Solutions for Introduction to Experimental Mathematics 1st Edition by Eilers

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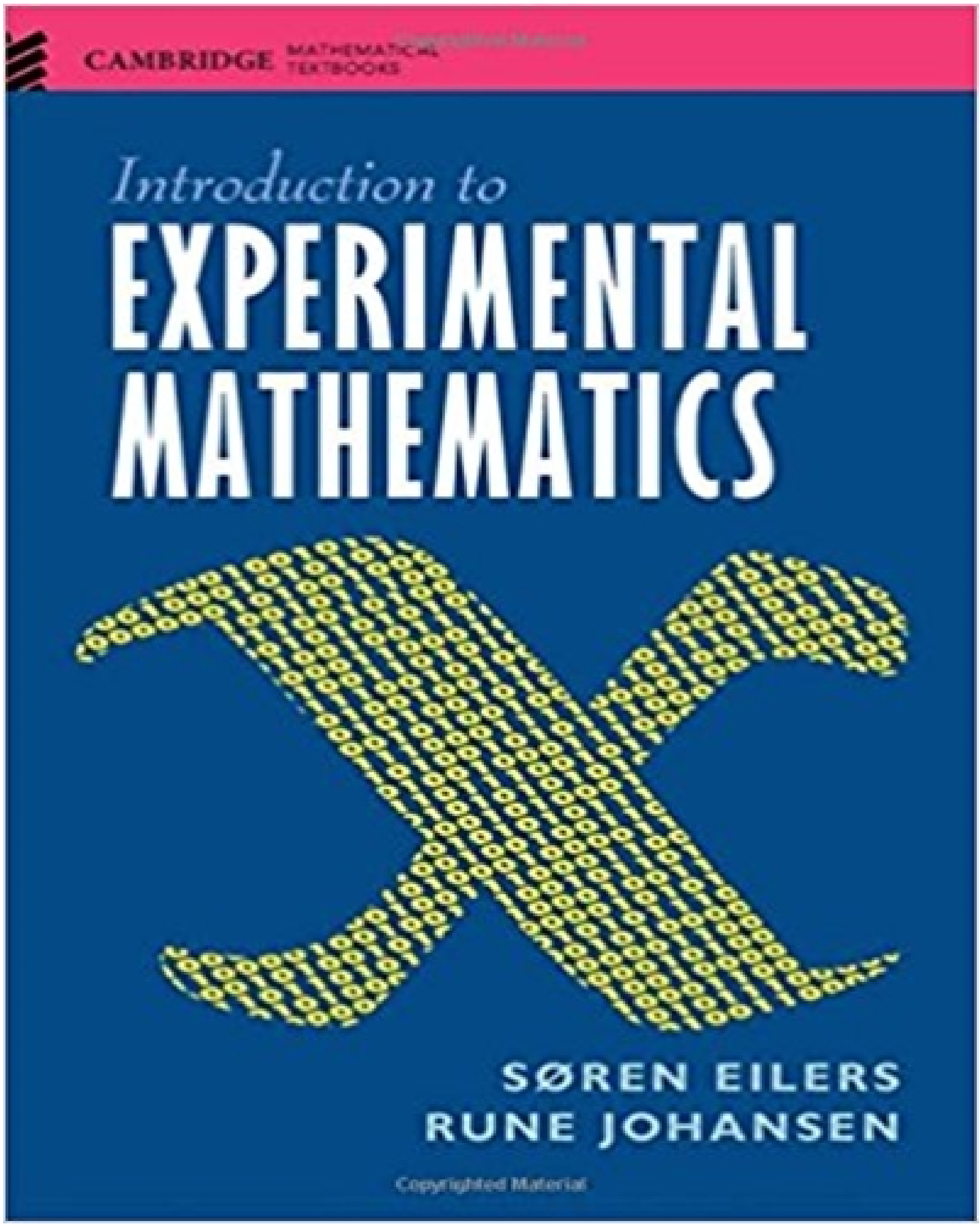

Solutions

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Material for Chapter 2

Example 2.5.7

These sequences are known as *Sturmian sequences*. We have refrained from mentioning the name in the example to make it a little harder for students to look up the answers instead of formulating their own hypotheses. Note, however, that the name and a reference are given in the final section of the chapter for the sake of completeness. Exercises 2.19, 2.20, 2.21 and 2.41 all investigate the structure of these sequences.

