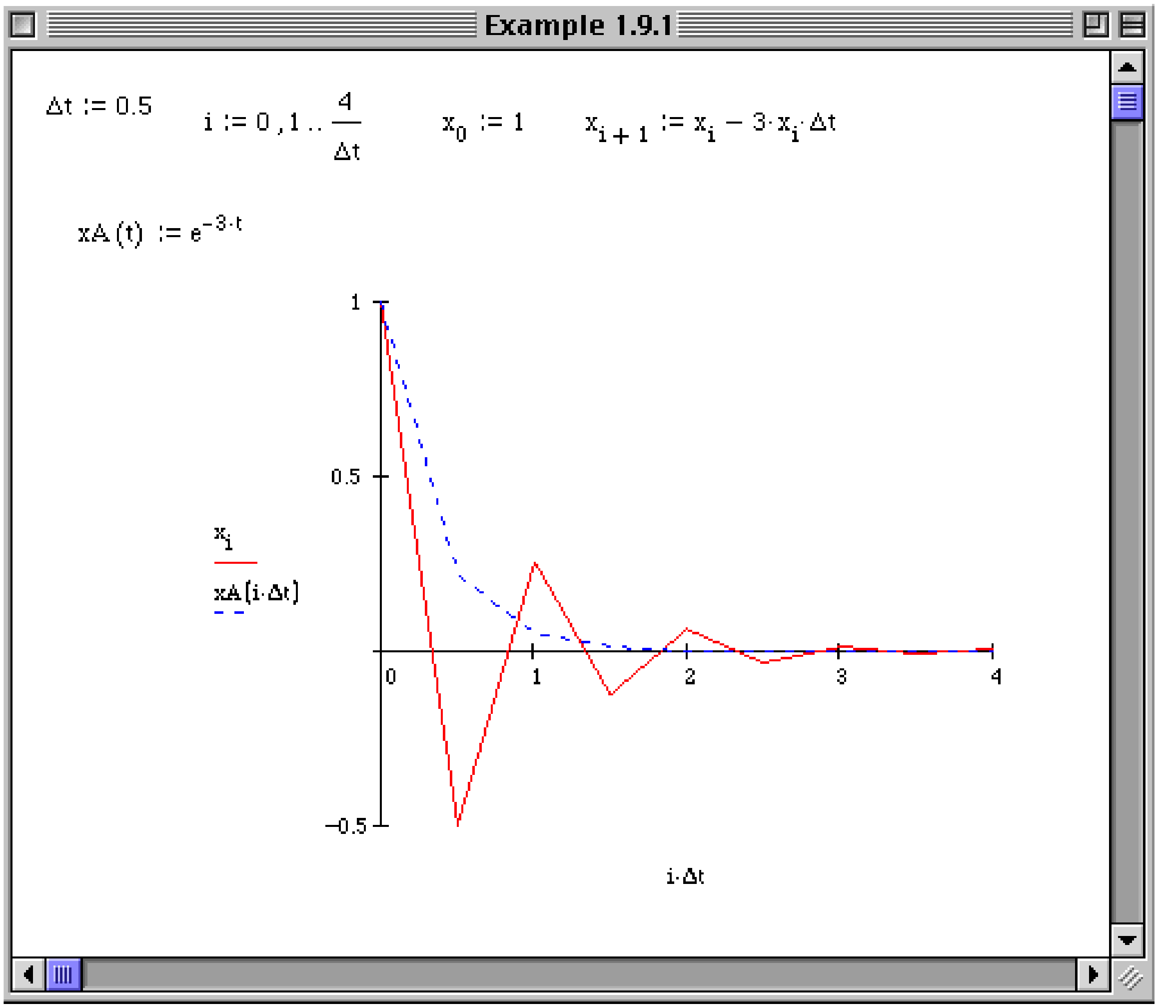
$\zeta 6 := 0.01$ $\quad x6(t) := e^{-10\cdot\zeta 6\cdot t}\cdot\sin\left(10\cdot t\sqrt{1-\zeta 6^2}\right)$



Problems and Solutions Section 1.9 (1.128 through 1.134)

1.128* Compute and plot the response to $\dot{x} = -3x,\ x(0) = 1$ using Euler's method for time steps of 0.1 and 0.5. Also plot the exact solution and hence reproduce Figure 1.43.

Solution: The code is given here in Mathcad, which can be run repeatedly with different Δt to see the importance of step size. Matlab and Mathematica can also be used to show this.



1.129* Use numerical integration to solve the system of Example 1.7.3 with m = 1361 kg, k = 2.688 x 10^5 N/m, c = 3.81 x 10^3 kg/s subject to the initial conditions $x(0)$ = 0 and $v(0)$ = 0.01 mm/s. Compare your result using numerical integration to just plotting the analytical solution (using the appropriate formula from Section 1.3) by plotting both on the same graph.

Solution: The solution is shown here in Mathcad using an Euler integration. This can also been done in the other codes or the Toolbox:

1.84

$k := 2.688\cdot 10^5 \quad m := 1361 \quad x_0 := 0.0 \quad v_0 := 0.01 \quad c := 3.81\cdot 10^3$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2\cdot\pi} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$$

$$A := \frac{1}{\omega d}\cdot\sqrt{\left(v_0+\zeta\cdot x_0\cdot\omega n\right)^2+\left(x_0\cdot\omega d\right)^2} \qquad T := \frac{2\cdot\pi}{\omega d}$$

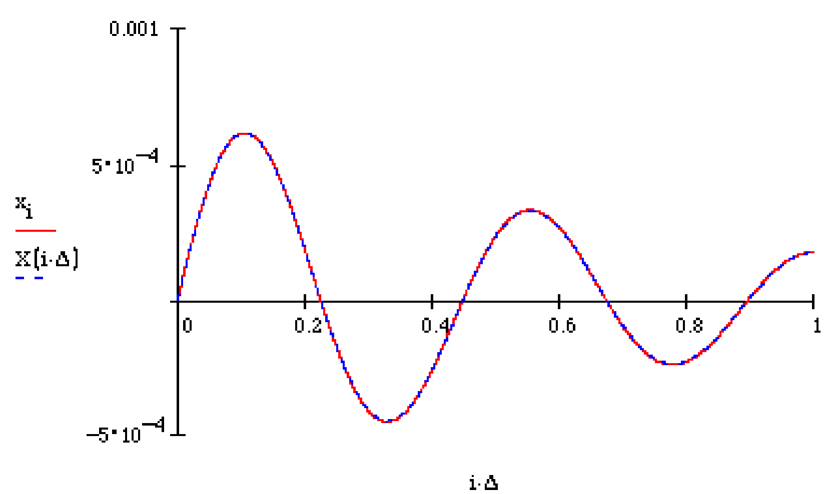
$$\phi := \operatorname{atan}\left[\frac{\omega d\cdot x_0}{v_0+\zeta\cdot\omega n\cdot x_0}\right] \qquad \zeta = 0.1$$

$$X(t) := A\cdot\sin\left(\omega d\cdot t+\phi\right)\cdot e^{-\zeta\cdot\omega n\cdot t}$$

$$N := 10000 \qquad \Delta := \frac{6\cdot\pi}{\omega n\cdot N} \qquad y := \begin{bmatrix}0\\0.01\end{bmatrix}$$

