Solutions for Numerical Methods with Chemical Engineering Applications 1st Edition by Dorfman

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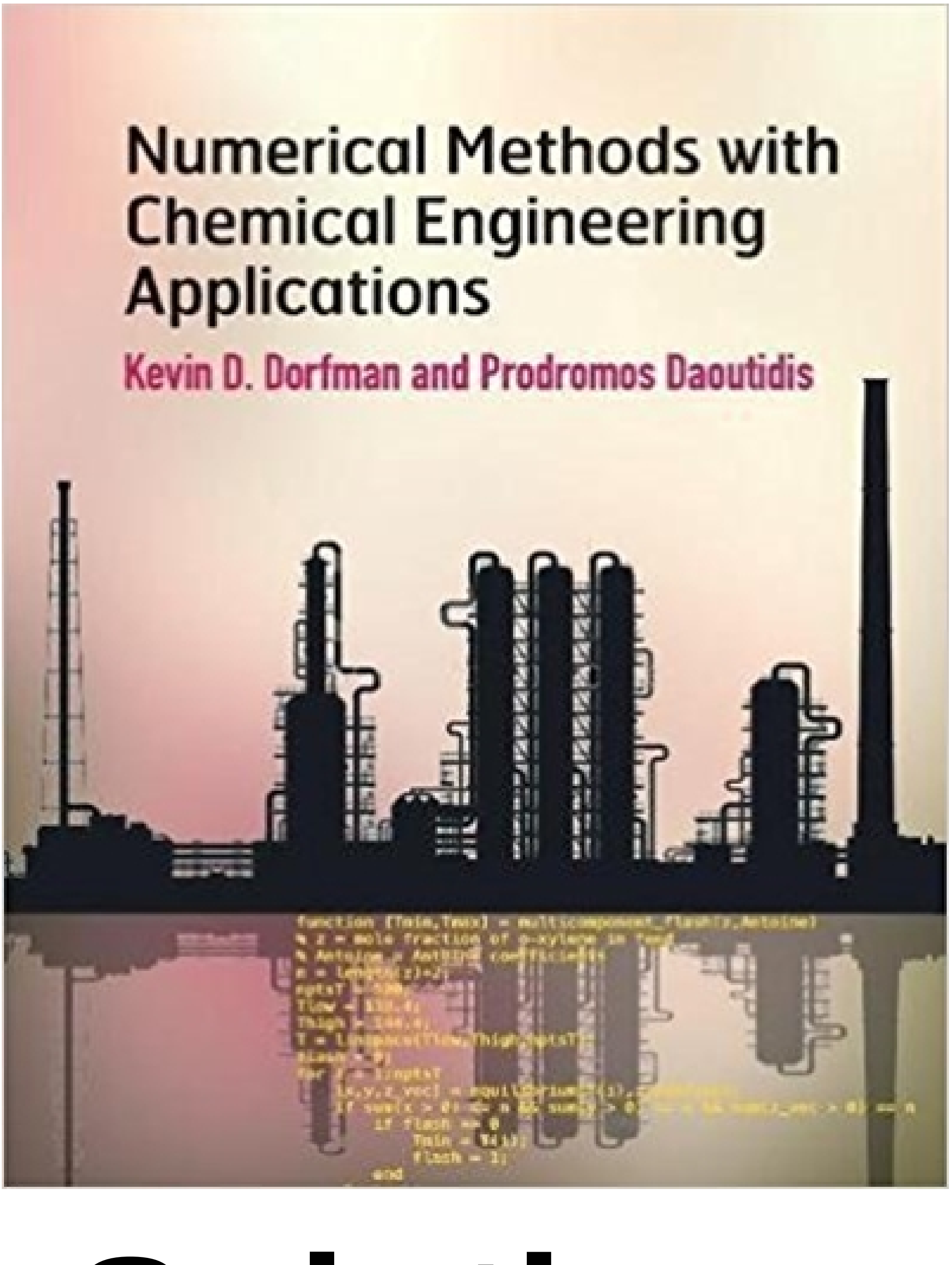


Solutions

2 Linear equations

Problems

(2.1) The form of the co-factor expansion is

$$\begin{aligned}\det \mathbf{A} &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= 2(-1)^4 \det\begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix} + 3(-1)^5 \det\begin{bmatrix} 6 & 5 \\ 2 & 5 \end{bmatrix} + 2(-1)^6 \det\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \\ &= 2 \det\begin{bmatrix} 3 & 5 \\ 1 & 5 \end{bmatrix} - 3 \det\begin{bmatrix} 6 & 5 \\ 2 & 5 \end{bmatrix} + 2 \det\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\end{aligned}$$

(2.2) $-2 \det\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} - \det\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} + 3 \det\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$

(2.3) -10 $(4 \times 1 \times -1/4 \times 10 = -10)$

(2.4) Cofactor expansion yields

$$\begin{aligned}\det \mathbf{A} &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 2(-1)^3 \det\begin{bmatrix} 3 & 5 \\ 3 & 2 \end{bmatrix} + (1)(-1)^4 \det\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix} + 5(-1)^5 \det\begin{bmatrix} 6 & 3 \\ 2 & 3 \end{bmatrix} \\ &= -2 \det\begin{bmatrix} 3 & 5 \\ 3 & 2 \end{bmatrix} + \det\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix} - 5 \det\begin{bmatrix} 6 & 3 \\ 2 & 3 \end{bmatrix}\end{aligned}$$

(2.5) Cofactor expansion yields $\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$. The terms are $a_{11} = 3(-1)^{1+1} = 3$, $a_{12} = 5(-1)^{1+2} = -5$, $a_{13} = 6(-1)^{1+3} = 6$ and

$$A_{11} = \det\begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix} = 6$$

$$A_{12} = \det\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = 3$$

$$A_{13} = \det\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = 0$$

Making the substitution gives $\det \mathbf{A} = 3$.

(2.6) The fastest choice is co-factor expansion on the 4th row:

$$\det \mathbf{A} = -2 \det\begin{bmatrix} 6 & -2 & 3 \\ 2 & 4 & -1 \\ 1 & 1 & -1 \end{bmatrix} - 2 \det\begin{bmatrix} 6 & 1 & 3 \\ 2 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix} + \det\begin{bmatrix} 6 & 1 & -2 \\ 2 & -2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

These remaining determinants can be evaluated using the formula for the 3x3 matrix to give $\det \mathbf{A} = 52$.

(2.7) The determinant is -904

(2.8) Cofactor expansion for $i = 3$ has the most zero entries, giving

$$\det \mathbf{A} = (-1)(-1)^{(3+1)} \det \begin{bmatrix} 2 & 1 & 0 & 1 \\ -3 & 2 & 1 & 4 \\ -7 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$
$$+(0)(-1)^{(3+2)} \det \begin{bmatrix} 3 & 1 & 0 & 1 \\ 5 & 2 & 1 & 4 \\ 8 & 2 & 3 & 4 \\ -4 & 4 & 1 & 2 \end{bmatrix}$$
$$+(1)(-1)^{(3+3)} \det \begin{bmatrix} 3 & 2 & 0 & 1 \\ 5 & -3 & 1 & 4 \\ 8 & -7 & 3 & 4 \\ -4 & 3 & 1 & 2 \end{bmatrix}$$
$$+(0)(-1)^{(3+4)} \det \begin{bmatrix} 3 & 2 & 1 & 1 \\ 5 & -3 & 2 & 4 \\ 8 & -7 & 2 & 4 \\ -4 & 3 & 4 & 2 \end{bmatrix}$$
$$+(3)(-1)^{(3+5)} \det \begin{bmatrix} 3 & 2 & 1 & 0 \\ 5 & -3 & 2 & 1 \\ 8 & -7 & 2 & 3 \\ -4 & 3 & 4 & 1 \end{bmatrix}$$

which reduces to

$$\det \mathbf{A} = -\det \begin{bmatrix} 2 & 1 & 0 & 1 \\ -3 & 2 & 1 & 4 \\ -7 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix} + \det \begin{bmatrix} 3 & 2 & 0 & 1 \\ 5 & -3 & 1 & 4 \\ 8 & -7 & 3 & 4 \\ -4 & 3 & 1 & 2 \end{bmatrix} + 3 \det \begin{bmatrix} 3 & 2 & 1 & 0 \\ 5 & -3 & 2 & 1 \\ 8 & -7 & 2 & 3 \\ -4 & 3 & 4 & 1 \end{bmatrix}$$

You could also expand on column $j = 4$ instead. Recall that $\det \mathbf{A} = \det \mathbf{A}^{\dagger}$, so expansions on columns are the same as expansions on rows.

(2.9) No solution (rows 2 and 3 are inconsistent)

(2.10) Use co-factor expansion on the last row of $\mathbf{U}$. The only non-zero entry is $U_{n,n}$. As a result, the determinant after one co-factor expansion is

$$\det \mathbf{U} = U_{n,n}(-1)^{2n} \det \mathbf{U}_{n-1}$$

where $\mathbf{U}_{n-1}$ is an $(n-1) \times (n-1)$ upper triangular matrix formed by removing the last row and last column from $\mathbf{U}$. Since $2n$ is even for any n, we have

$$\det \mathbf{U} = U_{n,n} \det \mathbf{U}_{n-1}$$

If we now do the same co-factor expansion on the last row of $\mathbf{U}_{n-1}$, we have

$$\det \mathbf{U} = U_{n,n} U_{n-1,n-1}(-1)^{2(n-1)} \det \mathbf{U}_{n-2}$$

where $\mathbf{U}_{n-2}$ is an $(n-2) \times (n-2)$ upper triangular matrix formed by removing the last row and last column from $\mathbf{U}_{n-1}$. The sign is again positive,

$$\det \mathbf{U} = U_{n,n} U_{n-1,n-1} \det \mathbf{U}_{n-2}$$

If we continue at each step $j = 0, 1, \ldots, n-1$ with co-factor expansion on the $(n-j)$th row of $\mathbf{U}_{n-j}$, then we get the desired result:

$$\det \mathbf{U} = \prod_{i=1}^{n} U_{i,i}$$

(2.11) The matrix determinants are

$$\det \mathbf{A} = \det \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} = 5$$

$$\det \mathbf{A}_1 = \det \begin{bmatrix} 13 & 1 & 3 \\ 8 & 2 & 1 \\ 20 & 3 & 4 \end{bmatrix} = 5$$

$$\det \mathbf{A}_2 = \det \begin{bmatrix} 2 & 13 & 3 \\ 1 & 8 & 1 \\ 2 & 20 & 4 \end{bmatrix} = 10$$

$$\det \mathbf{A}_3 = \det \begin{bmatrix} 2 & 1 & 13 \\ 1 & 2 & 8 \\ 2 & 3 & 20 \end{bmatrix} = 15$$

Cramer's rule says that $x_i = \det \mathbf{A}_i / \det \mathbf{A}$, so we have $x = 1$, $y = 2$ and $z = 3$.