In our teaching, Exercise 2.41 has often been used as a homework assignment to give the students the opportunity to complete an independent experimental investigation that can reveal both superficial and quite deep structure. In the following, we give some useful background theory for these sequences.

Definition 2.1. A sequence $(x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ is said to be

- *Sturmian* if the number of distinct subwords of length k is $k+1$ for all $k \in \mathbb{N}$.
- *balanced* if for each $k \in \mathbb{N}$ there exists $w_k \in \mathbb{N}$ such that each subword of length k contains either precisely w_k or precisely $w_k + 1$ ones.
- *eventually periodic* if there exist $k, N \in \mathbb{N}$ such that $x_n = x_{n+k}$ for all $n \geq N$.

For a more detailed introduction to Sturmian systems, see [Fog02]. The following two theorems give fundamental descriptions of Sturmian sequences.

Theorem 2.2. *A sequence $(x_n)_{n\in\mathbb{N}} \subset \{0,1\}^{\mathbb{N}}$ is Sturmian if and only if it is balanced and not eventually periodic.*

Theorem 2.3. *A sequence $(x_n)_{n\in\mathbb{N}} \subset \{0,1\}^{\mathbb{N}}$ is Sturmian if and only if there exist $\alpha \in]0;1[\setminus\mathbb{Q}$ and $\rho \in \mathbb{R}$ such that*

$$x_n = \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor.$$

An intuitive way to construct such a Sturmian sequence is to consider a coordinate system and a line with irrational slope α. Build a sequence by considering the intersections with the grid lines of the positive integers: For each n, the nth entry should be 1 if the line crosses a horizontal line between $n-1$ and n, otherwise it should be 0. This is sketched in Figure 1. Note that the irrationality of α guarantees that this gives a well-defined sequence, and that this sequence has the form specified in Theorem 2.3.

Sturmian sequences have the following properties:

- A Sturmian sequence has the property that either 00 or 11 is not a subword.
- The number of distinct subwords of length n in an unbalanced aperiodic sequence grows exponentially with n. The growth rate of a Sturmian sequence is the smallest possible for an aperiodic sequence (in other words, a Sturmian sequence has minimal entropy).
- The frequency of ones in the N first elements of a Sturmian sequence given by α and ρ converges to α as N goes to infinity.

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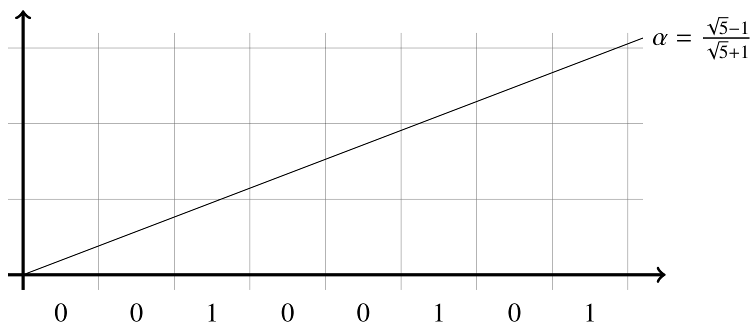

Figure 1: Construction of a Sturmian sequence with $\alpha = \frac{\sqrt{5}-1}{\sqrt{5}+1}$ and $\rho = 0$.

- A Sturmian sequence where 00 is not a subword can be recoded by replacing each block of ones by the length of the block and throwing away the zeroes. Since a Sturmian sequence is balanced, these blocks will have length k and $k+1$ for some $k \in \mathbb{N}$.