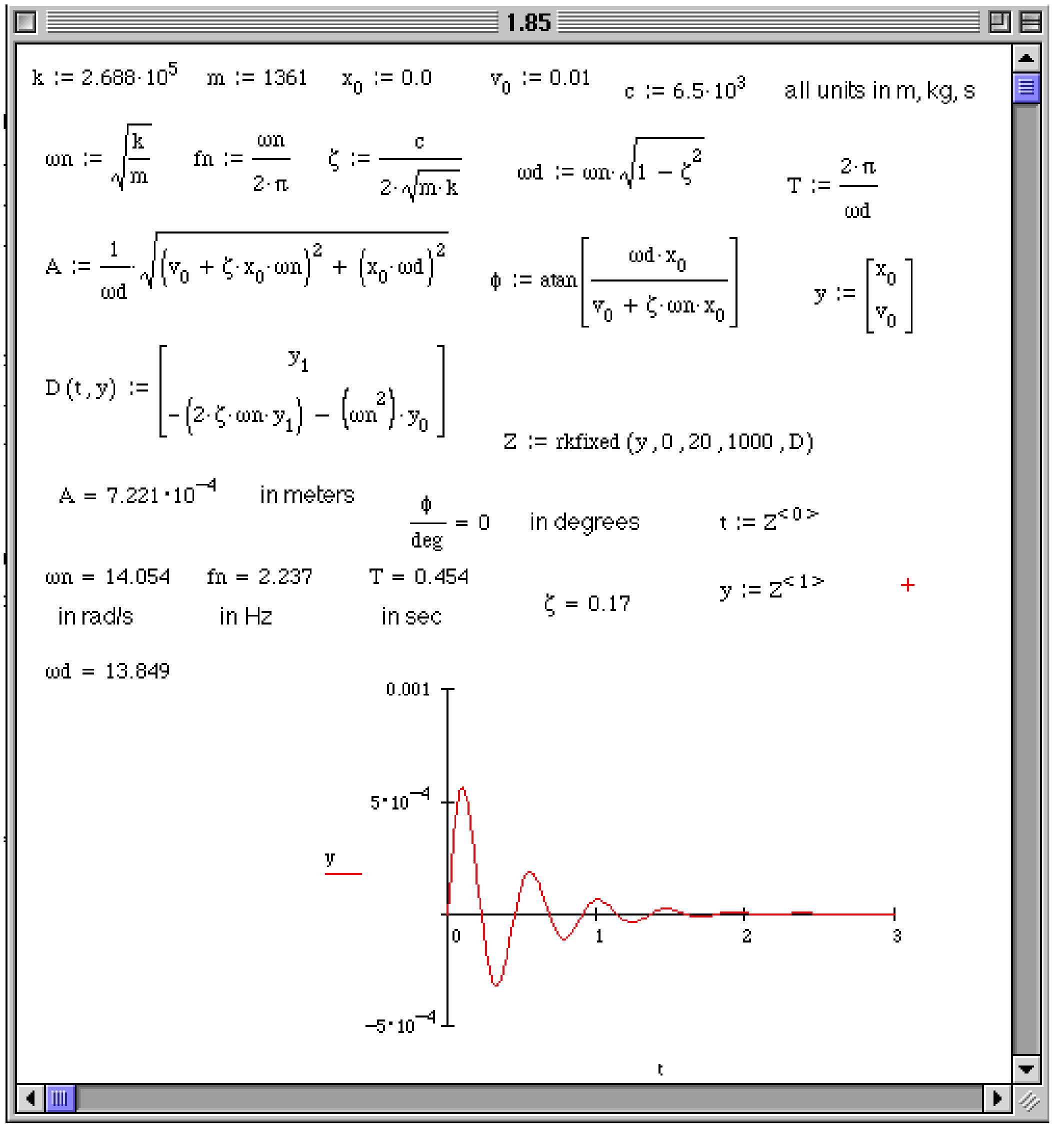
$$i := 0..N$$

$$\begin{bmatrix}x_{i+1}\\v_{i+1}\end{bmatrix} := \begin{bmatrix}v_i\cdot\Delta+x_i\\-\omega n^2\cdot(\Delta)\cdot x_i-2\cdot\zeta\cdot\omega n\cdot v_i\cdot\Delta+v_i\end{bmatrix}$$



1.130* Consider again the damped system of Problem 1.117 and design a damper (that is calculate a value for *c*) such that the oscillation dies out after 2 seconds. There are at least two ways to do this. Here it is intended to solve for the response numerically, following Examples 1.9.2 or 1.9.3, using different values of the damping parameter *c* until the desired response is achieved. Compare this to using the settling time definition of Example 1.3.5.

Solution: Working directly in Mathcad (or use one of the other codes). Changing *c* until the response dies out within about 2 sec yields *c* =6500 kg/s or $\zeta = 0.17$.