(2.12) The solutions are

$$x = \frac{\det \begin{bmatrix} 1 & 1 & -1 \\ 4 & 0 & 2 \\ 0 & -2 & 3 \end{bmatrix}}{\det \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 4 & -2 & 3 \end{bmatrix}} = 0$$

$$y = \frac{\det \begin{bmatrix} 2 & 1 & -1 \\ -1 & 4 & 2 \\ 4 & 0 & 3 \end{bmatrix}}{\det \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 4 & -2 & 3 \end{bmatrix}} = \frac{51}{17} = 3$$

$$z = \frac{\det\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 4 \\ 4 & -2 & 0 \end{bmatrix}}{\det\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 4 & -2 & 3 \end{bmatrix}} = \frac{34}{17} = 2$$

(2.13) We need to compute a bunch of 3×3 determinants:

$$\det\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & -3 \end{bmatrix} = 20$$

$$\det\begin{bmatrix} 5 & 2 & 1 \\ 5 & -1 & 2 \\ 1 & 0 & -3 \end{bmatrix} = 40$$

$$\det\begin{bmatrix} 3 & 5 & 1 \\ 1 & 5 & 2 \\ 1 & 1 & -3 \end{bmatrix} = -20$$

$$\det\begin{bmatrix} 3 & 2 & 5 \\ 1 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = 20$$

This gives $x = 2$, $y = -1$ and $z = 1$

(2.14) 1 To solve the system by Cramers rule we must compute 4 determinants first

$$|A| = det\begin{bmatrix} 3 & 2 & 3 \\ 4 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = 44$$

Next

$$|A_1| = det\begin{bmatrix} 8 & 2 & 3 \\ 2 & 2 & 2 \\ 9 & 1 & 2 \end{bmatrix} = 44$$

and

$$|A_2| = det\begin{bmatrix} 3 & 8 & 3 \\ 4 & 2 & 2 \\ 1 & 9 & 2 \end{bmatrix} = 88$$

Finally

$$|A_3| = det\begin{bmatrix} 3 & 2 & 8 \\ 4 & 2 & 2 \\ 1 & 1 & 9 \end{bmatrix} = 132$$

Then we find the values of x_1,x_2,and x_3 as follows

$$x_1 = \frac{|A_1|}{|A|} = 1$$
$$x_2 = \frac{|A_2|}{|A|} = 2$$
$$x_3 = \frac{|A_3|}{|A|} = 3$$

2 For Gauss elimination first we right the augmented matrix:

$$\begin{bmatrix} 3 & 2 & 3 & 8 \\ 4 & 2 & 2 & 2 \\ 1 & 1 & 2 & 9 \end{bmatrix}$$

eliminating the first row yeilds

$$\begin{bmatrix} 3 & 2 & 3 & 8 \\ 0 & \frac{14}{3} & -6 & \frac{-26}{3} \\ 0 & \frac{5}{3} & 1 & \frac{19}{3} \end{bmatrix}$$

and the second

$$\begin{bmatrix} 3 & 2 & 3 & 8 \\ 0 & \frac{14}{3} & -6 & \frac{-26}{3} \\ 0 & 0 & \frac{22}{7} & \frac{66}{7} \end{bmatrix}$$

forward elimination yeilds$x_3 = 3$, $x_2 = 2$, and $x_1 = 1$. The same solution found using Cramers rule.

(2.15) The number of operations for each step are:

(a) The determinant of a 2×2 matrix is $ad - bc$. This calculation can be completed with two multiplications and a subtraction. So this requires 3 operations. For later use, let's call this value q.

(b) For cofactor expansion of a 3×3 matrix, we end up with 3 matrices of size 2×2 and a pre-factor that multiplies the determinant of each of these 2×2 matrices. So we have to do $3 \times (1 + 3) = 12$ operations. In terms of q, this is $3 \times (1 + q)$ operations.

(c) For cofactor expansion of a 4×4 matrix, we end up with 4 matrices of size 3×3 and a pre-factor that multiplies the determinant of each of the 3×3 matrices. So we have to do $4 \times (1 + 12) = 52$ operations. To make the next step easier, it is convenient to write this as $4 \times (1 + [3 \times (1 + 3)])$. If we rewrite in terms of the starting value of $n = 4$, this becomes $n \times [1 + (n - 1) \times (1 + q)]$.

(d) For cofactor expansion of a 5×5 matrix ($n = 5$,) the last equation will now be $n \times [1 + (n - 1) \times [1 + (n - 2) \times (1 + q)]]$. To see how this is the answer, we need to see that (i) $(n - 2) \times (1 + q)$ is the effort required to compute the determinant of a 3×3 matrix, (ii) there are $(n-1) = 4$ of these matrices and 1 pre-factor multiplication per matrix, and (iii) the first cofactor expansion will produce an additional pre-factor for each of the five 4×4 matrices.

(e) For $n \to \infty$, we do not need to worry about the effort for the pre-factors (the terms of $+1$ in the last step). If we do so for $n = 5$ from the last step, we see that the effort requires computing $n \times (n-1) \times (n-2)$ determinants of size 2×2, which requires $n \times (n-1) \times (n-2) \times q = n! \times q$ steps. As n gets larger and larger, the estimate of calculating $n!$ determinants of size 2×2 becomes increasingly more accurate.

(2.16) The Gauss elimination steps gives the following set of matrices:

$$\left[\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 0 \\ 3 & 2 & 1 & 0 & -3 \\ 0 & 2 & 3 & 5 & 4 \\ 4 & -2 & 3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 0 \\ 0 & 1/2 & -5/4 & -3/4 & -3 \\ 0 & 2 & 3 & 5 & 4 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right]$$

Although you might want to swap rows at this point, it is not allowed in Gauss elimination unless you are pivoting. So we continue and get

$$\left[\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 0 \\ 0 & 1/2 & -5/4 & -3/4 & -3 \\ 0 & 0 & 8 & 8 & 16 \\ 0 & 0 & -10 & -6 & -24 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 0 \\ 0 & 1/2 & -5/4 & -3/4 & -3 \\ 0 & 0 & 8 & 8 & 16 \\ 0 & 0 & 0 & 4 & -4 \end{array}\right]$$

Applying the back substitution formulas gives

$$z = -1$$
$$y = \frac{16 - 8z}{8} = 3$$
$$x = \frac{-3 + 3z/4 + 5y/4}{1/2} = 0$$
$$w = \frac{-z - 3y - 2x}{4} = -2$$

We could also have done this problem setting the diagonal entries unity. While you get the same answer, this is not used in the numerical implementation of Gauss elimination. If asked to do Gauss elimination, you should not do the following even though it works:

$$\left[\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 0 \\ 3 & 2 & 1 & 0 & -3 \\ 0 & 2 & 3 & 5 & 4 \\ 4 & -2 & 3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 3 & 2 & 1 & 0 & -3 \\ 0 & 2 & 3 & 5 & 4 \\ 4 & -2 & 3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & -5/4 & -3/4 & -3 \\ 0 & 2 & 3 & 5 & 4 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 0 & 1 & -5/2 & -3/2 & -6 \\ 0 & 2 & 3 & 5 & 4 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 0 & 1 & -5/2 & -3/2 & -6 \\ 0 & 0 & 8 & 8 & 16 \\ 0 & 0 & -10 & -6 & -24 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 0 & 1 & -5/2 & -3/2 & -6 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -10 & -6 & -24 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 3/4 & 1/4 & 0 \\ 0 & 1 & -5/2 & -3/2 & -6 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 & -4 \end{array}\right]$$

Applying the back substitution formulas gives

$$\begin{aligned} z &= -1 \\ y &= (2 - z) = 3 \\ x &= -6 - (-3z/2 - 5y/2) = 0 \\ w &= -(z/4 + 3y/4 + x/2) = -2 \end{aligned}$$

(2.17)

$$a_{4,3}^{(2)} = 4 - \frac{3}{2}(3) = \frac{8}{2} - \frac{9}{2} = -\frac{1}{2}$$

(2.18) First write the augmented matrix

$$\begin{bmatrix} 2 & -3 & 1 & 7 \\ 1 & -1 & -2 & -2 \\ 3 & 1 & -1 & 0 \end{bmatrix}$$

Then eliminate the first column

$$\begin{bmatrix} 2 & -3 & 1 & 7 \\ 0 & 1 & -5 & -11 \\ 0 & 11 & -5 & -21 \end{bmatrix}$$

Then the second column

$$\begin{bmatrix} 2 & -3 & 1 & 7 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 50 & 100 \end{bmatrix}$$

Now back sub

$$x_3 = 2$$

$$x_2 = \frac{-11 - (-5)(2)}{-1} = -1$$

$$x_1 = \frac{7 - (1)(2) - (-3)(-1)}{2} = 1$$

(2.19) Starting with the augmented matrix

$$\begin{bmatrix} 5 & 2 & 1 & 3 & 2 \\ 4 & 1 & -1 & -3 & -2 \\ -2 & 3 & -3 & 5 & 8 \end{bmatrix}$$

eliminate the first row

$$\begin{bmatrix} 5 & 2 & 1 & 3 & 2 \\ 0 & -3 & -9 & -27 & -18 \\ 0 & 19 & -13 & 31 & 44 \end{bmatrix}$$

then the second

$$\begin{bmatrix} 5 & 2 & 1 & 3 & 2 \\ 0 & -3 & -9 & -27 & -18 \\ 0 & 0 & -14 & -28 & -14 \end{bmatrix}$$

now backwards elimination first for the solution in column 4

$$\begin{aligned} x_3 &= 2 \\ x_2 &= \frac{-27 - (-9)(2)}{-3} = 3 \\ x_1 &= \frac{3 - (1)(2) - (2)(3)}{5} = -1 \end{aligned}$$

and now for column 5

$$\begin{aligned} x_3 &= 1 \\ x_2 &= \frac{-18 - (-9)(1)}{-3} = 3 \\ x_1 &= \frac{2 - (1)(1) - (2)(3)}{5} = -1 \end{aligned}$$