- Replacing each k by a 0 and each $k+1$ by a 1 (or vice versa) in the sequence constructed in the preceding entry gives a new Sturmian sequence in $\{0,1\}^{\mathbb{N}}$. Iterating the process yields a sequence of Sturmian sequences. For each Sturmian sequence, one can consider the length of the basic blocks of zeroes/ones and these numbers are given by the continued fraction expansion of the α which defines the original sequence. In short, this allows the Sturmian sequence to give an optimally efficient approximation of α. We touch on this theme in Exercise 5.23.

Example 2.6.2

For odd n, Player B should always choose the opposite of whatever digit Player A last played. It is not hard to prove that this is a winning strategy. However, it is much harder to find winning strategies for even n. Indeed, we do not know who can win the game at even $n \geq 8$. See also the discussion in Example 6.5.5. Note that Exercise 2.43 asks students to carry out an independent investigation of this problem. We have tried to make it clear that some values of n are untractable.

Example 2.8.1

In our experience, it is surprisingly hard to get students to make thorough automatic tests. This simple example has been chosen to show how such a test may be set up, but when students solve the accompanying exercises (Exercise 2.41 in particular), they often fail to carry out such tests.

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Example 2.8.2

Let $\beta \in \mathbb{R}$ with $\beta > 1$. For each $t \in [0;1]$ define sequences $(r_n(t))_{n\in\mathbb{N}}$ and $(x_n(t))_{n\in\mathbb{N}}$ by

$$r_1(t) = \langle \beta t \rangle, \qquad r_n(t) = \langle \beta r_{n-1}(t) \rangle,$$
$$x_1(t) = \lfloor \beta t \rfloor, \qquad x_n(t) = \lfloor \beta r_{n-1}(t) \rfloor,$$

where $\lfloor y \rfloor$ is the integer part – and $\langle y \rangle$ is the fractional part – of $y \in \mathbb{R}$. The sequence $x_1(t)x_2(t)\cdots$ is said to be the *β-expansion* of t, and Rényi has proved that

$$t = \sum_{n=1}^{\infty} \frac{x_n(t)}{\beta^n}.$$

The β-expansion of 1 is denoted $e(\beta)$. The sequences considered in the example are such expansions of 1 and knowing this greatly facilitates the study. Such sequences may be finite, periodic, eventually periodic or aperiodic depending on β. For example, $e(\beta) = 11000\cdots$ for $\beta = (1+\sqrt{5})/2$.

Define $B_\beta = \{x_n(t)\cdots x_m(t) \mid 1 \le n \le m, t \in [0;1]\}$. This set B_β is the *language* of a kind of *shift space* known as a *beta-shift*. The alphabet $\mathcal{A}_\beta$ consisting of the symbols that appear in the words in B_β is either $\{0,\ldots,\beta-1\}$ or $\{0,\ldots,\lfloor\beta\rfloor\}$ depending on whether β is an integer or not.

The following theorem is one of the properties that experimentation may reveal.

Theorem 2.4 (Parry)**.** *A sequence $a_1a_2\cdots$ is the β-expansion of 1 for some $\beta > 1$ if and only if $a_ka_{k+1}\cdots \le a_1a_2\cdots$ (in lexicographic order) for all $k \in \mathbb{N}$. Such a β is uniquely given by the expansion of 1.*

Let $a_1a_2\cdots$ be the β-expansion of 1 for some $\beta > 1$. For each $n \in \mathbb{N}$ let c_n be the number of words in $\mathcal{A}_\beta^n$ that are smaller than $a_1a_2\cdots a_n$ in lexicograpic order. Then c_n grows exponentially with n and the growth rate is given by β. Note that c_n counts the number of elements of B_β of length n.

Exercise 2.19–2.21

See the discussion of Examples 2.5.7 and 2.8.1 above.