From Example 1.3.5

$$T_S = \frac{4}{\zeta\omega_n} \text{ and from Eq. 1.30: } c = 2\zeta m\omega_n$$

Setting T_s=2 s implies $\zeta\omega_n = 2$ so that

$$c = 2m(\zeta\omega_n) = 4m = 4(1361) = 5{,}444 \text{ kg/s}$$

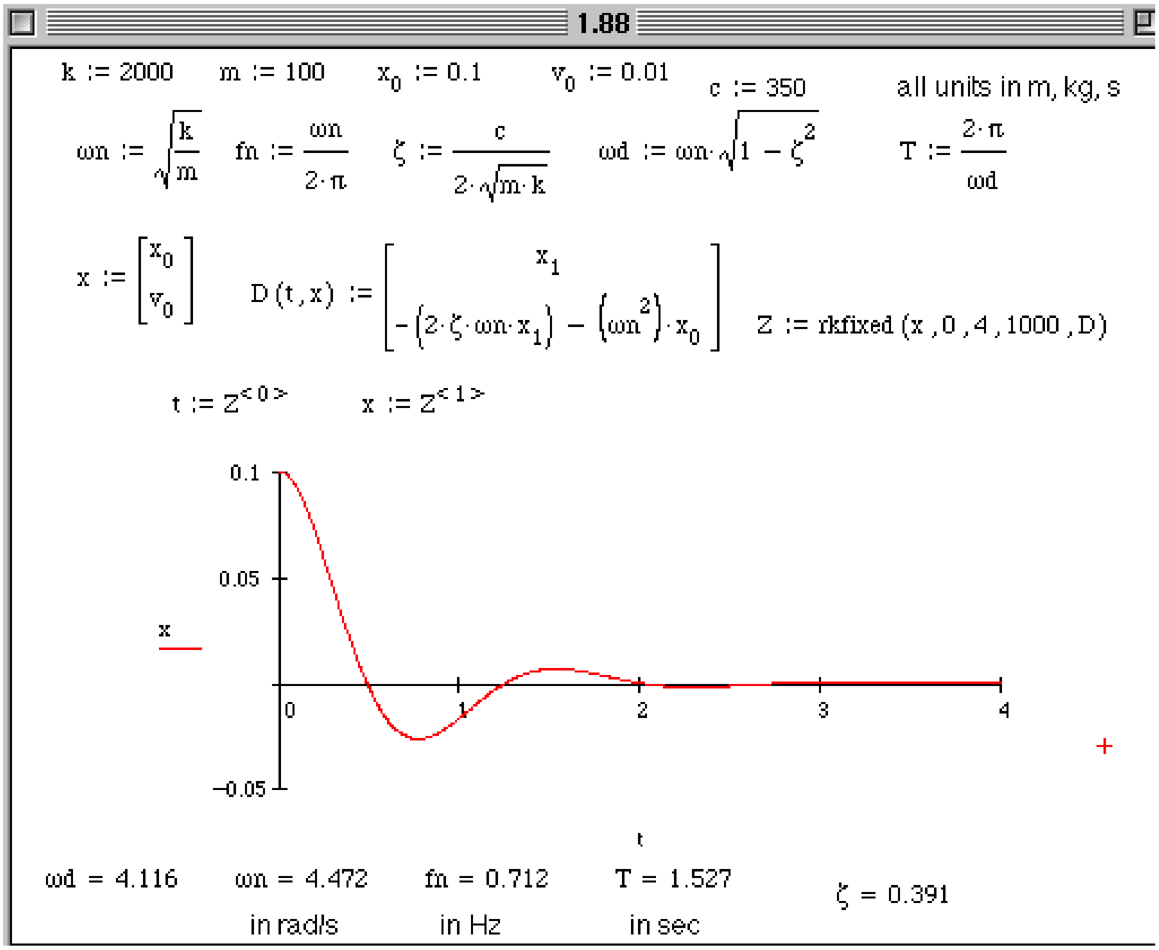
This is consistent with the numerical search but a little tighter.

1.131* Repeat Problem 1.130 for the initial conditions $x(0)$ = 0.1 m and $v(0)$ = 0.01 mm/s.

Solution: Trick question: changing the initial conditions does not change the settling time, which is just a function of ζ and ω_n. Hence the values determined in the previous problem will still reduce the response within 2 seconds.

1.132* A spring and damper are attached to a mass of 100 kg in the arrangement given in Figure 1.10. The system is given the initial conditions $x(0)$ = 0.1 m and $v(0)$ = 1 mm/s. Design the spring and damper (i.e. choose k and c) such that the system will come to rest in 2 s and not oscillate more than two complete cycles. Try to keep c as small as possible. Also compute ζ.

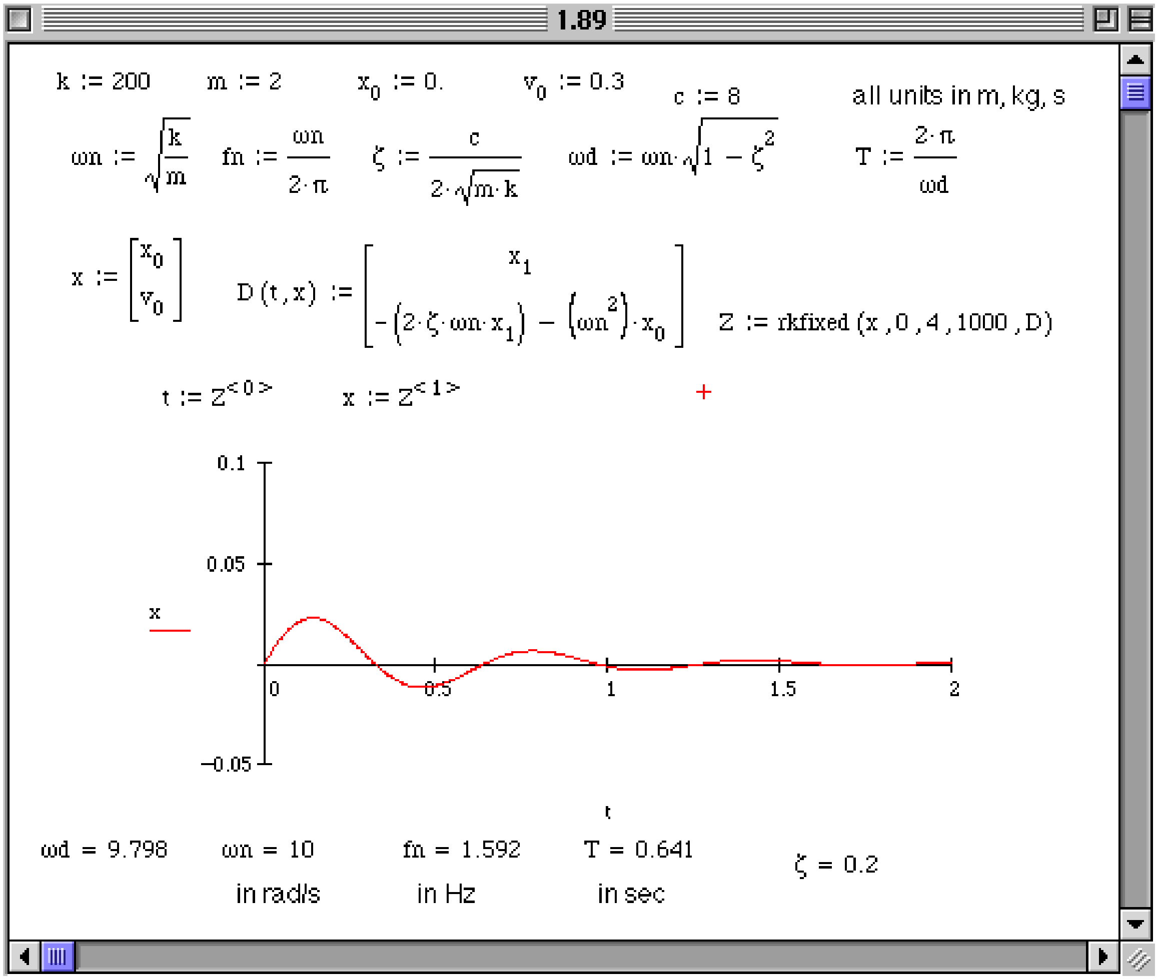
Solution: In performing this numerical search on two parameters, several underdamped solutions are possible. Students will note that increasing k will decrease ζ. But increasing k also increases the number of cycles, which is limited to two. A solution with c = 350 kg/s and k =2000 N/m is illustrated.