(2.20) The augemented matrix for this problem is

$$\left[\begin{array}{ccc|ccc} 5 & 6 & -3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 4 & 2 & -6 & 0 & 0 & 1 \end{array}\right]$$

eliminating the first row

$$\left[\begin{array}{ccc|ccc} 5 & 6 & -3 & 1 & 0 & 0 \\ 0 & 9/5 & 8/5 & -1/5 & 1 & 0 \\ 0 & -14/5 & -18/5 & -4/5 & 0 & 1 \end{array}\right]$$

Then the second

$$\left[\begin{array}{ccc|ccc} 5 & 6 & -3 & 1 & 0 & 0 \\ 0 & 9/5 & 8/5 & -1/5 & 1 & 0 \\ 0 & 0 & -10/9 & -10/9 & 14/9 & 1 \end{array}\right]$$

Solving for all three by back substitution give the following

$$A^{-1} = \left[\begin{array}{ccc} 2 & -3 & -3/5 \\ 1 & 9/5 & 4/5 \\ 1 & -7/5 & -9/10 \end{array}\right]$$

(2.21) The second and fourth rows are switched to give

$$\begin{bmatrix} 1 & 2 & 7 & 2 & 4 \\ 0 & 4 & 1 & 8 & 4 \\ 0 & 3 & 4 & 1 & 5 \\ 0 & 2 & 3 & 1 & 8 \end{bmatrix}$$

(2.22) Swap row 6 with row 3

(2.23) There is no pivoting at the first iteration (note that we are adding an extra step to make the diagonal entries unity; this is not used in the numerical implementation of Gauss elimination:

$$\left[\begin{array}{cccc|c} 4 & 2 & 2 & 1 & -2 \\ 2 & 1 & 3 & 0 & -9 \\ 0 & 2 & 2 & 3 & 2 \\ -1 & 3 & 2 & 3 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 2 & 1 & 3 & 0 & -9 \\ 0 & 2 & 2 & 3 & 2 \\ -1 & 3 & 2 & 3 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 0 & 2 & -1/2 & -8 \\ 0 & 2 & 2 & 3 & 2 \\ 0 & 7/2 & 5/2 & 13/4 & -3/2 \end{array}\right]$$

Make a partial pivot step and eliminate:

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 7/2 & 5/2 & 13/4 & -3/2 \\ 0 & 2 & 2 & 3 & 2 \\ 0 & 0 & 2 & -1/2 & -8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 5/7 & 13/14 & -3/7 \\ 0 & 2 & 2 & 3 & 2 \\ 0 & 0 & 2 & -1/2 & -8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 5/7 & 13/14 & -3/7 \\ 0 & 0 & 4/7 & 8/7 & 20/7 \\ 0 & 0 & 2 & -1/2 & -8 \end{array}\right]$$

Make another partial pivot step and eliminate:

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 5/7 & 13/14 & -3/7 \\ 0 & 0 & 2 & -1/2 & -8 \\ 0 & 0 & 4/7 & 8/7 & 20/7 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 5/7 & 13/14 & -3/7 \\ 0 & 0 & 1 & -1/4 & -4 \\ 0 & 0 & 4/7 & 8/7 & 20/7 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 1/4 & -1/2 \\ 0 & 1 & 5/7 & 13/14 & -3/7 \\ 0 & 0 & 1 & -1/4 & -4 \\ 0 & 0 & 0 & 9/7 & 36/7 \end{array}\right]$$

Applying the back substitution formula gives

$$\begin{aligned} z &= (36/7)/(9/7) = 4 \\ y &= -4 + z/4 = -3 \\ x &= -3/7 - (13z/14 + 5y/7) = -2 \\ w &= -1/2 - (z/4 + y/2 + x/2) = 1 \end{aligned}$$

(2.24) Need to swap the first and second rows to get

$$\begin{bmatrix} 10 & 5 & 2 & 3 & -5 & 11 \\ -1 & 3 & -2 & -3 & 2 & -5 \\ -3 & 2 & 9 & 0 & 4 & 1 \\ 3 & 1 & -3 & 2 & 7 & 14 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}$$

Elimination gives

$$\begin{bmatrix} 10 & 5 & 2 & 3 & -5 & 11 \\ 0 & 3.5 & -1.8 & -2.7 & 1.5 & -3.9 \\ 0 & 3.5 & 9.6 & 0.9 & 2.5 & 4.3 \\ 0 & -0.5 & -3.6 & 1.1 & 8.5 & 10.7 \\ 0 & 0.5 & 0.8 & 0.7 & 1.5 & 2.9 \end{bmatrix}$$

The biggest entry is on the diagonal, so we can eliminate to get

$$\begin{bmatrix} 10 & 5 & 2 & 3 & -5 & 11 \\ 0 & 3.5 & -1.8 & -2.7 & 1.5 & -3.9 \\ 0 & 0 & 11.4 & 3.6 & 1 & 8.2 \\ 0 & 0 & -3.8571 & 0.7143 & 8.7143 & 10.1429 \\ 0 & 0 & 1.0571 & 1.0857 & 1.2857 & 3.4571 \end{bmatrix}$$

The biggest entry is on the diagonal, so we can eliminate to get

$$\begin{bmatrix} 10 & 5 & 2 & 3 & -5 & 11 \\ 0 & 3.5 & -1.8 & -2.7 & 1.5 & -3.9 \\ 0 & 0 & 11.4 & 3.6 & 1 & 8.2 \\ 0 & 0 & 0 & 1.9323 & 9.0526 & 12.9173 \\ 0 & 0 & 0 & 0.7519 & 1.1930 & 2.6967 \end{bmatrix}$$

The biggest entry is on the diagonal, so we can eliminate to get

$$\begin{bmatrix} 10 & 5 & 2 & 3 & -5 & 11 \\ 0 & 3.5 & -1.8 & -2.7 & 1.5 & -3.9 \\ 0 & 0 & 11.4 & 3.6 & 1 & 8.2 \\ 0 & 0 & 0 & 1.9323 & 9.0526 & 12.9173 \\ 0 & 0 & 0 & 0 & -2.3294 & -2.3294 \end{bmatrix}$$

Back substitution gives

$$\begin{bmatrix} u \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(2.25) For the Gauss elimination with pivoting, first write as an augmented matrix

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 3 & 4 & 1 & 2 & 8 \\ -1 & 0 & 3 & -2 & 2 \\ -2 & 4 & 3 & 1 & 14 \end{array}\right]$$

There is no pivoting on the first step, so we just eliminate to get

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 0 & 2 & -3 & 7 & 8 \\ 0 & 2/3 & 13/3 & -11/3 & 2 \\ 0 & 16/3 & 17/3 & -7/3 & 14 \end{array}\right]$$

We now pivot by exchanging rows 2 and 4:

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 0 & 16/3 & 17/3 & -7/3 & 14 \\ 0 & 2/3 & 13/3 & -11/3 & 2 \\ 0 & 2 & -3 & 7 & 8 \end{array}\right]$$

and do the elimination:

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 0 & 16/3 & 17/3 & -7/3 & 14 \\ 0 & 0 & 87/24 & -81/24 & 1/4 \\ 0 & 0 & -16 & 18 & 2 \end{array}\right]$$

We need to pivot again:

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 0 & 16/3 & 17/3 & -7/3 & 14 \\ 0 & 0 & -16 & 18 & 2 \\ 0 & 0 & 87/24 & -81/24 & 1/4 \end{array}\right]$$

and do the elimination

$$\left[\begin{array}{cccc|c} 3 & 2 & 4 & -5 & 0 \\ 0 & 16/3 & 17/3 & -7/3 & 14 \\ 0 & 0 & -16 & 18 & 2 \\ 0 & 0 & 0 & 270/384 & 270/384 \end{array}\right]$$

We now use back substitution to get $x_4 = 1$, $x_3 = 1$, $x_2 = 2$ and $x_1 = -1$. The determinant comes from the product of the diagonals and the number of pivots, det **A** = $(-1)^2(-180) = -180$.

(2.26) Using the formula for an upper triangular matrix with pivoting gives

$$\det \mathbf{A} = (-1)^p \prod_i U_{ii} = (-1)^3 4 = -4$$

(2.27) The first step is a pivot to give

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 4 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

The elimination factors are $m_{21} = 1/4$ and $m_{31} = 1/2$. These give the upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} 4 & 6 & 4 \\ 0 & -1/2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Using the formula for the determinant with 1 pivot,

$$\det \mathbf{A} = (-1)^1(4)(-1/2)(2) = 4$$

(2.28) B = n - p
C = k + 1
D = k + p
E = k + 1
F = k + p
G = n - p + 1
H = k + 1
L = n
M = k+1
N = n

(2.29) A matrix is banded with size p and q (and bandwidth $p+q+1$ if $a_{ij}=0$ for $j>i+p$ and $i>j+q$. Analyze the p and q values for each diagonal and take the maximum values

i	p	q
1	4	0
2	3	0
3	3	0
4	2	3
5	1	2
6	1	1
7	0	0

The maximum values are $p=4$ and $q=3$ so the bandwidth is 8.

(2.30) 12 ($p=7$, $q=4$)

(2.31) $p=3$, $q=3$, bandwidth = 7

(2.32) $\mathbf{LU}=\mathbf{A}$ gives $6+L_{3,2}=9$ so $L_{3,2}=3$.

(2.33)

$$U=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(2.34)

$$L=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$U=\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

(2.35) Using $\mathbf{Ly}=\mathbf{b}$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & L_{3,2} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

From forward substitution, we get

$$y_1=5$$
$$y_2=10-2y_1=0$$
$$y_3=-y_1-L_{3,2}y_2=-5$$

Using $\mathbf{Ux}=\mathbf{y}$ gives

$$\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

From back substitution, we get

$$x_3 = -5/3$$
$$x_2 = -4x_3 = 20/3$$

(2.36) The first round of elimination gives

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & \frac{11}{2} & \frac{-5}{2} \end{bmatrix}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & ? & 1 \end{bmatrix}$$

The second round of elimination gives

$$\mathbf{U} = \begin{bmatrix} 2 & -3 & 1 \\ 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & 0 & 25 \end{bmatrix}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 11 & 1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 11 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -5.5 \\ 50 \end{bmatrix}$$

then solve

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & \frac{1}{2} & \frac{-5}{2} \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -5.5 \\ 50 \end{bmatrix}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(2.37) The **U** matrix is

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 4 & -2 & -4 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the **L** matrix is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 1 & 0.25 & 1 & 0 \\ 1 & 0.25 & -3 & 1 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 0 \\ 4 & -1 & -3 \\ 5 & 0 & -5 \end{bmatrix}$$

(2.38) We need to create **U** and **L** from Gauss elimination:

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

For the first forcing function, we get $\mathbf{y} = [0, 0, -1]$ and $\mathbf{x} = [-1, 0, 1]$. For the second forcing function, we get $\mathbf{y} = [9, -3, -4]$ and $\mathbf{x} = [2, 3, 4]$. For the third forcing function, we get $\mathbf{y} = [0, -3, 2]$ and $\mathbf{x} = [-1, 3 - 2]$.