Exercise 2.22

When we gave this exercise in our course recently, we provided an example

$$\begin{aligned}\mathbf{b}[5/2,1] = [&2,1,0,1,1,1,0,0,0,0,1,1,0,1,2,1,0,0,0,1,1,\\ &1,1,1,2,1,0,0,0,1,0,0,2,0,0,0,1,0,0,1,2,0,\\ &1,0,0,1,1,1,1,1,0,0,0,1,1,0,0,1,2,0,0,0,1,\\ &2,0,0,1,2,0,1,1,1,1,0,0,0,2,0,1,0,0,2,0,0,\\ &1,0,1,0,1,1,0,1,1,0,1,1,0,1,2,0,0,1,1,1,1,\\ &2,0,1,1,0,0,0,1,1,2,0,0,0,0,2,0,2,0,0,0,1,\ldots].\end{aligned}$$

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and explicitly asked the students to test their program against it. Nevertheless, a vast majority of students had made a program using `evalf` to speed up computations, and this would cause their program to return

$$\mathbf{b}[5/2, 1] = [2, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 2, 1, 0, 0, 0, 1, 1, \\ 1, 1, 1, 2, 1, 0, 0, 0, 1, 0, 0, \underline{1, 2, 0, 1, 0, 1, \ldots}]$$

at standard precision. Interestingly, **none** of the students computing the incorrect values had noted it – they happily concluded that their program worked after visually inspecting, one may assume, the first line or so. This of course provided for a very useful discussion afterwards, and the students were taught a lesson they are nok likely to forget! We strongly recommend that you set up your students for this or a similar experience.

See also the discussion of Example 2.8.2 above.

Exercise 2.23–2.24

See discussion of Example 2.8.2 above.

Exercise 2.25

See discussion of Example 1.6.2 above.

Exercise 2.26–2.40

All of these exercises concern very hard problems, some of which are open. They are included here to give students the opportunity to get experience with experimental investigation and the formulation of hypotheses. Clearly, this will not result in new information about the Goldbach conjecture, and in several cases the students will formulate false hypotheses because extremely extensive computations are needed to find the first counterexample. However, we believe that these classical problems can still give excellent training. Indeed, formulating a conjecture based on experimental evidence and later realizing that the conjecture is false may be a very important lesson.

Note that Table 2.1 gives references to relevant literature and a summary of the state of each problem. However, we believe that the exercises will be more interesting if the students try to solve them without first consulting the table.

Exercise 2.41

See the discussion of Example 2.5.7 and 2.8.1 above. In our teaching, this exercise has often been used as part of the first homework assignment.

Exercise 2.43

See discussion of Example 2.6.2 above.

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Exercise 2.44

See discussion of Example 2.8.2 above for a list of interesting features that may be revealed by such an investigation. Inspired by Example 2.5.7 and 2.8.1, students may attempt to find a systematic dependency between β and the frequency of zeroes, ones and twos. However, such an investigation is non-trivial due to problems with rounding errors and can easily lead to false hypotheses. For more on this subject, see the discussion of Exercise 4.39 below.

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Material for Chapter 3

Exercise 3.1

The answer is 97. This is probably most easily seen by bisecting the intervals on which the program is run (and using `forget` repeatedly).

Exercise 3.3

The zeroes are listed in A28488.

Exercise 3.5

The matrices can be created with `MM(9)`, `MM(13)` and `MM(29)` where we define `MM` as below.

```
[> MM:=n->Matrix(n,n,convert(map(x->(x[1]+1,x[2]+1)=1,
            map(x->subs(x,[c,d]),[msolve(a^4+b^4+c^4=d^4,n)])),set));
```

Exercise 3.12

With

```
[> t:=proc(m,N)
      if m=0 then
         return 1;
      else
         return (N-1)*t(m-1,N)+(N+1)^(m-1);
      end if;
   end proc;
```

we get the correct output which is given in Exercise 5.14. The closed form is

$$\frac{(N-1)^m+(N+1)^m}{2}$$

which may be implemented as

```
[> tt:=(m,N)->((N-1)^m+(N+1)^m)/2;
```

The test computes all differences and returns 0 precisely if they all agree.

Exercise 3.13

We have used

```
[> isPower:=proc(y,n,k) local x;
      if(k=1) then return is(y=floor(y^(1/n))^n); end if;
      for x from max(1,floor((y/k)^(1/n))) to floor(y^(1/n)) do
         if(isPower(y-x^n,n,k-1)) then
```