1.133* Repeat Example 1.7.1 by using the numerical approach of the previous problems.

Solution: The following Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1.

CLICK HERE TO ACCESS THE COMPLETE Solutions



1.134* Repeat Example 1.7.1 for the initial conditions $x(0) = 0.01$ m and $v(0) = 1$ mm/s.

Solution: The above Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1. For the given values of initial conditions, the solution to Problem 1.133 also works in this case. Note that if $x(0)$ gets too large, this problem will not have a solution.

Problems and Solutions Section 1.10 (1.135 through 1.136)

1.135 A 2-kg mass connected to a spring of stiffness 10^3 N/m has a dry sliding friction force (F_c) of 3 N. As the mass oscillates, its amplitude decreases 20 cm. How long does this take?

Solution: With m = 2kg, and k = 1000 N/m the natural frequency is just

$$\omega_n = \sqrt{\frac{1000}{2}} = 22.36 \text{ rad/s}$$

From equation (1.101): slope = $\dfrac{-2\mu mg\omega_n}{\pi k} = \dfrac{-2F_c\omega_n}{\pi k} = \dfrac{\Delta x}{\Delta t}$

Solving the last equality for Δt yields:

$$\Delta t = \frac{-\Delta x \pi k}{2 f_c \omega_n} = \frac{-(0.20)(\pi)(10^3)}{2(3)(22.36)} = \underline{4.68 \text{ s}}$$

1.136 Consider the system of Figure 1.46 with m = 5 kg and $k = 9 \times 10^3$ N/m with a friction force of magnitude 6 N. If the initial amplitude is 4 cm, determine the amplitude one cycle later as well as the damped frequency.

Solution: Given $m = 5$ kg, $k = 9\times 10^3$ N/m, $f_c = 6$ N, $x_0 = 0.04$ m, the amplitude after one cycle is $x_1 = x_0 - \dfrac{4f_c}{k} = 0.04 - \dfrac{(4)(6)}{9\times 10^3} = \underline{0.0373 \text{ m}}$

Note that the damped natural frequency is the same as the natural frequency in the case of Coulomb damping, hence $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9\times 10^3}{5}} = \underline{42.43 \text{ rad/s}}$

1.137* Compute and plot the response of the system of Figure P1.137 for the case where x_0 = 0.1 m, v_0 = 0.1 m/s, μ_K = 0.05, m = 250 kg, $\theta = 20°$ and k =3000 N/m. How long does it take for the vibration to die out?

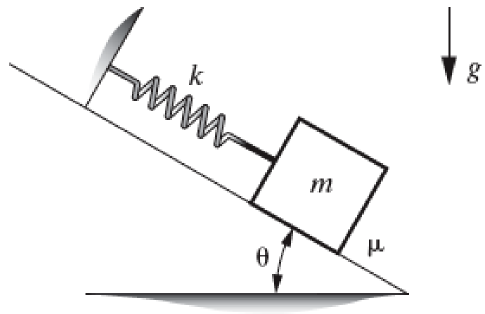
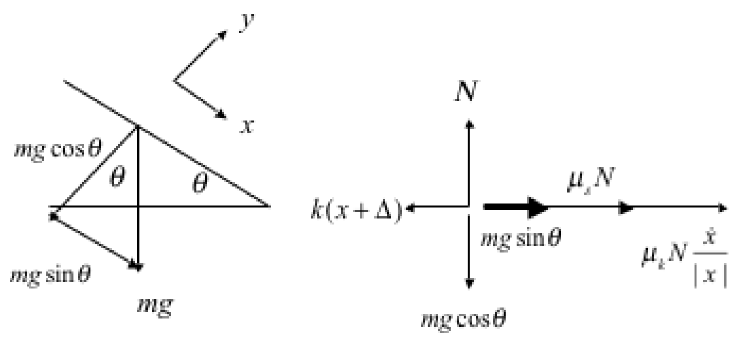


Figure P1.137

Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:



For the static case

$$\sum F_x = 0 \Rightarrow \; k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow \; N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + \mu_s N + mg\sin\theta - \mu_k N\frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta\frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

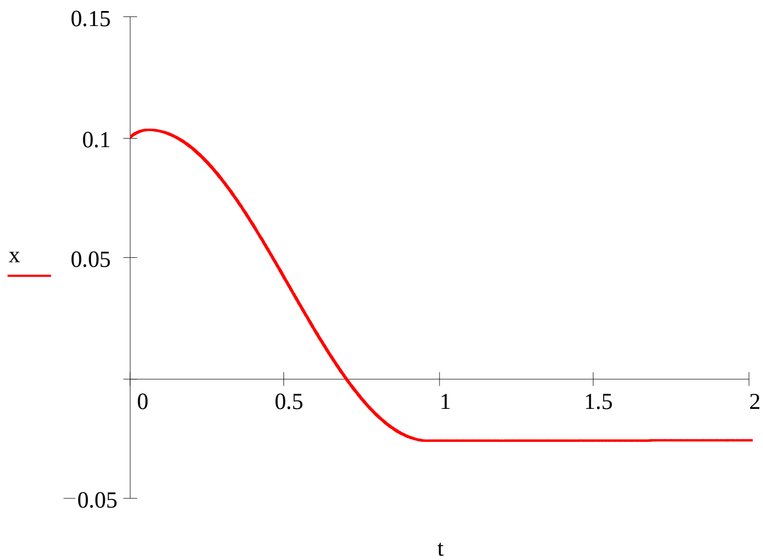
Answer: The oscillation dies out after about 0.9 s. This is illustrated in the following Mathcad code and plot.