(2.39) The original matrix is

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

which gives

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ -2 & -4 & -3 \end{bmatrix}$$

The forcing function is

$$\mathbf{b} = \mathbf{Ax} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

which gives

$$\mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

So the original problem was

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

During LU decomposition, we would have

$$\mathbf{Ly} = \mathbf{b}$$

which gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

The solution by forward substitution gives

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

(2.40) First perform the LU decomposition with the U matrix on the left and the L matrix on the right:

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 2 & -2 & 3 \\ 2 & 1 & 3 & 1 \\ 1 & 5 & -5 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ ? & ? & 1 & 0 \\ ? & ? & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 3 & 1 & -3 \\ 0 & 6 & -6 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & ? & 1 & 0 \\ 1 & ? & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

For the first forcing function, the equation for **y** is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ 15 \end{bmatrix}$$

Using forward substitution, we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 5 \end{bmatrix}$$

The corresponding equation for **x** is

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 5 \end{bmatrix}$$

Using back substitution, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

For the second forcing function, the equation for **y** is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ -2 \\ 42 \end{bmatrix}$$

Using forward substitution, we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ -20 \\ 15 \end{bmatrix}$$

The corresponding equation for **x** is

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ -20 \\ 15 \end{bmatrix}$$

Using back substitution, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \end{bmatrix}$$

For the third forcing function, the equation for $\mathbf{y}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 10 \\ -2 \end{bmatrix}$$

Using forward substitution, we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ 0 \\ 5 \end{bmatrix}$$

The corresponding equation for $\mathbf{x}$ is

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ 0 \\ 5 \end{bmatrix}$$

Using back substitution, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

(2.41) Starting from the matrix of constants A

$$\underline{\underline{\mathbf{A}}} = \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

The first round of elimination gives

$$\underline{\underline{\mathbf{A}}} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix}$$

with

$$\underline{\underline{\mathbf{L}}} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & ? & 1 \end{bmatrix}$$

The second round of elimination yields

$$\underline{\underline{\mathbf{U}}} = \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix}$$

with

$$\underline{\underline{\mathbf{L}}} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -.5 & 1 \end{bmatrix}$$

Now we solve $\underline{\underline{\mathbf{L}}}\,\underline{b} = \underline{y}$ the forward elimination step yeilds

$$\underline{y} = \begin{bmatrix} 1 \\ 2.5 \\ 4 \end{bmatrix}$$

Then we back eliminate to find

$$\underline{x} = \begin{bmatrix} 0.482 \\ -0.153 \\ 0.471 \end{bmatrix}$$

(2.42) The first round of elimination gives

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -1 \end{bmatrix}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & ? & 1 \end{bmatrix}$$

The second round of elimination gives

$$\mathbf{U} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

For the first column of the inverse, we first solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We then solve

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11/40 \\ 9/20 \\ -1/4 \end{bmatrix}$$

For the second column of the inverse, we first solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We then solve

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/40 \\ -1/20 \\ 1/4 \end{bmatrix}$$

For the third column of the inverse, we first solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We then solve

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/8 \\ -1/4 \\ 1/4 \end{bmatrix}$$

So the inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -11/40 & -1/40 & 3/8 \\ 9/20 & -1/20 & -1/4 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}$$

or, if you prefer nicer numbers

$$\mathbf{A}^{-1} = \frac{1}{40} \begin{bmatrix} -11 & -1 & 15 \\ 18 & -2 & -10 \\ -10 & 10 & 10 \end{bmatrix}$$

(2.43) While you do Gauss elimination, you can get the entries to **L**. The augmented matrix is

$$\begin{bmatrix} 2 & 2 & 3 & 5 \\ 4 & 5 & 7 & 11 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

For the first step of forward elimination, $m_{21} = 2$ and $m_{31} = 1$, which gives

$$\begin{bmatrix} 2 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{bmatrix}$$

For the second step of elimination, $m_{32} = 2$ and we get the upper-triangular matrix

$$\begin{bmatrix} 2 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Back-substitution gives

$$\begin{aligned} x_3 &= 1 \\ x_2 &= 1 - x_3 = 0 \\ x_1 &= \frac{5 - 3x_3 - 2x_2}{2} = 1 \end{aligned}$$

From Gauss elimination, we know that

$$\mathbf{U} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

The first equation to solve is $\mathbf{Ly} = \mathbf{b}$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

which we compute by forward substitution to get

$$y_1 = 5$$
$$y_2 = 11 - 2y_1 = 11 - 10 = 1$$
$$y_3 = 8 - y_1 - 2y_2 = 8 - 5 - 2 = 1$$

The back substitution for $\mathbf{Ux} = \mathbf{y}$ is identical to Gauss elimination.

(2.44) (a) We need to compute 3 determinants:

$$\det \mathbf{A} = 4 - 3 = 1$$

$$\det \mathbf{A}_1 = \det \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} = 0$$

$$\det \mathbf{A}_2 = \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = 1$$

The unknowns are

$$x = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = 0$$

and

$$y = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = 1$$

(b) There is no pivot so we just need to eliminate

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 2 & 2 \end{array}\right]$$

which gives

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & 1/2 & 1/2 \end{array}\right]$$

Using back substitution gives

$$y = \frac{1/2}{1/2} = 1$$

and

$$x = \frac{3-3}{2} = 0$$

(c) For LU decomposition, we use the result from (b) to know that

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$$

and

$$\mathbf{U} = \begin{bmatrix} 2 & 3 \\ 0 & 1/2 \end{bmatrix}$$

The forward substitution of

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

gives

$$y_1 = 3$$

and

$$y_2 = 2 - \frac{y_1}{2} = 1/2$$

The rest of the problem is identical to part (b), using back substitution to solve

$$\mathbf{U} = \begin{bmatrix} 2 & 3 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$

(2.45) (a) For naive Gauss elimination, the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 1 & 3 & 2 & 6 \end{array}\right]$$

One step of elimination gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -3 & -6 & -9 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

The second step gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

Back substitution gives

$$z = \frac{-3}{-3} = 1$$

and

$$y = \frac{-9 + 6z}{-3} = 1$$

and

$$x = \frac{6 - 3z - 2y}{1} = 1$$

(b) For Gauss elimination with pivoting, we start with

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 1 & 3 & 2 & 6 \end{array}\right]$$

The biggest entry is the second row, so we swap

$$\left[\begin{array}{ccc|c} 4 & 5 & 6 & 15 \\ 1 & 2 & 3 & 6 \\ 1 & 3 & 2 & 6 \end{array}\right]$$

and then eliminate the first column

$$\left[\begin{array}{ccc|c} 4 & 5 & 6 & 15 \\ 0 & 3/4 & 3/2 & 3/2 \\ 0 & 7/4 & 1/2 & 3/2 \end{array}\right]$$

In the second column, we see that we have to pivot again because the bottom entry is bigger than the second row. (Remember that you don't *have* to pivot, only if the diagonal is the biggest entry.)

$$\left[\begin{array}{ccc|c} 4 & 5 & 6 & 15 \\ 0 & 7/4 & 1/2 & 3/2 \\ 0 & 3/4 & 3/2 & 3/2 \end{array}\right]$$

Eliminating the second column gives

$$\left[\begin{array}{ccc|c} 4 & 5 & 6 & 15 \\ 0 & 7/4 & 1/2 & 3/2 \\ 0 & 0 & 9/7 & 9/7 \end{array}\right]$$

The back-substitution gives the same result as above.

(c) For LU decomposition, we just need to know the values we used for the naive Gauss elimination. For the first elimination we used $m_{21} = 4/1$ and $m_{31} = 1/1$. For the second elimination we used $m_{32} = 1/-3$. This means that the lower triangular matrix is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & -1/3 & 1 \end{bmatrix}$$

We already know the upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \end{bmatrix}$$

If you want to be sure that this is correct, just check that

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & -1/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

For the forward substitution we have

$$y_1 = 6$$

and

$$y_2 = 15 - 4y_1 = -9$$

and

$$y_3 = 6 - y_1 + \frac{y_2}{3} = -3$$

From the back substitution we get back the result from above. Indeed, it should be clear from the method that the vector **y** is just the augmented matrix at the end of naive Gauss elimination. This is the advantage of LU decomposition if you have many forcing functions — you only need to do the elimination of **A** and then you can quickly compute the **y** vectors for every forcing function.

(2.46) $\|\mathbf{A}\|_1 = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max(8, 28, 20) = 28$

(2.47) To find the condition number, we need to compute

$$\|\mathbf{A}\|_2 = 5.4772$$

We also need to find the inverse of **A**, which is

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.15 & 0.3 & 0.25 \\ 0.25 & -0.5 & -0.25 \\ 0.05 & 0.1 & -0.25 \end{bmatrix}$$

which has a Euclidian norm of 0.7906. The condition number is then 4.3301.

(2.48) For the Euclidian norm, we get

$$\|\mathbf{A}\| = \sqrt{2^2 + 4^2 + (-1)^2 + x^2} = \sqrt{x^2 + 21} \equiv z^{1/2}$$

For later use, I have defined $z \equiv x^2 + 21$. Note that this does not depend on the sign of x.

For the 2-norm, we first need to compute

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 \\ x & 4 \end{bmatrix}\begin{bmatrix} 2 & x \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2x-4 \\ 2x-4 & x^2+16 \end{bmatrix}$$

Now we need the eigenvalues

$$\det \begin{bmatrix} 5-\lambda & 2x-4 \\ 2x-4 & x^2+16-\lambda \end{bmatrix} = 0$$

Computing the determinant gives

$$(5-\lambda)(x^2+16-\lambda) - (2x-4)^2 = 0$$

Working out the products yields

$$(5x^2 + 80 - 5\lambda - \lambda x^2 - 16\lambda + \lambda^2) - (4x^2 - 16x + 16) = 0$$

and grouping the terms gives

$$\lambda^2 - \lambda(x^2 + 21) + (x^2 + 16x + 64) = 0$$

which we could also write as

$$\lambda^2 - z\lambda + (x+8)^2 = 0$$

Using the quadratic formula

$$\lambda = \frac{z \pm \sqrt{z^2 - 4(x+8)^2}}{2}$$

The largest eigenvalue is the positive one since $z > 0$. For simplicity, let's also define $b \equiv 4(x+8)^2 > 0$, independent of the sign of x. So the 2-norm is

$$||\mathbf{A}||_2 = \left(\frac{z + \sqrt{z^2 - b}}{2}\right)^{1/2}$$

We can rewrite this in the form

$$||\mathbf{A}||_2 = \left(z + \frac{\sqrt{z^2 - b} - z}{2}\right)^{1/2}$$

Since $b > 0$, we know that

$$\sqrt{z^2 - b} - z < 0$$

As a result, the Euclidian norm is always larger than the 2-norm for any value of x.

(2.49) $||x||_1 = \sum_{i=1}^{n} |x_i| = 10 + 3 + 4 + 1 + 5 = 23$

$||x||_2 = \sqrt{\sum_i x_i^2} = \sqrt{100 + 9 + 16 + 1 + 25} = \sqrt{151} = 12.3$

$||x||_\infty = \max |x_i| = 10$

(2.50) $L_1 = 9$, $L_e = \sqrt{88} \approx 9.38$, $L_\infty = 12$. The largest norm is L_∞ and the smallest norm is L_1.

(2.51) $L_1 = 19$, $L_e \approx 19.799$, $L_\infty = 26$. The largest norm is L_∞ and the smallest norm is L_1.

(2.52) $||\mathbf{A}||_1 \max_j \sum_i |a_{ij}| = \max(17, 12, 15, 13) = 17$

$||\mathbf{A}||_e = \sqrt{\sum_i \sum_j a_{ij}^2} = \sqrt{291} = 17.05$

$||\mathbf{A}||_\infty = \max_i \sum_j |a_{ij}| = \max(8, 18, 12, 19) = 19$

(2.53) The 1-norm of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 0 & 2 \end{bmatrix}$$

is 7 (from the first column). We now need to compute the inverse:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

Eliminate the first column:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -4 & -2 & -4 & 0 & 1 \end{array}\right]$$

Put 1 on the diagonal:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & -4 & -2 & -4 & 0 & 1 \end{array}\right]$$

Eliminate the 2nd column:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & 4 & -4 & 1 \end{array}\right]$$

Put 1 on the diagonal:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1/2 \end{array}\right]$$

Eliminate up the third column:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & 1/2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1/2 \end{array}\right]$$

Eliminate up the second column:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1/2 \end{array}\right]$$

The inverse is then

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 2 & -1 & 0 \\ -2 & 2 & -1/2 \end{bmatrix}$$

and its 1-norm is 5 (from the first column). So

$$\text{cond}(\mathbf{A}) = ||\mathbf{A}||\,||\mathbf{A}^{-1}|| = 35$$

(2.54) To compute the condition number, we first need the inverse of the matrix. We can compute this using simultaneous Gauss elimination on the identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 5/2 & 1/2 & -1/2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3/5 & 2/5 & -4/5 \\ 0 & 1 & 0 & -2/5 & 2/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2/5 & 2/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right]$$

So we have the inverse

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc} 1 & 0 & -1 \\ -2/5 & 2/5 & 1/5 \\ 1/5 & -1/5 & 2/5 \end{array}\right]$$

The two Euclidian norms are $||\mathbf{A}||_e = \sqrt{22} = 4.69$ and $||\mathbf{A}^{-1}||_e = \sqrt{65}/5 = 1.61$ so the condition number is the product of these norms, $\text{cond}(\mathbf{A}) = 7.56$.

(2.55) We first need to compute the inverse of the matrix:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

so that

$$\mathbf{A}^{-1} = \left[\begin{array}{cc} 1 & -2 \\ 0 & 1 \end{array}\right]$$

The two Euclidian norms are

$$\begin{aligned} ||\mathbf{A}||_e &= \sqrt{1^2 + 2^2 + 1^2 + 0^2} = \sqrt{6} \\ ||\mathbf{A}^{-1}||_e &= \sqrt{1^2 + (-2)^2 + 1^2 + 0^2} = \sqrt{6} \end{aligned}$$

The condition number is thus

$$\text{cond}(\mathbf{A}) = ||\mathbf{A}||\,||\mathbf{A}^{-1}|| = 6$$

(2.56) The 1-norm is the maximum of (3,5), which is 5. The ∞-norm is the maximum of (5,3), which is also 5. The Euclidian norm is $\sqrt{4+9+1+4} = \sqrt{18} = 4.24$. For the spectral norm, we need to diagonalize:

$$\mathbf{A}^T\mathbf{A} = \left[\begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array}\right]\left[\begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array}\right] = \left[\begin{array}{cc} 5 & 8 \\ 8 & 13 \end{array}\right]$$

The trace of the matrix is $\tau = 18$ and its determinant is $\Delta = (5)(13) - 8^2 = 1$. The eigenvalues are

$$\lambda = \frac{18 \pm \sqrt{18^2 - 4(1)}}{2} = \frac{18 \pm \sqrt{320}}{2}$$

The largest eigenvalue is the positive root, $\lambda = 17.9444$. The spectral norm is the square root of this number, $||\mathbf{A}||_2 = 4.236$.

(2.57) The first step of Jacobi's method is

$$x_1^{(1)} = \frac{4 - x_2^{(0)}}{2} = \frac{4 - 0}{2} = 2$$

and the second step is

$$x_2^{(1)} = \frac{3 - x_1^{(0)}}{1} = \frac{3 - 1}{1} = 2$$

The value of $\mathbf{Ax}$ is

$$\mathbf{Ax} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

so the value of $\mathbf{Ax} - \mathbf{b}$ is

$$\mathbf{Ax} - \mathbf{b} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The 1-norm is the sum of the absolute value of the entries, which is 3.

(2.58) The first iteration of Jacobi's method is simple

$$\begin{aligned} x_1^{(1)} &= -9/6 \\ x_2^{(1)} &= 9/8 \\ x_3^{(1)} &= 17/10 \\ x_4^{(1)} &= 13/9 \end{aligned}$$

The next iteration is a bit more involved:

$$\begin{aligned} x_1^{(2)} &= \frac{-9 - [9/8 - 3(13/9)]}{6} = -0.965 \\ x_2^{(2)} &= \frac{9 - [-2(-9/6) + 2(17/10) + 3(13/9)]}{8} = -0.217 \\ x_3^{(2)} &= \frac{17 - [-9/6 - 6(9/8) - 2(13/9)]}{10} = 2.81 \\ x_4^{(2)} &= \frac{13 - [2(-9/6) + 9/8 + 3(17/10)]}{9} = 1.09 \end{aligned}$$

(2.59) The first iteration of Gauss-Seidel gives

$$x_1^{(1)} = -3/2$$
$$x_2^{(1)} = \frac{9 - [-2(-3/2)]}{8} = 3/4$$
$$x_3^{(1)} = \frac{17 - [-3/2 - 6(3/4)]}{10} = 23/10$$
$$x_4^{(1)} = \frac{13 - [2(-3/2) + 3/4 + 3(23/10)]}{9} = 0.928$$

In the second iteration, we get

$$x_1^{(2)} = \frac{-9 - [3/4 - 3(0.928)]}{6} = -1.16$$
$$x_2^{(2)} = \frac{9 - [-2(-1.16) + 2(23/10) + 3(0.928)]}{8} = -0.088$$
$$x_3^{(2)} = \frac{17 - [-1.16 - 6(-0.088) - 2(0.928)]}{10} = 1.95$$
$$x_4^{(2)} = \frac{13 - [2(-1.16) - 0.088 + 3(1.95)]}{9} = 1.06$$

(2.60) The first iteration of Gauss-Seidel gives

$$x_1^{(1)} = \frac{6 - 0}{3} = 2$$
$$x_2^{(1)} = \frac{3 - [(1)(2)]}{3} = \frac{1}{3}$$

(2.61) The first iteration of SOR gives

$$r_1^{(1)} = \frac{6}{3} = 2$$
$$x_1^{(1)} = 0 + \frac{3}{2}(2) = 3$$
$$r_2^{(1)} = \frac{3 - (1)(3)}{3} = 0$$
$$x_2^{(1)} = 0 + \frac{3}{2}(0) = 0$$

(2.62) Compute each r and the new term:

$$r_1 = \frac{-9 - [6(0.9) + 0.1 - 3(1.1)]}{6} = -0.067$$
$$x_1^{(1)} = -0.9 + 3/2(-0.067) = -1$$

$$r_2 = \frac{9 - [-2(-1) + 8(0.1) + 2(2.2) + 3(1.1)]}{8} = -0.1875$$
$$x_2^{(1)} = 0.1 + 3/2(-0.1875) = -0.181$$

$$r_3 = \frac{17 - [-1 - 6(-0.181) + 10(2.2) - 2(1.1)]}{10} = -0.289$$
$$x_3^{(1)} = 2.2 + 3/2(-0.289) = 1.767$$

$$r_4 = \frac{13 - [2(-1) - 0.181 + 3(1.767) + 9(1.1)]}{9} = -0.00218$$

$$x_4^{(1)} = 1.1 + 3/2(-0.00218) = 1.103$$

(2.63) No because $2 < 6 + 2 = 8$.

(2.64) To be diagonally dominant, we require that $|a_{ii}| > \sum j, j \neq i|a_{ij}|$ on each row i. For each row, we have

$$6 > 1 + 0 + 3 = 4 \quad (i = 1)$$
$$8 > 2 + 2 + 3 = 7 \quad (i = 2)$$
$$10 > 1 + 6 + 2 = 9 \quad (i = 3)$$
$$9 > 2 + 1 + 3 = 6 \quad (i = 4)$$

(2.65) The components for linear regression are $s_x = 9.8$, $s_{xx} = 20.9$, $s_y = 25.3$ and $s_{xy} = 55.04$ for $n = 5$ elements. The coefficients are

$$a_1 = \frac{ns_{xy} - s_x s_y}{ns_{xx} - s_x^2} = 3.22$$

and

$$a_0 = \frac{s_y}{n} - a_1 \frac{s_x}{n} = -1.25$$

so the linear regression is $y = 3.22x - 1.25$.