$$X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \qquad k := 3000 \qquad m := 250 \qquad m := 0.05$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \cos(20\ \deg) \cdot m \cdot g \cdot \frac{X_1}{\left|X_1\right|} \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 10, 5000, D)$$

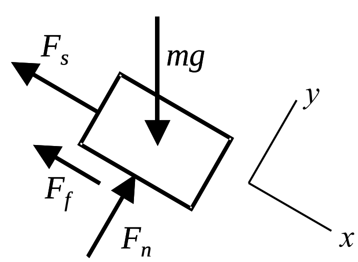
$$t := Z^{<0>} \qquad x := Z^{<1>}$$



Alternate Solution (Courtesy of Prof. Chin An Tan of Wayne State University):

<u>Static Analysis:</u>

In this problem, $x(t)$ is defined as the displacement of the mass from the equilibrium position of the spring-mass system under friction. Thus, the first issue to address is how to determine this equilibrium position, or what is this equilibrium position. In reality, the mass is attached onto an initially unstretched spring on the incline. The free body diagram of the system is as shown. The governing equation of motion is:



where is defined as the displacement measured from the unstretched position of the spring. Note that since the spring is initially unstretched, the spring force is zero initially. If the coefficient of static friction is sufficiently large, i.e., , then the mass remains stationary and the spring is unstretched with the mass-spring-friction in equilibrium. Also, in that case, the friction force , not necessarily equal to the maximum static friction. In other words, these situations may hold at equilibrium: (1) the maximum static friction may not be achieved; and (2) there may be no displacement in the spring at all. In this example, and one would expect that (not given) should be smaller than 0.364 since (very small). Thus, one would expect the mass to move downward initially (due to weight overcoming the maximum static friction). The mass will then likely oscillate and eventually settle into an equilibrium position with the spring stretched.

<u>Dynamic Analysis:</u>

The equation of motion for this system is:

$$m\ddot{x} = -kx - \mu mg\cos\theta \frac{\dot{x}}{|\dot{x}|}$$

where is the displacement measured from the equilibrium position. Define and . Employing the state-space formulation, we transform the original second-order ODE into a set of two first-order ODEs. The state-space equations (for MATLAB code) are:

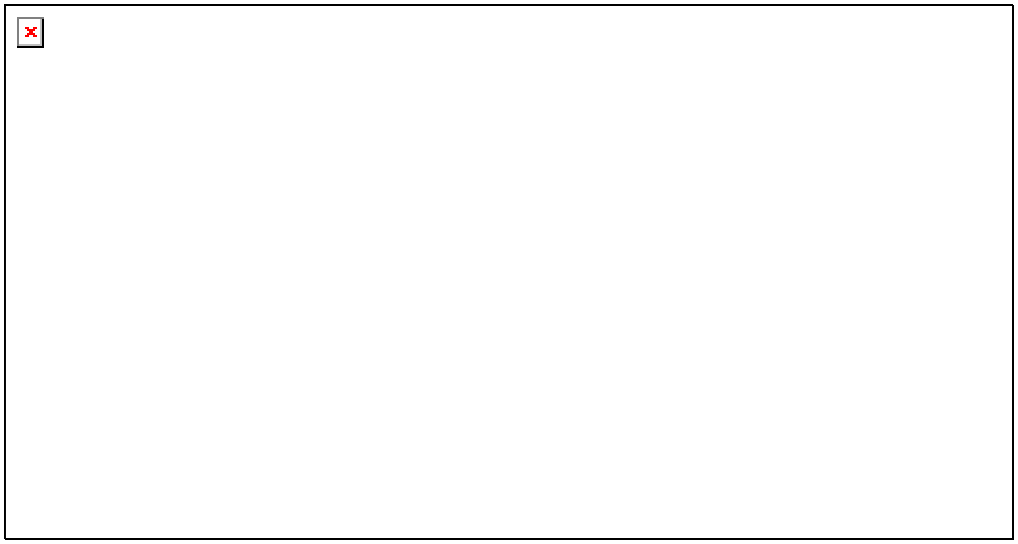
MATLAB Code:

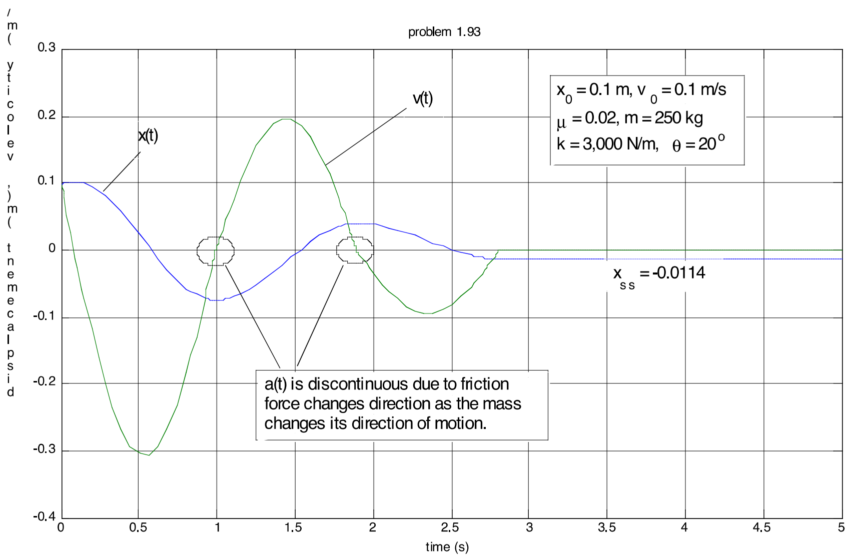
```
x0=[0.1, 0.1];
ts=[0, 5];
[t,x]=ode45('f1_93',ts,x0);
plot(t,x(:,1), t,x(:,2))
title('problem 1.93'); grid on;
xlabel('time (s)');ylabel('displacement (m), velocity (m/s)');

%--------------------------------------------
function xdot = f1_93(t,x)
% computes derivatives for the state-space ODEs
m=250; k=3000; mu=0.05; g=9.81;
angle = 20*pi/180;
xdot(1) = x(2);
xdot(2) = -k/m*x(1) - mu*g*cos(angle)*sign(x(2));
% use the sign function to improve computation time
xdot = [xdot(1); xdot(2)];
```

Plots for and cases are shown. From the simulation results, the oscillation dies out after about 0.96 seconds (using `ginput(1)` command to estimate). Note that the acceleration may be discontinuous at due to the nature of the friction force.

<u>Effects of μ</u>:

Comparing the figures, we see that reducing μ leads to more oscillations (takes longer time to dissipate the energy). Note that since there is a positive initial velocity, the mass is bounded to move down the incline initially. However, if μ is sufficiently large, there may be no oscillation at all and the mass will just come to a stop (as in the case of). This is analogous to an overdamped mass-damper-spring system. On the other hand, when μ is very small (say, close to zero), the mass will oscillate for a long time before it comes to a stop.





<u>Discussion on the ceasing of motion:</u>

Note that when motion ceases, the mass reaches another state of equilibrium. In both simulation cases, this occurs while the mass is moving upward (negative velocity). Note that the steady-state value of $x(t)$ is very small, suggesting that this is indeed the true equilibrium position, which represents a balance of the spring force, weight component along the incline, and the static friction.

1.138* Compute and plot the response of a system with Coulomb damping of equation (1.90) for the case where x_0 = 0.5 m, v_0 = 0, μ = 0.1, *m* = 100 kg and *k* =1500 N/m. How long does it take for the vibration to die out?

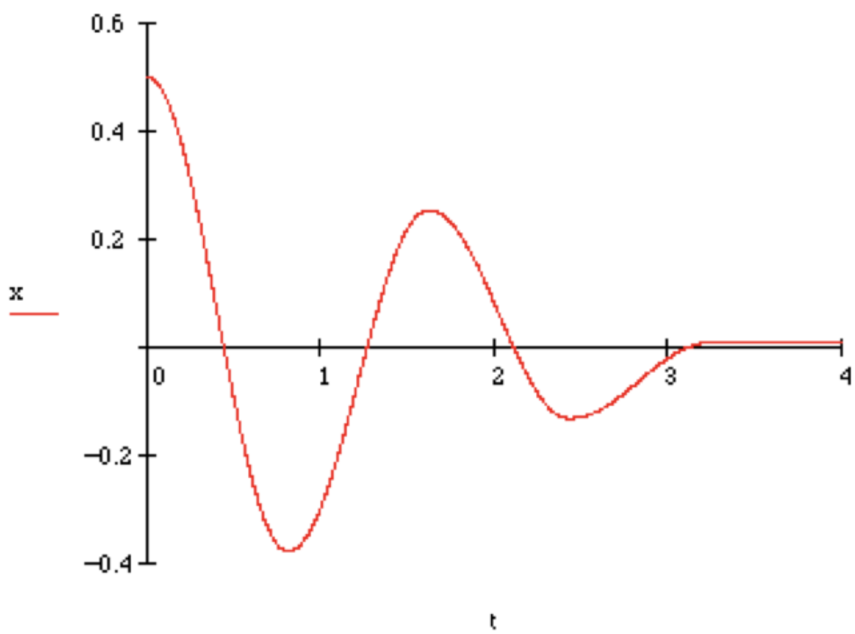
Solution: Here the solution is computed in Mathcad using the following code. Any of the codes may be used. The system dies out in about 3.2 sec.

$X := \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$ $\quad k := 1500 \quad m := 100 \quad \mu := 0.1$

$$D(t, X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \cos(20 \text{ deg}) \cdot \mu \cdot g \cdot \frac{X_1}{|X_1|} \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 5000, D)$

$t := Z^{\langle 0 \rangle} \quad x := Z^{\langle 1 \rangle}$



1.139* A mass moves in a fluid against sliding friction as illustrated in Figure P1.139. Model the damping force as a slow fluid (i.e., linear viscous damping) plus Coulomb friction because of the sliding, with the following parameters: m = 250 kg, μ =0.01, c = 25 kg/s and k =3000 N/m . a) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 0.1 m/s. b) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 1 m/s. How long does it take for the vibration to die out in each case?

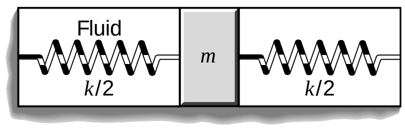
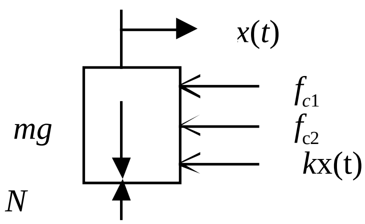


Figure P1.139

Solution: A free-body diagram yields the equation of motion.



$$m\ddot{x}(t)+\mu mg\,\mathrm{sgn}(\dot{x})+c\dot{x}(t)+kx(t)=0$$

where the vertical sum of forces gives the magnitude $\mu N=\mu mg$ for the Coulomb force as in figure 1.43.

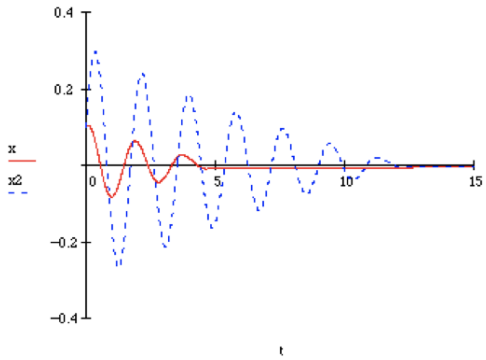
The equation of motion can be solved by using any of the codes mentioned or by using the toolbox. Here a Mathcad session is presented using a fixed order Runge Kutta integration. Note that the oscillations die out after 4.8 seconds for v_0=0.1 m/s for the larger initial velocity of v_0=1 m/s the oscillations go on quite a bit longer ending only after about 13 seconds. While the next problem shows that the viscous damping can be changed to reduce the settling time, this example shows how dependent the response is on the value of the initial conditions. In a linear system the settling time, or time it takes to die out is only dependent on the system parameters, not the initial conditions. This makes design much more difficult for nonlinear systems.

$c := 25 \quad k := 3000 \quad m := 250 \quad \mu := 0.01 \quad X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad X2 := \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$

$$D(t,X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \mu \cdot g \cdot \frac{X_1}{|X_1|} - \frac{c}{m} \cdot X_1 \end{bmatrix} \qquad D2(t,X2) := \begin{bmatrix} X2_1 \\ \frac{-k}{m} \cdot X2_0 - \mu \cdot g \cdot \frac{X2_1}{|X2_1|} - \frac{c}{m} \cdot X2_1 \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 15, 3000, D) \qquad Z2 := \text{rkfixed}(X2, 0, 15, 3000, D2)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad x2 := Z2^{\langle 1 \rangle}$



1.140* Consider the system of Problem 1.39 part (a), and compute a new damping coefficient, *c*, that will cause the vibration to die out after one oscillation.

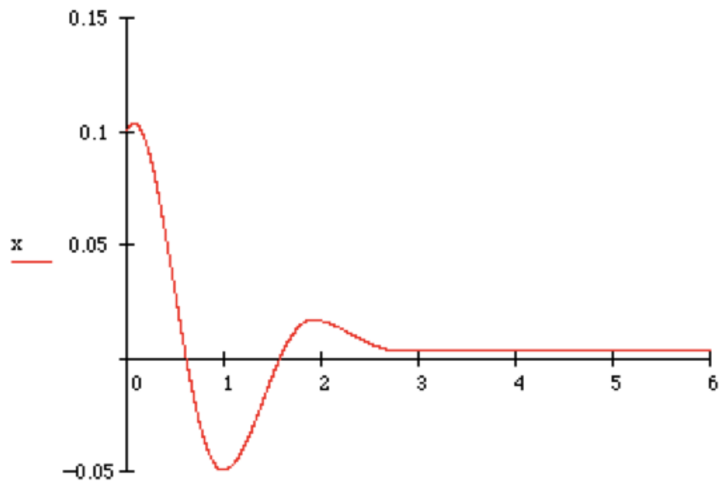
Solution: Working in any of the codes, use the simulation from the last problem and change the damping coefficient *c* until the desired response is obtained. A Mathcad solution is given which requires an order of magnitude higher damping coefficient,

$$\underline{c = 275 \text{ kg/s}}$$

$X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ $\quad k := 3000 \quad m := 250 \quad \mu := 0.01 \quad c := 275$

$$D(t, X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \mu \cdot g \cdot \frac{X_1}{|X_1|} - \frac{c}{m} \cdot X_1 \end{bmatrix} \qquad Z := \text{rkfixed}(X, 0, 6, 3000, D)$$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$



1.141 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x + \beta x^2 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 - \beta x_1^2$$

The equilibrium points are computed from:

$$x_2 = 0$$
$$-\omega_n^2 x_1 - \beta x_1^2 = 0$$

Solving yields the two equilibrium points:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\omega_n^2/\beta \\ 0 \end{bmatrix}$$

1.142 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x - \beta^2 x^3 + \gamma x^5 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5$$

The equilibrium points are computed from:

$$x_2 = 0$$
$$-\omega_n^2 x_1 + \beta^2 x_1^2 - \gamma x_1^5 = 0$$

Solving yields the five equilibrium points (one for each root of the previous equation). The first equilibrium (the linear case) is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Next divide by x_1 to obtain:

which is quadratic in x_1^2 and has the following roots which define the remaining four equilibrium points: $x_2 = 0$ and

$$x_1 = \pm\sqrt{\frac{-\beta^2 + \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}$$

$$x_1 = \pm\sqrt{\frac{-\beta^2 - \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}$$

Thus there are 5 equilibrium. Of course some disappear for certain combinations of the coefficients.

1.143* Consider the pendulum example 1.10.3 with length of 1 m an initial conditions of $\theta_0 = \pi/10$ rad and $\dot{\theta}_0 = 0$. Compare the difference between the response of the linear version of the pendulum equation (i.e. with $\sin(\theta) = \theta$) and the response of the nonlinear version of the pendulum equation by plotting the response of both for four periods.

Solution: First consider the linear solution. Using the formula's given in the text the solution of the linear system is just: $q(t) = 0.314\sin(3.132t + p/2)$. The following Mathcad code, plots the linear solution on the same plot as a numerical solution of the nonlinear system.

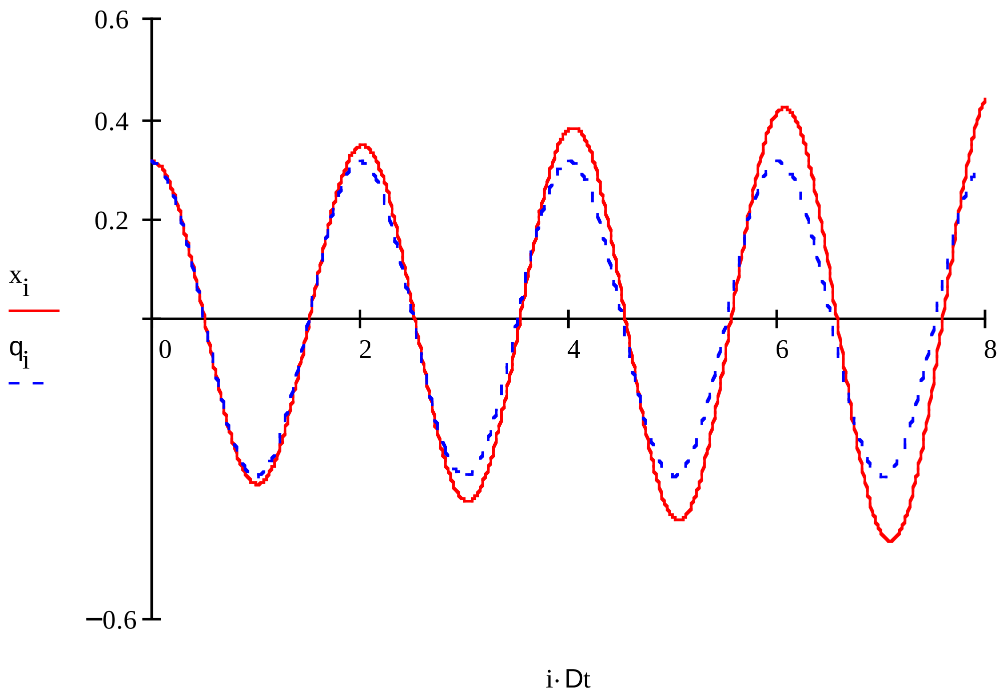
$$i := 0 .. 800$$

$$Dt := 0.01$$

$$q_i := 0.314 \cdot \sin\left(3.132 \cdot Dt \cdot i + \frac{p}{2}\right)$$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} \frac{p}{10} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot Dt \\ v_i - Dt \cdot \left(\sin\left(x_i\right)\right) 9.81 \end{bmatrix}$$



Note how the amplitude of the nonlinear system is growing. The difference between the linear and the nonlinear plots are a function of the ration of the linear spring stiffness and the nonlinear coefficient, and of course the size of the initial condition. It is work it to investigate the various possibilities, to learn just when the linear approximation completely fails.

1.144* Repeat Problem 1.143 if the initial displacement is $\theta_0 = \pi/2$ rad.

Solution: The solution in Mathcad is:

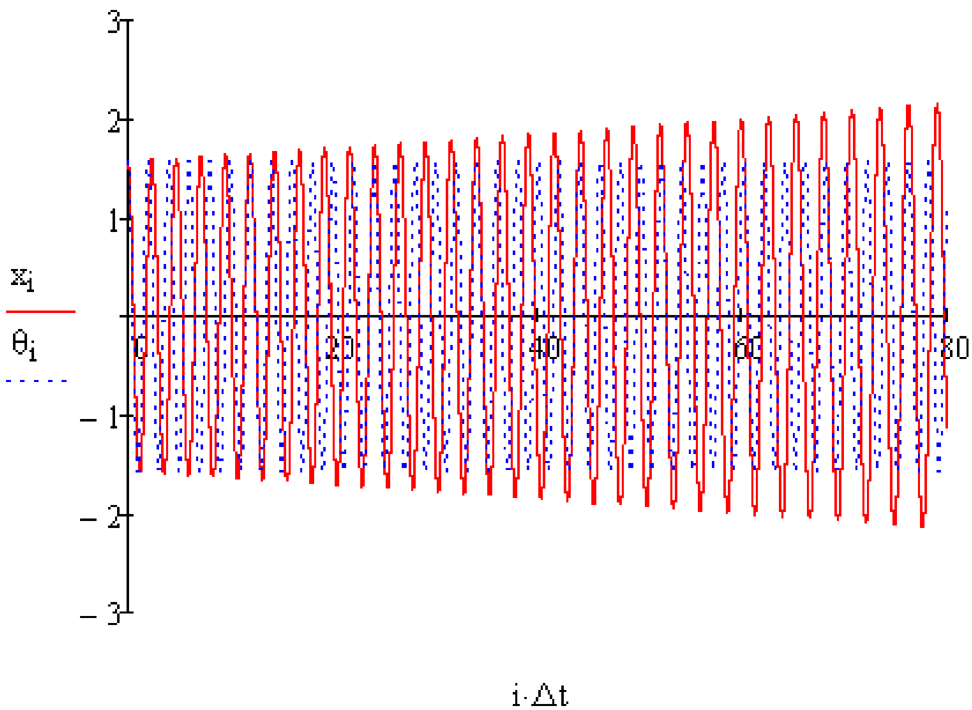
$$i := 0..80000 \qquad \omega := \sqrt{9.81}$$

$$\Delta t := 0.001$$

$$\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} := \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} \qquad \theta_i := \frac{\pi}{2} \cdot \sin\left(\omega \cdot \Delta t \cdot i + \frac{\pi}{2}\right)$$

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin\left(x_i\right)\right) 9.81 \end{bmatrix}$$

Here both solutions oscillate around the “stable” equilibrium, but the nonlinear solution is not oscillating at the natural frequency and is increasing in amplitude.



1.145 If the pendulum of Example 1.10.3 is given an initial condition near the equilibrium position of $\theta_0 = \pi$ rad and $\dot{\theta}_0 = 0$, does it oscillate around this equilibrium?

Solution The pendulum will not oscillate around this equilibrium as it is unstable. Rather it will "wind" around the equilibrium as indicated in the solution to Example 1.10.4.

1.46* Calculate the response of the system of Problem 1.132 for the initial conditions of $x_0 = 0.01$ m, $v_0 = 0$, and a natural frequency of 3 rad/s and for $\beta = 100$, $\gamma = 0$.

Solution: In Mathcad the solution is given using a simple Euler integration as follows:

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \qquad w := 3 \qquad A := \frac{1}{w} \cdot \sqrt{w^2 \cdot (x_0)^2}$$

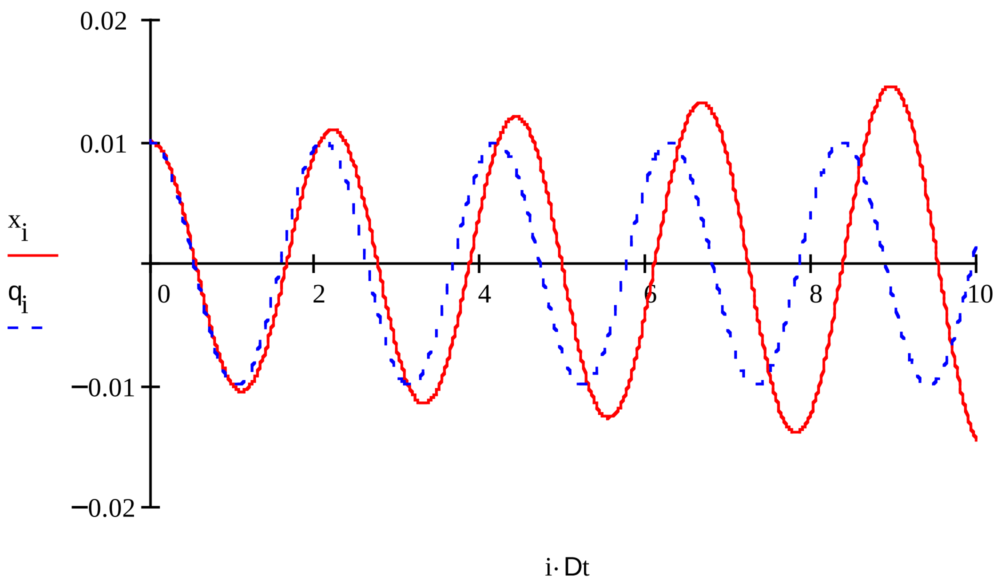
$\beta := 100$

$i := 0 .. 1000$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot Dt \\ v_i - Dt \cdot \left[w^2 \cdot x_i - b^2 \cdot (x_i)^3 \right] \end{bmatrix}$$

$$q_i := A \cdot \sin\left(3 \cdot Dt \cdot i + \frac{p}{2}\right)$$

This is the linear solution $\theta(t)$



The other codes may be used to compute this solution as well.

1.147* Repeat problem 1.146 and plot the response of the linear version of the system (β =0) on the same plot to compare the difference between the linear and nonlinear versions of this equation of motion.

Solution: The solution is computed and plotted in the solution of Problem 1.146. Note that the linear solution starts out very close to the nonlinear solution. The two solutions however diverge. They look similar, but the nonlinear solution is growing in amplitude and period.