(2.66) (a) First we need to calculate how many numbers we can store in the memory

$$1MB * \frac{1024\ KB}{1\ MB} * \frac{1024\ bytes}{1\ KB} * \frac{8\ bits}{1\ byte} * \frac{1\ number}{64\ bits} = 131,072\ numbers$$

That is the total number of entries we can have in A and b to determine how many equations we can store we take the sum of the space requirements $n^2 + n$ and set it equal to the total number of numbers we can store and solve

$$131072 = n^2 + n$$
$$n = 361.54$$

We must round down as storing half an equation is no use so 361 equations can be stored by the Mac/SE30

(b) For the Cray supercomputer the same basic principle is employed

$$500\ MW * \frac{1024^2\ words}{1\ MW} = n^2 + n$$
$$n^2 + n = 524,288,000$$
$$n \approx 22896\ equations$$

(c) We need to calculate the total number of operations we can do in one hour and set it equal to the amount of time to solve the equation

$$\frac{.01 * 10^6 \ floatingpointoperations}{1 \ second} * \frac{3600 \ seconds}{1 hour} = \frac{n^3}{3}$$

Solving yields $n = 476$ equations

(d) similarly for the Cray

$$\frac{100 * 10^6 \ floatingpointoperations}{1 \ second} * \frac{3600 \ seconds}{1 hour} = \frac{n^3}{3}$$

This gives a significantly larger answer $n = 10,260$ equations

(e) Based on the above numbers the Mac seems limited by the memory since it can solve more equations in a reasonable time than it can store. The Cray is the opposite it an solve more equations than it can store in a similar amount of time so it is limited by its processor speed.

(2.67) 10^4 ($p = 2$, ratio of time is $n^3/np^2 = (n/p)^2$)

(2.68) For a banded matrix, Jacobi's method requires n values of x_i, p evaluations per sum, and k iterations. So the scaling is $t \sim npk$. It is OK if they write something more accurate like $t \sim n(p + q + 1)k$, but that will change the numbers.

Gauss elimination scales like n^3, so the ratio is

$$\frac{t_{Gauss}}{t_{Jacobi}} \sim \frac{n^3}{npk} \sim \frac{n^2}{pk}$$

Putting in the numbers gives

$$t_{Jacobi} = \frac{pk}{n^2} t_{Gauss} = \frac{(2)(5)}{(1000)^2}(2 \text{ sec})$$

which gives $t_{Jacobi} = 2 \times 10^{-5}$ sec. Jacobi's method is preferred.

(2.69) (a) n^3 scaling gives $t = 10^3 = 1000$ sec.

(b) np^2 scaling gives

$$t = 1000 \left(\frac{p}{n}\right)^2 = 1000 \left(\frac{4}{1000^2}\right) = \frac{1}{250} \text{ sec}$$

(c) n^2k scaling gives

$$t = 1\frac{k}{n} = 0.1 \text{ sec}$$

(2.70) Gauss elimination is n^3 and Jacobi's method is n^2k. So we need $k \ll n$. OK if they say $k < n$.

(2.71) $y = 4$ ($x + 3y = 5x$ with $x = 3$)

(2.72) The trace is $\tau = 2$ and the determinant is $\Delta = 3$. So the eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4 - 4(3)}}{2} = 1 \pm \imath \sqrt{2}$$

Both eigenvalues have the same magnitude,

$$|\lambda| = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$$

(2.73) (a) 1

(b) $p = 4$, $q = 0$, so the bandwidth is $p + q + 1 = 5$

(c) Column $j = 3$ has the biggest sum: $53 + \pi = 56.14157$

(d) There are 5 eigenvalues. For an upper triangular matrix, it is easy to show that the entries on the diagonal are the eigenvalues. The eigenvalues satisfy

$$\det \lambda \mathbf{I} - \mathbf{A} = 0$$

If $\lambda = A_{ii}$ for any i, then

$$\prod_i \lambda \mathbf{I} - \mathbf{A} = 0$$

(e) No

(2.74) (a) 15.

(b) No because it is not diagonally dominant.

(c) After pivoting the rows, $a_{3,2}^{(2)} = 2 - (4/2) = 0$.

(2.75) Forward elimination without pivoting, Gauss elimination without pivoting or naive Gauss elimination

(2.76) The system of equations is

$$\begin{bmatrix} 2 & 3 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

The method of solution is Gauss-Seidel. To check for convergence, there are several steps:

(a) Is there a unique solution?

Need to see if $\det \mathbf{A} = 0$.

$$\det \begin{bmatrix} 2 & 3 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} = 30 + 6 - 4 - (3 + 8 - 30) = 32 + 19 = 51 \neq 0$$

So there is a unique solution.

(b) Is the matrix diagonally dominant?

Only row 3 is diagonally dominant.

Since diagonal dominance is sufficient but not necessary, the answer to this question is that we do not know if the program will converge.

(2.77) (a) This program solves the linear algebraic system

$$3x_1 + 2x_2 = 2$$
$$x_1 - x_2 = -1$$

(b) This problem uses Jacobi's method

(c) The criteria for stopping is that the absolute value of the residual is less than or equal to 10^{-4} or the number of iterations is greater than 20.

(d) The initial residual is

$$R = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and the one norm is $\|r\|_1 = 2 + 1 = 3$. The formatted output to the screen is

```
0    0.000000    0.000000    3.0000e0
```

(e) After one iteration of Jacobi's method, $x_1 = 2/3$ and $x_2 = 1$. The residual is then

$$R = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix}$$

and the one norm is $\|r\|_1 = 2 + 2/3 = 2.6667$. The formatted output to the screen is

```
1    0.666667    1.000000    2.6667e0
```

(2.78) (a)

$$\det \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

If they say det **A** that's OK too.

(b) Gauss elimination with pivoting

(c) The size of the matrix

(d) The value of the determinant

(e) If there was a pivot (row swap) during this step of elimination. Another OK answer is that s keeps track of the sign of the determinant.

(f) Forward elimination or Gauss elimination. Another OK answer is calculation of the product of the diagonal elements.

(g) We just need to do the forward elimination. There is a pivot on the first step ($s = -1$) to give

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Doing the forward elimination gives

$$\begin{bmatrix} 4 & 2 & 1 \\ 0 & 0 & 2.5 \\ 0 & 0.5 & 0.75 \end{bmatrix}$$

We really need to pivot on the next step ($s = 1$ now) because the diagonal is zero!

$$\begin{bmatrix} 4 & 2 & 1 \\ 0 & 0.5 & 0.75 \\ 0 & 0 & 2.5 \end{bmatrix}$$

There is nothing left to do. The product of the diagonal is 5. Since we pivoted twice, there is no sign change. The output of the last line is

```
q = 5.000000
```

Be sure to have the right number of decimal places from the formatting command.

(2.79) (a) The problem is

$$\begin{bmatrix} 3 & 10 & -5 \\ -4 & 1 & 2 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(b) successive relaxation

(c) The initial condition is

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) The criteria for convergence is

$$||\mathbf{A}x^{(k)} - \mathbf{b}|| \leq 10^{-7}$$

OK if they use $<$ instead of $\leq$ or do not put the superscript on **x**. The actual value k or $k+1$ depends how you count, so that's not something worth checking.

(e) Dampen oscillations

(f) Becomes Gauss-Seidel

(g) Diverges since $\omega > 2$.

(h) The sum will be computed incorrectly. The value from the previous iteration will carry over to the next iteration.

(i) The matrix is not diagonally dominant. You can make it diagonally dominant by swapping the first and second equations.

Computer Problems

(2.80) (a) The files for this problem are contained in the folder `s15c4p2_matlab`. The Matlab script is:

```
function s15h4p2
clc

A = zeros(12);
b = zeros(12,1);

%balance on #1
A(1,1) = -1;
A(1,2) = 1;
A(1,3) = 1;
A(1,4) = 1;
A(1,5) = 1;
b(1) = 0;

%balance on #2
A(2,2) = -1;
A(2,9) = 1;
A(2,10) = 1;
```

```
A(2,11) = 1;
b(2) = 0;

%balance on #3
A(3,5) = -1;
A(3,6) = 1;
A(3,7) = 1;
A(3,8) = 1;
b(3) = 0;

%balance on #4
A(4,4) = 1;
A(4,7) = 1;
A(4,11) = 1;
A(4,12) = -1;
b(4) = 0;

%spec on m1
A(5,1) = 1;
b(5) = 100;

%spec on 8 and 5
A(6,5) = 1;
A(6,8) = -5;
b(6) = 0;

%spec on 4, 7, 12
A(7,4) = 1;
A(7,7) = 1;
A(7,12) = -0.84;
b(7) = 0;

%spec on 1, 2, 3
A(8,1) = -0.7;
A(8,2) = 1;
A(8,3) = 1;
b(8) = 0;

%spec on 1, 9, 12
A(9,1) = -0.55;
A(9,9) = 1;
A(9,12) = 1;
b(9) = 0;

%spec on 6, 10
A(10,9) = -0.2;
A(10,10) = 1;
b(10) = 0;

%spec on 2, 11, 9
A(11,2) = -0.85;
A(11,9) = 1;
A(11,11) = 1;
b(11) = 0;

%spec on 6, 7, 8
```

```
A(12,6) = 3.2;
A(12,7) = -1;
A(12,8) = -1;
b(12) = 0;

fprintf('The matrix A = \n')
for i = 1:12
    for j = 1:12
        fprintf('%4.2f\t',A(i,j))
    end
    fprintf('\n')
end

fprintf('\n\n')
fprintf('The vector b = \n')
for i = 1:12
    fprintf('%4.2f\n',b(i))
end
```

(b) The files for this problem are contained in the folder `s15c4p3_matlab`.
The Matlab script is:

```
function s15h4p3
clc

[A,b] = writeAB; %use program from problem 2

xGaussNaive = linear_ngaussel(A,b) %solve with naive Gauss ...
    elimination

xGaussPivot = linear_gauss_pivot(A,b) %solve with Gauss + pivot

xMatlab = A\b %solve with Matlab solve

function [A,b] = writeAB
clc

A = zeros(12);
b = zeros(12,1);

%balance on #1
A(1,1) = -1;
A(1,2) = 1;
A(1,3) = 1;
A(1,4) = 1;
A(1,5) = 1;
b(1) = 0;

%balance on #2
A(2,2) = -1;
A(2,9) = 1;
A(2,10) = 1;
A(2,11) = 1;
b(2) = 0;

```

```
%balance on #3
A(3,5) = -1;
A(3,6) = 1;
A(3,7) = 1;
A(3,8) = 1;
b(3) = 0;

%balance on #4
A(4,4) = 1;
A(4,7) = 1;
A(4,11) = 1;
A(4,12) = -1;
b(4) = 0;

%spec on m1
A(5,1) = 1;
b(5) = 100;

%spec on 8 and 5
A(6,5) = 1;
A(6,8) = -5;
b(6) = 0;

%spec on 4, 7, 12
A(7,4) = 1;
A(7,7) = 1;
A(7,12) = -0.84;
b(7) = 0;

%spec on 1, 2, 3
A(8,1) = -0.7;
A(8,2) = 1;
A(8,3) = 1;
b(8) = 0;

%spec on 1, 9, 12
A(9,1) = -0.55;
A(9,9) = 1;
A(9,12) = 1;
b(9) = 0;

%spec on 6, 10
A(10,9) = -0.2;
A(10,10) = 1;
b(10) = 0;

%spec on 2, 11, 9
A(11,2) = -0.85;
A(11,9) = 1;
A(11,11) = 1;
b(11) = 0;

%spec on 6, 7, 8
A(12,6) = 3.2;
A(12,7) = -1;
A(12,8) = -1;
```

```
b(12) = 0;

fprintf('The matrix A = \n')
for i = 1:12
    for j = 1:12
        fprintf('%4.2f\t',A(i,j))
    end
    fprintf('\n')
end

fprintf('\n\n')
fprintf('The vector b = \n')
for i = 1:12
    fprintf('%4.2f\n',b(i))
end




function x = linear_gauss_pivot(A,b)
%  A = n x n matrix
%  b = column vector, n x 1
n=length(b);
x=zeros(n,1);
% Perform the forward elimination
for k=1:n-1
   %see if there is a bigger pivot
   Amax = abs(A(k,k)); %line changed from book
   swap_row = k;
   for i = k+1:n
       if abs(A(i,k)) > abs(Amax) %line changed from book
           Amax = A(i,k);
           swap_row = i;
       end
   end
   %exchange rows if true
   if swap_row ~= k
       old_pivot(1,:) = A(k,:);
       old_b = b(k);
       A(k,:) = A(swap_row,:);
       A(swap_row,:) = old_pivot;
       b(k) = b(swap_row);
       b(swap_row) = old_b;
   end
   %%%%%%%%%
   %This block is added just to show A as we go along
   fprintf('On elimination step %2d the matrix is now:\n',k)
   for i = 1:12
       for j = 1:12
            fprintf('%4.2f\t',A(i,j))
       end
       fprintf('\n')
   end
   fprintf('\n\n\n\n')
   %%%%%%%%%%%%

```

```
   %eliminate
   for i=k+1:n
      m=A(i,k)/A(k,k);
      for j=k+1:n
         A(i,j)=A(i,j)-m*A(k,j);
      end
      b(i)=b(i)-m*b(k);
   end
end
% Perform the back substitution
x(n)=b(n)/A(n,n);
for i=n-1:-1:1
   S=b(i);
   for j=i+1:n
       S=S-A(i,j)*x(j);
    end
    x(i)=S/A(i,i);
end

function x = linear_ngaussel(A,b)
%  A = n x n matrix
%  b = column vector, n x 1
n=length(b);
x=zeros(n,1);
% Perform the forward elimination
for k=1:n-1
   for i=k+1:n
      m=A(i,k)/A(k,k);
      for j=k+1:n
         A(i,j)=A(i,j)-m*A(k,j);
      end
      b(i)=b(i)-m*b(k);
   end
end
% Perform the back substitution
x(n)=b(n)/A(n,n);
for i=n-1:-1:1
   S=b(i);
   for j=i+1:n
       S=S-A(i,j)*x(j);
    end
    x(i)=S/A(i,i);
end
```

You need to solve this problem with pivoting to avoid a zero. Note that the program in the book was incorrect and needed to be fixed in two places to use absolute value for the pivoting!

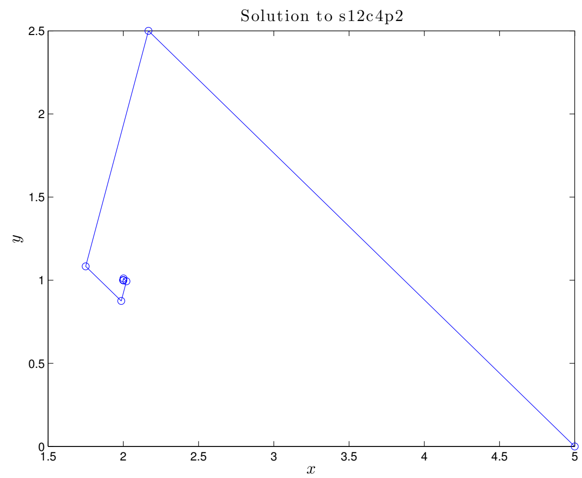
(2.81) The files for this problem are contained in the folder `s12c4p2_matlab`.

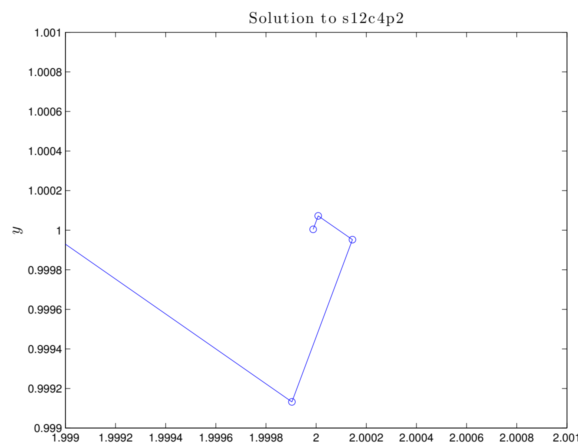
The Matlab script is:

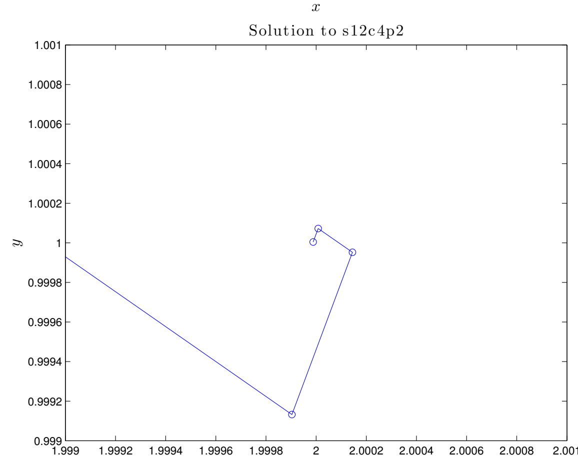
```
function s12c4p2
close all
set(0,'defaulttextinterpreter','latex')

```

```
A = [6, 1; 4, -8];
b = [13;0];
x_solve = A\b;

x = [5,0];
[xplot,yplot,err_plot,k] = jacobi(A,x,b);
xplot = xplot;
yplot = yplot;

%make the original figure
h = figure;
plot(xplot,yplot,'-ob')
xlabel('$x$','FontSize',14), ylabel('$y$','FontSize',14)
title('Solution to s12c4p2','FontSize',14)
saveas(h,'s12c4p2_solution_figure1.eps','psc2')

%make the zoom figure
small = 0.001;
axis([x_solve(1)-small,x_solve(1)+small,x_solve(2)-small,...
x_solve(2)+small])
saveas(h,'s12c4p2_solution_figure2.eps','psc2')


%make the error figure
g = figure;
semilogy(err_plot,'-ob')
xlabel('Iteration','FontSize',14), ylabel('Error','FontSize',14)
title('Solution to s12c4p2','FontSize',14)
saveas(h,'s12c4p2_solution_figure3.eps','psc2')


function [x_plot,y_plot,err_plot,k]=jacobi(A,x,b)
x_old = x;
err = 100;
x_plot(1) = x(1);
y_plot(1) = x(2);
k = 1;
while err > 10^-4
    k = k+1;
    x(1) = (b(1)-A(1,2)*x_old(2))/A(1,1);
    x(2) = (b(2)-A(2,1)*x_old(1))/A(2,2);
    err = norm(x-x_old);
    x_old = x;
    err_plot(k-1) = err;
    x_plot(k) = x(1);
    y_plot(k) = x(2);
end
```

The output files are:





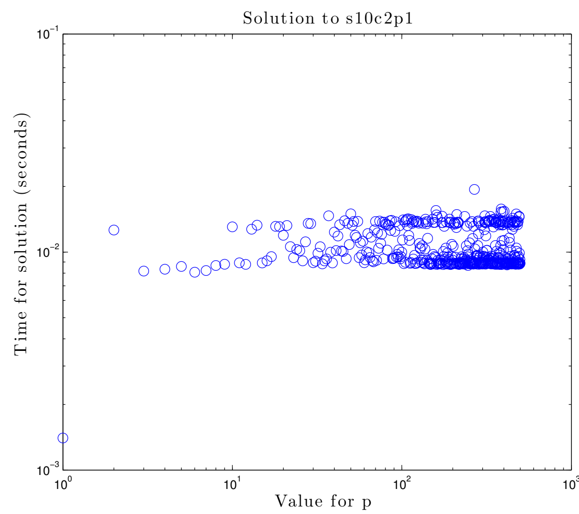


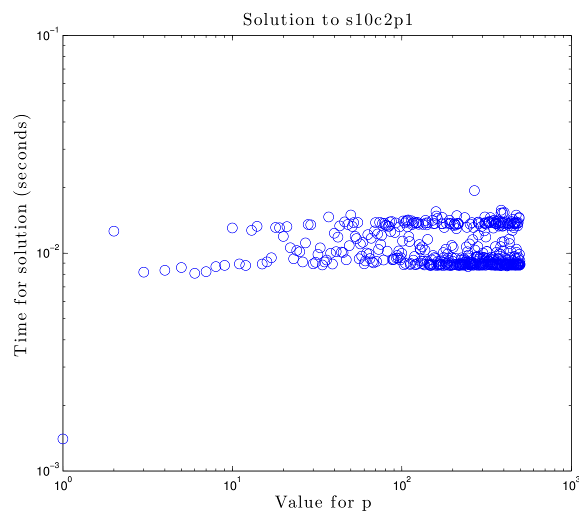
(2.82) The files for this problem are in the folder `s10c2p1_matlab`.

The Matlab script is:

```
function s10c2p1
close all
set(0,'defaulttextinterpreter','latex')

 n = 500;
 output = zeros(3,n); %initialize the output. row 1 = p, row 2 ...
     = time, row 3 = number of non-zero elements
    for p = 1:n-1
         if mod(p,50)==0
            disp(p) %write p to screen to check progress
         end
         A = make_sparse(n,p); %make the matrix with bandwidth ...
             p and q = 0.
         output(3,p) = size(find(A),1); %find the number of ...
             non-zero elements
         b = rand(n,1); %generate the forcing function
         tic;
         x = A\b;
         output(1,p) = p; %put p in the output
         output(2,p) = toc; %put the time in the output
    end
    h = figure;
    loglog(output(1,:),output(2,:),'o','MarkerSize',8)
    xlabel('Value for p','FontSize',14)
    ylabel('Time for solution (seconds)','FontSize',14)
    title('Solution to s10c2p1','FontSize',14)
    saveas(h,'s10c2p1_solution_figure1.eps','psc2')
    figure
    plot(output(1,:),output(3,:),'o','MarkerSize',8)
    xlabel('Value of p','FontSize',14)
    ylabel('Number of non-zero elements in A','FontSize',14)
    title('Solution to s10c2p1','FontSize',14)
    saveas(h,'s10c2p1_solution_figure2.eps','psc2')

 out = 1;


function out = make_sparse(n,p,q)
A = zeros(n,n); %make an nxn matrix with random elements

for i = 1:n %loop through the rows
    %fill in the current row up to the value of p or the end ...
        of the matrix
    j_end = min(n,i+p);
    for j = i:j_end
        A(i,j) = rand();
    end
    %fill in the current column up to the value of q or the ...
        end of the
    %matrix
    for j = i:j_end
        A(j,i) = rand();
    end
end

out = A;
```

The output files are:





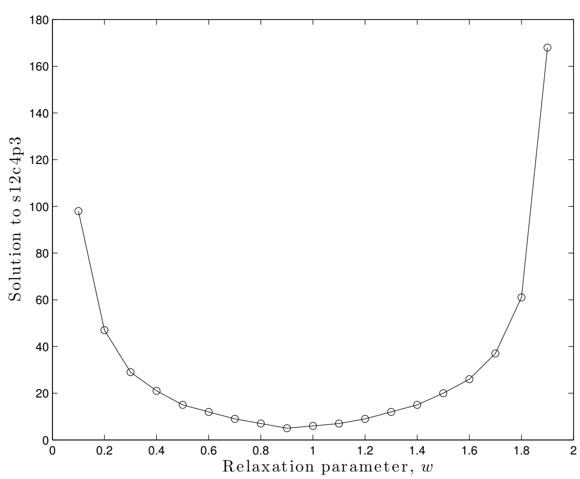
(2.83) The files for this problem are contained in the folder `s12c4p3_matlab`.

The Matlab script is:

```
function s12c4p3
clc
close all

set(0,'defaulttextinterpreter','latex')

load s12c4p3_data
n = 100;


```

```
%solve the system using SOR with the desired value of w
for z = 1:19
    w = 0.1*z;
    w_plot(z) = w;
    fprintf('\n*****************************************\n')
    fprintf('\n \n Starting calculation for w = %3.1f \n',w)
    k = 0;
    x = zeros(n,1);
    err = norm(A*x-b);
    err_plot = err;

    while err > 10^-4
        k = k+1;
        for i = 1:n
            s = 0; %keep track of the sum
            for j = 1:n
                s = s + A(i,j)*x(j);
            end
            r = (b(i) - s)/A(i,i);
            x(i) = x(i) + w*r;
        end
        err = norm(A*x-b);
        if k >= 200
            fprintf('Did not converge! \n \n')
            err = 0;
            n_iter(z) = -1;
        end
    end
    fprintf('k = %4d \t Error = %8.6e \n',k,err)
    n_iter(z) = k;
end
h = figure;
plot(w_plot,n_iter,'-ok')
xlabel('Relaxation parameter, $w$','FontSize',14),
ylabel('Solution to s12c4p3','FontSize',14)
saveas(h,'s12c4p3_solution_figure.eps','psc2')
```

The output file is:



The number of iterations increase dramatically at extreme values of w but it is rather flat around $w = 1$. This behavior indicates that the linear system is relatively well behaved and does not benefit much from SOR — the result for Gauss-Seidel ($w = 1$) is very fast. Recall that SOR is only stable for $0 < w < 1.9$. If you play around with values of w very close to the stability limits, the time for the solution increases dramatically.

(2.84) The files for this problem are contained in the folder `s11c3p1_matlab`.

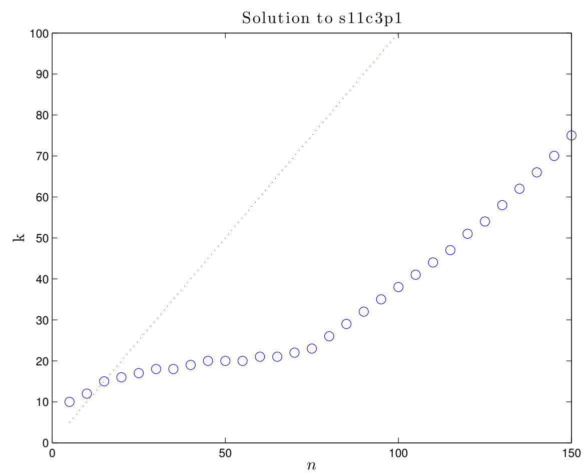
The Matlab script is:

```
function s11c3p1
clc
close all
set(0,'defaulttextinterpreter','latex')

npts = 30;

%initialize the output files
n_output = zeros(npts,1);
iter_output = zeros(npts,1);
diag_output = zeros(npts,1);
err_output = zeros(npts,1);
write_ouptut = zeros(npts,3);


for k = 1:npts
    n = 5*k;
    n_output(k) = n; %store the matrix size
    fprintf('\n\n================ n = %3d =============\n',n)
    %initialize the matrix
    A = zeros(n); b = zeros(n,1);
    for i = 1:n
        for j = 1:n
            if i == j
                A(i,j) = 8+0.2*i;
            else
                A(i,j) = (i+j)/i/j;
            end
        end
        b(i,1) = i*n;
    end


    %check the matrix for diagonal dominance
    diag_check = 0;
    for i = 1:n
        sum_row = 0;
        for j = 1:n
            sum_row = sum_row + A(i,j); %sum up the row
        end
        sum_row = sum_row - A(i,i); %remove the middle element
        if sum_row > A(i,i)
            diag_check = diag_check + 1;
        end
```

```
    end

    if diag_check > 0
        diag_output(k) = 0; %not diagonally dominant
        fprintf('The matrix is not diagonally dominant!\n')
    else
        diag_output(k) = 1; %diagonally dominant
        fprintf('The matrix is diagonally dominant.\n')
    end


    %set the parameters for Gauss-Seidel
    tol = 1e-8;
    x = zeros(n,1);
    iterations = 0;
    err = 1000;

    while err > tol
        %increment the iteration counter
        iterations = iterations+1;

        for i = 1:n
            sum_term = 0;
            for j = 1:n
                sum_term = sum_term + A(i,j)*x(j);
            end
            sum_term = sum_term - A(i,i)*x(i); %remove diagonal
            x(i) = (b(i) - sum_term)/A(i,i); %update x
        end


        %check the solution
        err = norm(A*x-b);

        %output the error to screen
        fprintf('The error after iteration %3d is %8.6e ...
            \n',iterations,err)


        %check to see if there are too many iterations
        if iterations > 100
            fprintf('Failed to converge.\n')
            err = -1000;
        end
    end
    %save the number of iterations and the error
    iter_output(k) = iterations;
    err_output(k) = err;
end

%we want to compare to a line of slope n
h = figure;
plot(n_output,iter_output,'ob',n_output,n_output,':b',...
'MarkerSize',8)
axis([0,150,0,100])
xlabel('$n$','FontSize',14)
```

```
ylabel('k','FontSize',14)
title('Solution to s11c3p1','FontSize',14)
saveas(h,'s11c3p1_solution_figure.eps','psc2')

%output the data in the correct format
write_output(:,1) = n_output;
write_output(:,2) = diag_output;
write_output(:,3) = err_output;
write_output(:,4) = iter_output;

dlmwrite('s11c3p1_output.txt',write_output)
```

The output figure is:



The text output is:

```
5,1,1.8466e-09,10
10,0,6.8661e-09,12
15,0,7.4406e-10,15
20,0,1.6447e-09,16
25,0,2.4971e-09,17
30,0,2.8452e-09,18
35,0,8.615e-09,18
40,0,2.9148e-09,19
45,0,1.4053e-09,20
50,0,8.6347e-09,20
55,0,4.1406e-09,20
60,0,8.1553e-09,21
65,0,3.6799e-09,21
70,0,8.7412e-09,22
75,0,4.8083e-09,23
80,0,4.7562e-09,26
85,0,8.0456e-09,29
90,0,8.029e-09,32
95,0,7.4525e-09,35
100,0,7.0643e-09,38
105,0,7.067e-09,41
110,0,7.5676e-09,44
```

```
115,0,8.6687e-09,47
120,0,5.9007e-09,51
125,0,8.0926e-09,54
130,0,7.0545e-09,58
135,0,6.9511e-09,62
140,0,7.6793e-09,66
145,0,9.521e-09,70
150,0,8.8e-09,75
```

For the small values of n, it looks like the scaling for the work required by Gauss-Seidel is less than Gauss elimination. As we get to larger values of n, the Gauss-Seidel starts to look like a linear scaling. Note that the absolute value of the line $k = n$ does not reflect the total work required by Gauss elimination. Rather, it is the way that the work scales with increasing n. So when you are comparing the two methods, you want to look at the slope of the lines rather than their absolute value. The absolute value will depend on both the scaling of the algorithm and how well you write your program. It is possible to have a method that does not scale well be the faster method if the program is written in a very efficient manner.

(2.85) The files for this problem are contained in the folder `s10c3p1_matlab`.

The Matlab script for this problem is:

```
function s10c3p1

close all
set(0,'defaulttextinterpreter','latex')

%initialize the matrix and forcing function
A = [3 1 2; 6 3 3; 3 1 2;];
b = [7;10;8];

%increment through the powers of epsilon
for i = 1:15
    epsilon = 10^(-i/1);
    A(3,3) = A(1,3)+epsilon; %change the last entry
    epsilon_out(i) = epsilon;%store epsilon
    condition_out(i) = cond(A);%compute the condition number
    x = A\b;%solve the system
    error_out(i) = abs(epsilon*x(3) - 1);%compute the error. ...
        the true value is 1/epsilon
end
%make the plots
h = figure;
loglog(epsilon_out,condition_out,'--o','MarkerSize',8)
xlabel('$epsilon$','FontSize',14)
ylabel('Condition Number','FontSize',14)
title('Solution to s10c3p1','FontSize',14)
saveas(h,'s10c3p1_solution_figure1.eps','psc2')
h = figure;
loglog(condition_out,error_out,'--o','MarkerSize',8)
xlabel('Condition Number','FontSize',14)
ylabel('Relative error in $x_3$','FontSize',14)
title('Solution to s10c3p1','FontSize',14)
```

```
saveas(h,'s10c3p1_solution_figure2.eps','psc2')
```

The output files are: