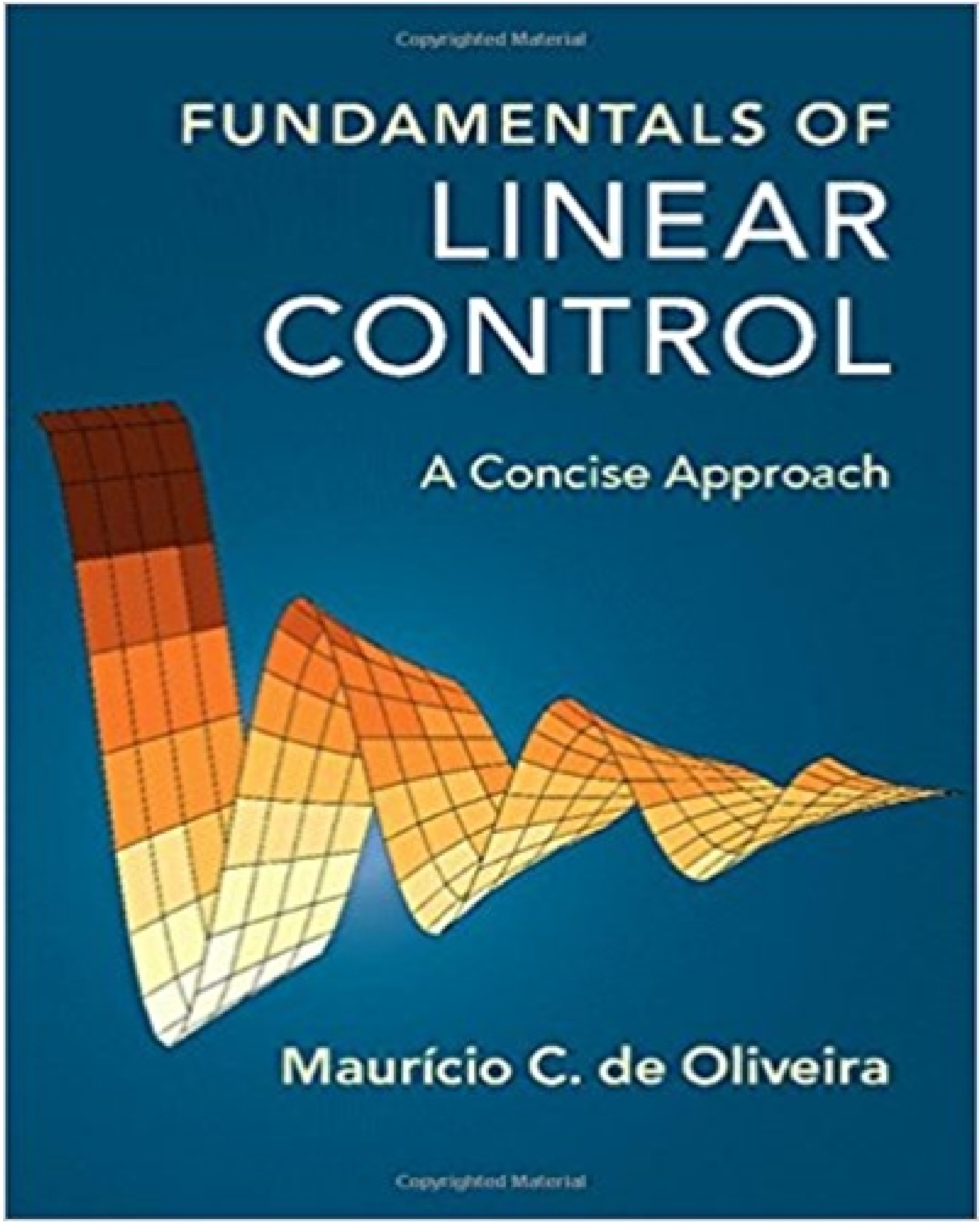
Solutions for Fundamentals of Linear Control A Concise Approach 1st Edition by Oliveira

CLICK HERE TO ACCESS COMPLETE Solutions



Solutions

Appendix B Chapter 2

P2.1. In SI units, $v(t)$ is in m/s and $\dot{v}(t)$ is in m/s^2, and $m\dot{v}$ and $bv(t)$ are in N = kg m/s^2, m is in kg and b is in kg/s. Therefore $\lambda = -b/m$ is in s^{-1} and $\tau = -\lambda^{-1}$ is in s.

P2.2. About $x = 0$ the function $f(x) = \tan^{-1}(x)$ admits the Taylor Series expansion $f(x) = x + O(x^3)$ hence

$$c\tan^{-1}(\alpha^{-1}y) = c\alpha^{-1}y + O(y^3) \approx c\alpha^{-1}y = by.$$

P2.3. If $u(t) = (b/p)\bar{y}$ and $w(t) = \bar{w}$ then

$$\dot{y}(t) + \frac{b}{m}y(t) = \frac{p}{m}u(t) + \frac{p}{m}w(t) = \frac{b}{m}\bar{y} + \frac{p}{m}\bar{w}$$

and

$$y(t) = \tilde{y}(1 - e^{\lambda t}) + y_0 e^{\lambda t}$$

where

$$\lambda = -\frac{b}{m}, \qquad \tilde{y} = \bar{y} + \frac{p}{b}\bar{w} = \bar{y} + G(0)\,\bar{w}.$$

Consequently with $y_0 = \bar{y}$

$$y(t) = (\bar{y} + G(0)\,\bar{w})\,(1 - e^{-(b/m)t}) + \bar{y}\,e^{-(b/m)t} = \bar{y} + (1 - e^{-(b/m)t})\,G(0)\,\bar{w}$$

and

$$\Delta y(t) = y(t) - y_0 = y(t) - \bar{y} = (1 - e^{-(b/m)t})\,G(0)\,\bar{w}.$$

P2.4. With Newton's second law help we write

$$m\dot{v} = F = m\,g - bv,$$

or $m\dot{v} + bv = mg$.

P2.5. The solution to the first-order differential equation from P2.4 is

$$v(t) = \tilde{v}\,(1 - e^{\lambda t}) + v(0)\,e^{-\lambda t}, \quad \lambda = -\frac{b}{m}, \quad \tilde{v} = \frac{mg}{b}.$$

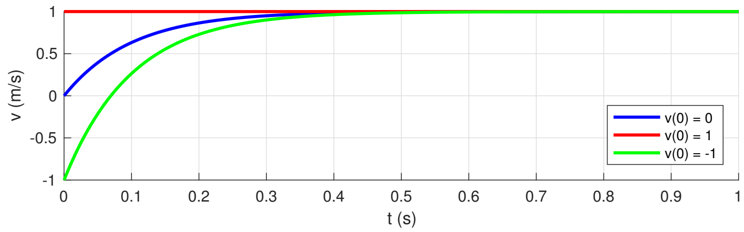
If $m = 1$kg, $g = 10$m/s^2, $b = 10$kg/s then

$$\tilde{v} = \frac{mg}{b} = 1\text{m/s}, \qquad \lambda = -\frac{b}{m} = -10\text{s}^{-1}$$

with which

$$v(t) = 1 + (v(0) - 1)\,e^{-10t}.$$

The responses when $v(0) = 0$, $v(0) = 1$m/s, and $v(0) = -1$m/s should be as in the following plot:



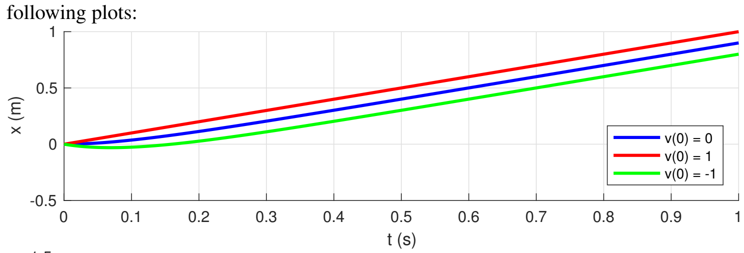
P2.6. The vertical position is obtained after integrating the solution to problem P2.5 as in

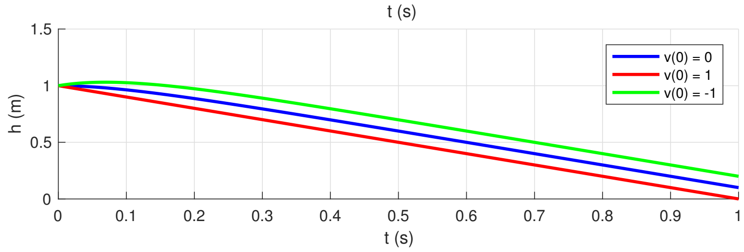
$$x(t) = x(0) + \int_0^t v(\tau)\,d\tau = x(0) + \int_0^t 1 + (v(0)-1)\,e^{-10\tau}\,d\tau$$
$$= x(0) + t + \frac{(v(0)-1)}{10}(1 - e^{-10t}).$$

It is related to the height measured from the ground through

$$h(t) = h(0) - x(t)$$

The responses when $x(0) = 0$, $h(0) = 1$ and $v(0) = 0$, $v(0) = 1\text{m/s}$, and $v(0) = -1\text{m/s}$ should be as in the following plots:





P2.7. Differentiate the given expression for $v(t)$ and substitute into the differential equation.

P2.8. The terminal velocity in the free-fall and the parachute phases can be calculated using

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \tilde{v}(1 - e^{\lambda t}) + v(0)\,e^{-\lambda t} = \tilde{v}$$

where

$$\lambda = -\frac{b}{m}, \qquad \tilde{v} = \frac{mg}{b}.$$

In this problem $m = 70\text{kg}$, $g = 10\text{m/s}^2$, and

$$\tilde{v}_f = 200\text{km/h} \approx 56\text{m/s}$$

during free-fall and

$$\tilde{v}_c = 20\text{km/h} \approx 5.6\text{m/s}$$

after the parachute opens. From these we calculate

$$b_f = \frac{mg}{\tilde{v}_f} = \frac{700}{56} \approx 12.6\text{kg/s} \qquad b_c = \frac{mg}{\tilde{v}_c} = \frac{700}{5.6} \approx 126\text{kg/s}$$

with which

$$\lambda_f = -\frac{b_f}{m} \approx -0.18\text{s}^{-1} \quad \lambda_c = -\frac{b_f}{m} \approx -1.8\text{s}^{-1}.$$

The time constants in each phase are

$$\tau_f = -\lambda_f^{-1} \approx 5.5\text{s}, \qquad \tau_c = -\lambda_c^{-1} \approx 0.55\text{s}.$$

Assuming that the speed at free-fall is close to $\tilde{v}_f$ when the parachute opens we can calculate

$$v_c(t) = \tilde{v}_c\,(1 - e^{\lambda_c t}) + \tilde{v}_f\, e^{-\lambda_c t}$$

from which the time necessary to reach 29km/h is

$$v_c(t^*) = \tilde{v}_c\,(1 - e^{\lambda_c t^*}) + \tilde{v}_f\, e^{-\lambda_c t^*} = 29\text{km/h} \approx 8\text{m/s}.$$

or

$$5.6 + (56 - 5.6)\, e^{-1.8t^*} \approx 8\text{m/s} \quad \Longrightarrow \quad t^* \approx -\frac{1}{1.8}\log\frac{8-5.6}{56-5.6} \approx 1.7\text{s}$$

so the parachute should be opened at least 1.7s prior to landing.

As for the height we calculate

$$h_c(t) = h_c(0) - x_c(t) = h_c(0) - x_c(0) - \tilde{v}_c t + \frac{(\tilde{v}_f - \tilde{v}_c)}{\lambda_c}(1 - e^{\lambda_c t})$$

where $t = 0$ is the moment the parachute opens and $x_c(0)$ can be considered equal to 0. If after $t^* \approx 1.7$s we have landed then $h_c(t^*) = 0$ and

$$h_c(0) = 5.6 \times 1.7 + \frac{(56 - 5.6)}{1.8}(1 - e^{-1.8\times 1.7}) \approx 5.6 \times 1.7 + 28 \times 0.95 \approx 36\text{m}$$

is the minimum height at which the parachute can be opened.

If the dive starts at 4km or 4000m with zero vertical velocity and the parachute is opened after 60s, at that point the diver should be at the heigth

$$\begin{aligned} h_f(60) &= h_f(0) - x_f(0) - \tilde{v}_f \times 60 + \frac{(v_f(0) - \tilde{v}_f)}{\lambda_f}(1 - e^{\lambda_f \times 60}) \\ &= 4000 - 0 - 56 \times 60 + \frac{56}{0.18}(1 - e^{-0.18\times 60}) \approx 975\text{m} \end{aligned}$$

From that height the fall will continue until

$$\begin{aligned} 0 &= h_c(0) - x_c(0) - \tilde{v}_c t + \frac{(\tilde{v}_f - \tilde{v}_c)}{\lambda_c}(1 - e^{\lambda_c t}) \\ &= 975 - 0 - 5.6t + \frac{(56 - 5.6)}{1.8}(1 - e^{-1.8\times t}) \end{aligned}$$

This is approximately $975/5.6 \approx 176$s. One can solve the nonlinear equation for a more accurate solution of about 181s. The total time airborne is approximately $60 + 181 = 240$s, that is 4 minutes.

P2.9. The terminal velocity in the free-fall and the parachute phases can be calculated using

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{1 + \alpha\, e^{\lambda t}}{1 - \alpha\, e^{\lambda t}}\, \tilde{v} = \tilde{v}.$$

As in P2.8, $m = 70$kg, $g = 10\text{m/s}^2$, and

$$\tilde{v}_f = 200\text{km/h} \approx 56\text{m/s}$$

during free-fall and

$$\tilde{v}_c = 20\text{km/h} \approx 5.6\text{m/s}$$

after the parachute opens. From these we calculate

$$b_f = \frac{mg}{\tilde{v}_f^2} = \frac{700}{56^2} \approx 0.227\text{kg/s}^2 \qquad b_c = \frac{mg}{\tilde{v}_c^2} = \frac{700}{5.6^2} \approx 22.7\text{kg/s}^2$$

with which

$$\lambda_f = -\frac{b_f}{m} \approx -0.36 \quad \lambda_c = -\frac{b_f}{m} \approx -3.6.$$

Because the system is now nonlinear we have to calculate the time constants based on the definition, that is τ is the time at which, starting from $v(0) = 0$,

$$v(t) = (1 - e^{-1})\,\tilde{v}$$

that is

$$\frac{1 + \alpha\, e^{\lambda t}}{1 - \alpha\, e^{\lambda t}} = 1 - e^{-1}, \qquad \alpha = \frac{v(0) - \tilde{v}}{v(0) + \tilde{v}} = -1,$$

which implies

$$\tau = -\lambda^{-1} \ln(2e - 1).$$

Using this formula the time constants in each phase are

$$\tau_f = -\lambda_f^{-1} \approx 4.1\text{s}, \qquad \tau_c = -\lambda_c^{-1} \approx 0.41\text{s}.$$

Assuming that the speed at free-fall is close to $\tilde{v}_f$ when the parachute opens we can calculate

$$v_c(t) = \frac{1 + \alpha\, e^{\lambda_c t}}{1 - \alpha\, e^{\lambda_c t}}\,\tilde{v}, \qquad \alpha = \frac{\tilde{v}_f - \tilde{v}_c}{\tilde{v}_f + \tilde{v}_c} \approx 0.82$$

from which the time necessary to reach $v(t^*) = 29\text{km/h}$ is

$$t^* = \lambda_c^{-1} \ln\left(\frac{v_c(t^*) - \tilde{v}_c}{(v_c(t^*) + \tilde{v}_c)\,\alpha}\right) \approx 0.42\text{s},$$

so the parachute should be opened at least 0.42s prior to landing.

In order to calculate the height we first integrate

$$x_c(t) = x_c(0) + \int_0^t v_c(\tau)\,d\tau = x_c(0) + \tilde{v}_c t + \frac{2\,\tilde{v}_c}{\lambda_c} \ln\left(\frac{1 - \alpha}{1 - \alpha\, e^{\lambda_c t}}\right)$$

then calculate for $t = t^*$, $h_c(t^*) = 0$, and $x_c(0) = 0$, the quantity

$$h_c(0) = h_c(t^*) + x_c(t^*) = x_c(t^*) \approx 6.9\text{m},$$

which is the minimum height at which the parachute can be opened.

If the dive starts at 4km or $h_f(0) = 4000\text{m}$ with zero vertical velocity and the parachute is opened after 60s, at that point the diver should be at the heigth

$$h_f(60) = h_f(0) - x_f(0) - \tilde{v}_f 60 + \frac{2\,\tilde{v}_f}{\lambda_f} \ln\left(\frac{1 - \alpha}{1 - \alpha\, e^{\lambda_f 60}}\right) \approx 880\text{m}$$

From that height the fall will continue until

$$\begin{aligned} 0 = h_c(t) &= h_c(0) - x_c(t) \\ &= 880 - 0 - \tilde{v}_c t + \frac{2\,\tilde{v}_c}{\lambda_c} \ln\left(\frac{1 - \alpha}{1 - \alpha\, e^{\lambda_c 60}}\right) \end{aligned}$$

This is approximately $880/5.6 \approx 159$s. One can solve the nonlinear equation for a more accurate solution of about 58s. The total time airborne is approximately $60 + 158 = 228$s, that is a bit less than 4 minutes.

P2.10. At the inertia J_1

$$J_1\,\dot{\omega}_1 = \tau + f_1\,r_1 - f_2\,r_1,$$

and at the inertia J_2

$$J_2\,\dot{\omega}_2 = f_2\,r_2 - f_1\,r_2.$$

Since the inertias are coupled by a belt, the linear speeds must be the same:

$$\omega_1 r_1 = \omega_2 r_2 \quad\Longrightarrow\quad \omega_2 = (r_1/r_2)\omega_1.$$

Multiplying the first equation by r_2 and the second by r_1 we obtain

$$\begin{aligned} r_2 J_1\,\dot{\omega}_1 &= r_2\tau + f_1\,r_1 r_2 - f_2\,r_1 r_2,\\ r_1 J_2\,\dot{\omega}_2 &= f_2\,r_1 r_2 - f_1\,r_1 r_2.\end{aligned}$$

Adding these together:

$$r_2 J_1\,\dot{\omega}_1 + r_1 J_2\,\dot{\omega}_2 = r_2\tau.$$

Subsitituting $\omega_2 = (r_1/r_2)\omega_1$:

$$r_2 J_1\,\dot{\omega}_1 + r_1^2/r_2 J_2\,\dot{\omega}_1 = r_2\tau,$$

and multiplying by r_2

$$(J_1 r_2^2 + J_2 r_1^2)\,\dot{\omega}_1 = r_2^2\tau.$$

P2.11. From P2.10

$$r_2(f_2 - f_1) = J_2\,\dot{\omega}_2$$

so $f_2 = f_1$ only if $\dot{\omega}_2 = (r_1/r_2)\,\dot{\omega}_1 = 0$, that is at constant speed.

P2.12. At the inertia J_1

$$J_1\,\dot{\omega}_1 + b_1\,\omega_1 = \tau + f_1\,r_1 - f_2\,r_1,$$

and at the inertia J_2

$$J_2\,\dot{\omega}_2 + b_2\omega_2 = f_2\,r_2 - f_1\,r_2.$$

Since the inertias are coupled by a belt, the linear speeds must be the same, that is

$$\omega_1 r_1 = \omega_2 r_2 \quad\Longrightarrow\quad \omega_2 = (r_1/r_2)\omega_1.$$

Multiplying the first equation by r_2 and the second by r_1 we obtain

$$\begin{aligned} r_2 J_1\,\dot{\omega}_1 + r_2 b_1\,\omega_1 &= r_2\tau + f_1\,r_1 r_2 - f_2\,r_1 r_2,\\ r_1 J_2\,\dot{\omega}_2 + r_1 b_2\omega_2 &= f_2\,r_1 r_2 - f_1\,r_1 r_2.\end{aligned}$$

Adding these together:

$$r_2 J_1\,\dot{\omega}_1 + r_1 J_2\,\dot{\omega}_2 + r_2 b_1\,\omega_1 + r_1 b_2\omega_2 = r_2\tau.$$

Substituting $\omega_2 = (r_1/r_2)\omega_1$,

$$r_2 J_1\,\dot{\omega}_1 + r_1^2/r_2 J_2\,\dot{\omega}_1 + r_2 b_1\,\omega_1 + r_1^2/r_2 b_2\omega_1 = r_2\tau,$$

and multiplying by r_2

$$(J_1 r_2^2 + J_2 r_1^2)\,\dot{\omega}_1 + (b_1 r_2^2 + b_2 r_1^2)\,\omega_1 = r_2^2\tau.$$

P2.13. The solution to the first-order differential equation from P2.12 when $\tau = \bar{\tau}$ is constant is

$$\omega_1(t) = \bar{\omega}_1 (1 - e^{\lambda t}) + \omega_1(0) e^{-\lambda t},$$

where

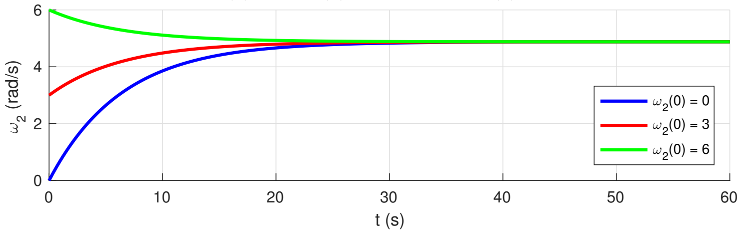
$$\lambda = -\frac{b_1 r_2^2 + b_2 r_1^2}{J_1 r_2^2 + J_2 r_1^2}, \qquad \bar{\omega}_1 = \frac{r_2^2}{b_1 r_2^2 + b_2 r_1^2} \bar{\tau}.$$

If $\tau = 1$N m, $r_1 = 25$mm, $r_2 = 500$mm, $b_1 = 0.01$kg m^2/s, $b_2 = 0.1$kg m^2/s, $J_1 = 0.0031$kg m^2, $J_2 = 25$kg m^2, then

$$\bar{\omega}_1 \approx 98\text{rad/s}, \qquad \lambda \approx -0.16\text{s}^{-1}.$$

Because we are interested in ω_2 we first calculate ω_1 then $\omega_2(t) = (r_1/r_2)\omega_1(t)$. Note that the initial condition must also be translated as $\omega_1(0) = (r_2/r_1)\omega_2(0)$.

The responses when $\omega_2(0) = 0$, $\omega_2(0) = 3$rad/s, and $\omega_2(0) = 6$rad/s should be as in the following plot:



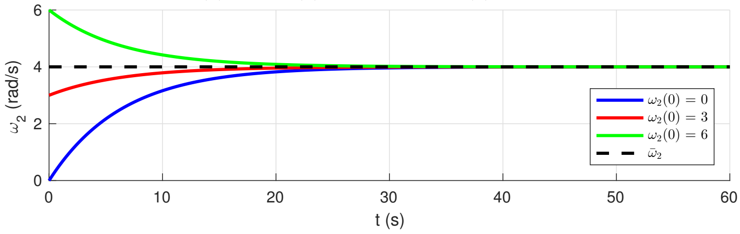
P2.14. In order to achieve a desired rotational speed $\bar{\omega}_2 = 4$rad/s in steady-state we set a desired rotational speed $\bar{\omega}_1 = (r_2/r_1)\bar{\omega}_2$ and calculate

$$\tau = \bar{\tau} = \frac{b_1 r_2^2 + b_2 r_1^2}{r_2^2} \bar{\omega}_1.$$

Substituting the data we obtain

$$\bar{\omega}_1 = (r_2/r_1)\bar{\omega}_2 = 80\text{rad/s}, \qquad \bar{\tau} = \frac{b_1 r_2^2 + b_2 r_1^2}{r_2^2} \bar{\omega}_1 \approx 0.82\text{N m}.$$

The responses when $\omega_2(0) = 0$, $\omega_2(0) = 3$rad/s, and $\omega_2(0) = 6$rad/s should be as in the following plot:



P2.15. In this case the control would continue to be

$$\tau = \bar{\tau} = \frac{b_1 r_2^2 + b_2 r_1^2}{r_2^2} \frac{r_2}{r_1} \bar{\omega}_2$$

while the velocity would converge to

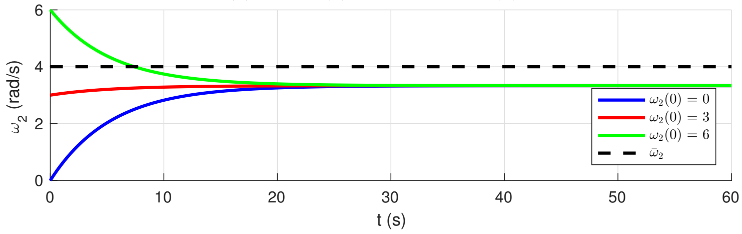
$$\bar{\omega}_2 = \frac{r_1}{r_2} \bar{\omega}_1 = \frac{r_2^2}{1.2(b_1 r_2^2 + b_2 r_1^2)} \bar{\tau} = \frac{1}{1.2} \bar{\omega}_2 \approx 0.83 \bar{\omega}_2 \approx 3.3\text{rad/s}.$$

The rate of convergence would also become faster since

$$\lambda = -1.2 \frac{b_1 r_2^2 + b_2 r_1^2}{J_1 r_2^2 + J_2 r_1^2},$$

would also be 20% larger.

The responses when $\omega_2(0) = 0$, $\omega_2(0) = 3$rad/s, and $\omega_2(0) = 6$rad/s should be as in the following plot:



P2.16. The connection of the model

$$(J_1 r_2^2 + J_2 r_1^2)\,\dot{\omega}_1 + (b_1 r_2^2 + b_2 r_1^2)\,\omega_1 = r_2^2 \tau$$

with the controller

$$\tau = K\,(\bar{\omega}_2 - \omega_2) = (r_1/r_2)\,K\,(\bar{\omega}_1 - \omega_1)$$

where $\bar{\omega}_1 = (r_2/r_1)\,\bar{\omega}_2$ produces the closed-loop differential equation

$$(J_1 r_2^2 + J_2 r_1^2)\,\dot{\omega}_1 + (b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K)\,\omega_1 = r_1 r_2 K\,\bar{\omega}_1.$$

The solution to this equation is

$$\omega_1(t) = \tilde{\omega}_1\,(1 - e^{\lambda t}) + \omega_1(0)\,e^{-\lambda t},$$

where

$$\tilde{\omega}_1 = \frac{r_1 r_2 K\,\bar{\omega}_1}{b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K}, \qquad \lambda = -\frac{b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K}{(J_1 r_2^2 + J_2 r_1^2)}.$$

The closed-loop time-constant is

$$\tau = -\lambda^{-1} = \frac{J_1 r_2^2 + J_2 r_1^2}{b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K}.$$

Be careful not to confuse the time-constant with the torque! We want to select the control gain K to set $\tau = 3$s. Using the data from P2.13

$$K = \frac{(J_1 r_2^2 + J_2 r_1^2)/\tau - (b_1 r_2^2 + b_2 r_1^2))}{r_1 r_2} \approx 0.23$$

In open-loop the time-constant is

$$\tau = -\lambda^{-1} = \frac{J_1 r_2^2 + J_2 r_1^2}{b_1 r_2^2 + b_2 r_1^2} \approx 6.4\text{s}$$

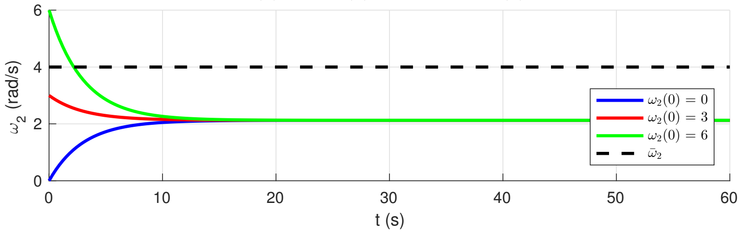
The steady state error is

$$\begin{aligned}
\bar{\omega}_2 - \tilde{\omega}_2 &= (r_1/r_2)\,(\bar{\omega}_1 - \tilde{\omega}_1) \\
&= (r_1/r_2)\left(\bar{\omega}_1 - \frac{r_1 r_2 K\,\bar{\omega}_1}{b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K}\right) \\
&= (r_1/r_2)\,\frac{b_1 r_2^2 + b_2 r_1^2}{b_1 r_2^2 + b_2 r_1^2 + r_1 r_2 K}\,\bar{\omega}_1 \\
&= (r_1/r_2)\,\frac{1}{1 + r_1 r_2 K/(b_1 r_2^2 + b_2 r_1^2)}\,\bar{\omega}_1
\end{aligned}$$

Substituting the data

$$\bar{\omega}_2 - \tilde{\omega}_2 \approx 1.9\text{rad/s}$$

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.17. In closed-loop the gain would remain the same and the steady-state error would be

$$\bar{\omega}_2 - \tilde{\omega}_2 = (r_1/r_2)\frac{1}{1 + r_1 r_2 K/(1.2(b_1 r_2^2 + b_2 r_1^2))}\bar{\omega}_1 \approx 2.1\text{rad/s}$$

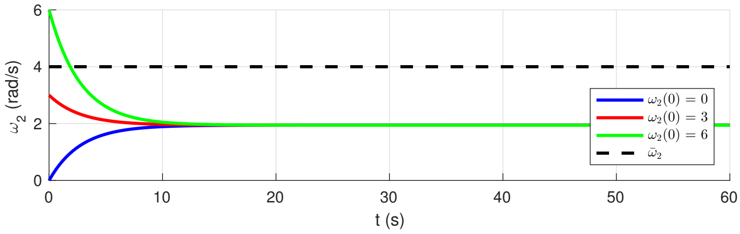
which is less than a 10% change.

The time-constant would also

$$\tau = \frac{J_1 r_2^2 + J_2 r_1^2}{1.2(b_1 r_2^2 + b_2 r_1^2) + r_1 r_2 K} \approx 2.7\text{s}$$

which is also less than a 10% change. As expected, in closed-loop the sensitivity to system parameter variation is reduced.

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.18. At the inertia J_1

$$J_1\,\dot{\omega}_1 + b_1\,\omega_1 = \tau + r(f_1 - f_2)$$

and at the inertia J_2

$$J_2\,\dot{\omega}_2 + b_2\,\omega_2 = r(f_4 - f_3).$$

Since the inertias are coupled by a belt, the linear speeds must be the same, that is

$$\omega_1 r = \omega_2 r \quad \Longrightarrow \quad \omega_2 = \omega_1 = \omega$$

Similarly

$$v_1 = \omega r, \quad v_2 = -\omega r$$

so that at the masses

$$r m_1 \dot{\omega} = m_1 \dot{v}_1 = m_1 g + f_3 - f_1,$$
$$r m_2 \dot{\omega} = -m_2 \dot{v}_2 = -m_2 g + f_2 - f_4.$$

Multiplying the last two equations by r and adding to the first two equations we obtain

$$(J_1 + J_2 + r^2 m_1 + r^2 m_2)\,\dot{\omega} + (b_1 + b_2)\,\omega = \tau + g\,r\,(m_1 - m_2).$$

P2.19. The solution to the first-order differential equation from P2.18 when $\tau = 0$ is

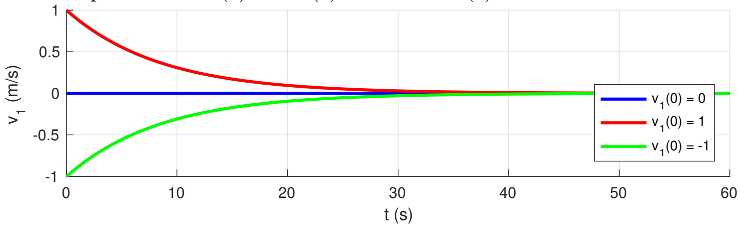
$$\omega(t) = \omega(0)\,e^{-\lambda t}, \qquad \lambda = -\frac{b_1 + b_2}{J_1 + J_2 + r^2(m_1 + m_2)}$$

If $\tau = 0$N m, $r = 1$m, $b_1 = b_2 = 120$kg m^2/s, $J_1 = J_2 = 20$kg m^2, then

$$\lambda \approx -0.16\text{s}^{-1}.$$

Because we are interested in v_1 we first calculate ω then $v_1(t) = r\omega(t)$. Note that the initial condition must also be translated as $\omega(0) = v_1(0)/r$.

The responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.20. In this case there is a net torque due to the difference between the masses m_1 and m_2. The solution to the first-order differential equation from P2.18 is then

$$\omega(t) = \tilde{\omega}\,(1 - e^{\lambda t}) + \omega(0)\,e^{-\lambda t},$$

where

$$\tilde{\omega} = gr\frac{m_1 - m_2}{b_1 + b_2}, \qquad \lambda = -\frac{b_1 + b_2}{J_1 + J_2 + r^2(m_1 + m_2)}.$$

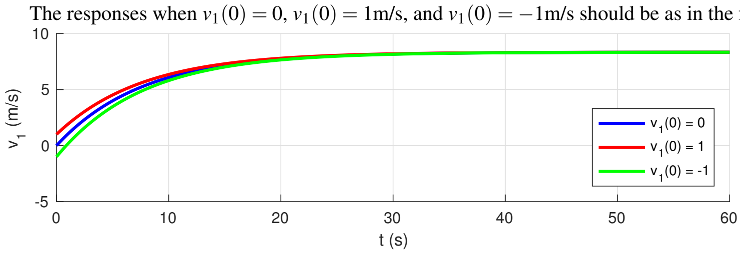
Substituting the problem data

$$\tilde{\omega} \approx 8.33\text{rad/s}, \qquad \lambda \approx -0.13\text{s}^{-1},$$

As before, we first calculate $\omega(t)$ then $v_1(t) = r\omega(t)$.

Note that because mass m_2 is now smaller than mass m_1 the masses will no longer converge to zero velocities. Without braking, the mass m_1 would move to the bottom of the elevator.

The responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.21. In order to achieve a desired linear speed $\bar{v}_1 = 2$m/s in steady-state we set a desired rotational speed $\omega = \bar{\omega} = \bar{v}_1/r$ and calculate

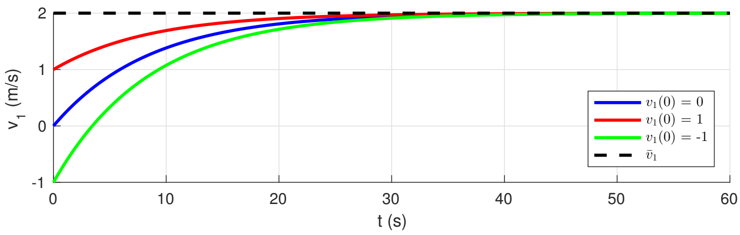
$$\tau = \bar{\tau} = (b_1 + b_2)\,\bar{\omega} - g\,r\,(m_1 - m_2).$$

Substituting the data we obtain

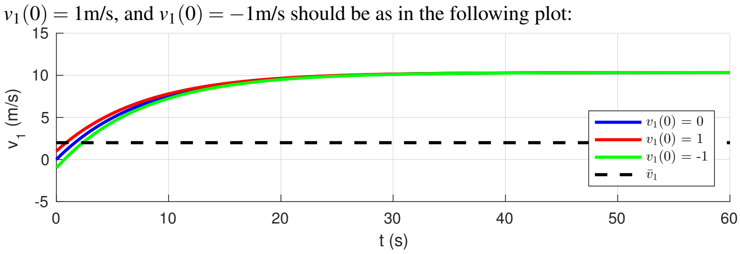
$$\bar{\tau} = (b_1 + b_2)\,\bar{\omega} = 480\text{N m}.$$

Note that the required torque is relatively small, since $m_2 = m_1$ and it only has to overcome friction.

The responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.22. Proceed as in P2.21 to show that the responses when $m_2 = 800$kg, τ is as in P2.21 and $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:

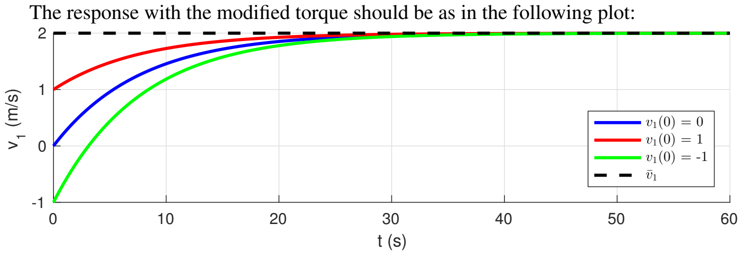


Note that the final velocity is no longer the desired velocity $\bar{v}_1 = 2$m/s. In order to achieve a desired speed we need to recalculate

$$\tau = \bar{\tau} = (b_1 + b_2)\,\bar{\omega} - g\,r\,(m_1 - m_2) = -1520\text{N m}.$$

The torque necessary to keep the velocity at 1m/s is now a breaking torque since m_2 is lighter than m_1. It is also now much higher when compared to P2.21 since it also has to support the mismatch between the masses m_1 and m_2.

The response with the modified torque should be as in the following plot:



that looks very much like the response obtained in P2.21 except for a slightly smaller $\lambda = -0.1304$ when compared with $\lambda = -0.1176$ from P2.21.

P2.23. The connection of the model

$$(J_1 + J_2 + r^2 m_1 + r^2 m_2)\,\dot{\omega} + (b_1 + b_2)\,\omega = \tau + g\,r\,(m_1 - m_2)$$

with the controller

$$\tau = K(\bar{v}_1 - v_1) = r\,K(\bar{\omega} - \omega)$$

where $\bar{\omega} = r\,\bar{v}_1$ produces the closed-loop differential equation

$$(J_1 + J_2 + r^2 m_1 + r^2 m_2)\,\dot{\omega} + (b_1 + b_2 + r\,K)\,\omega = r\,K\bar{\omega} + g\,r\,(m_1 - m_2).$$

The solution to this equation is

$$\omega(t) = \tilde{\omega}\,(1 - e^{\lambda t}) + \omega(0)\,e^{-\lambda t},$$

where

$$\tilde{\omega} = \frac{r\,K\bar{\omega} + g r\,(m_1 - m_2)}{b_1 + b_2 + r\,K}, \qquad \lambda = -\frac{b_1 + b_2 + r\,K}{J_1 + J_2 + r^2(m_1 + m_2)}.$$

The closed-loop time-constant is

$$\tau = -\lambda^{-1} = \frac{J_1 + J_2 + r^2(m_1 + m_2)}{b_1 + b_2 + rK}$$

Be careful not to confuse the time-constant with the torque! We want to select the control gain K to set $\tau = 5$s. Using the data from P2.19

$$K = \frac{[J_1 + J_2 + r^2(m_1 + m_2)]/\tau - (b_1 + b_2)}{r} \approx 168$$

In open-loop the time-constant is

$$\tau = -\lambda^{-1} = \frac{J_1 + J_2 + r^2(m_1 + m_2)}{b_1 + b_2} \approx 8.5\text{s}$$

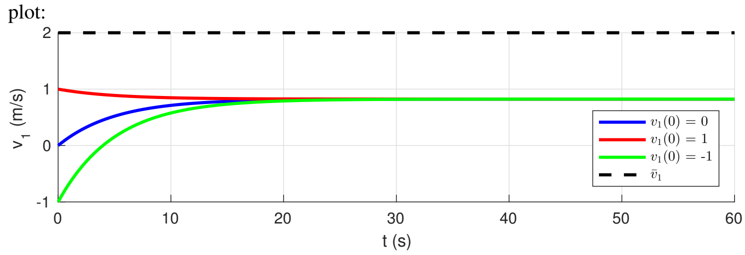
The steady state error is

$$\begin{aligned}
\bar{v}_1 - \tilde{v}_1 &= r(\bar{\omega} - \tilde{\omega}) \\
&= r\left(\bar{\omega} - \frac{rK\bar{\omega} + gr(m_1 - m_2)}{b_1 + b_2 + rK}\right) \\
&= r\frac{(b_1 + b_2)\bar{\omega} - gr(m_1 - m_2)}{b_1 + b_2 + rK} \\
&= r\frac{1}{1 + rK/(b_1 + b_2)}\bar{\omega} - gr^2\frac{(m_1 - m_2)/(b_1 + b_2)}{1 + rK/(b_1 + b_2)}
\end{aligned}$$

Substituting the data

$$\bar{v}_1 - \tilde{v}_1 \approx 1.18\text{m/s}$$

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:

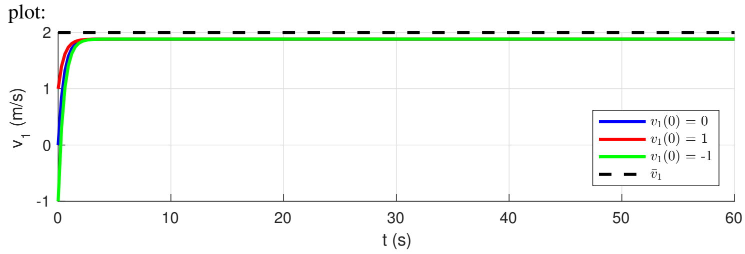


P2.24. The solution is as in P2.23 with $\tau = 0.5$s. The relevant quantities are:

$$\begin{aligned}
K &\approx 3840, \\
\tau &\approx 8.5\text{s} \qquad \text{(open-loop)}, \\
\bar{v}_1 - \tilde{v}_1 &\approx 0.12\text{m/s}
\end{aligned}$$

Note how high the gain is. This high gain will produce high closed-loop torques that the motors might have trouble delivering.

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.25. Using the data from P2.23 and $m_2 = 800$kg we recalculate the control gain

$$K = \frac{[J_1 + J_2 + r^2(m_1 + m_2)]/\tau - (b_1 + b_2)}{r} \approx 128$$

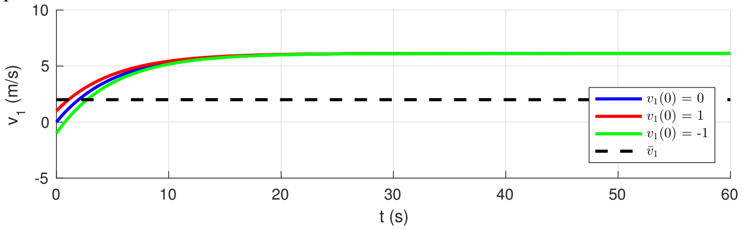
and the steady state error

$$\bar{v}_1 - \tilde{v}_1 = r\frac{1}{1 + rK/(b_1 + b_2)}\bar{\omega} - gr^2\frac{(m_1 - m_2)/(b_1 + b_2)}{1 + rK/(b_1 + b_2)} \approx -4.13\text{m/s}.$$

The open-loop time-constant is

$$\tau = -\lambda^{-1} = \frac{J_1 + J_2 + r^2(m_1 + m_2)}{b_1 + b_2} \approx 7.67\text{s}$$

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:

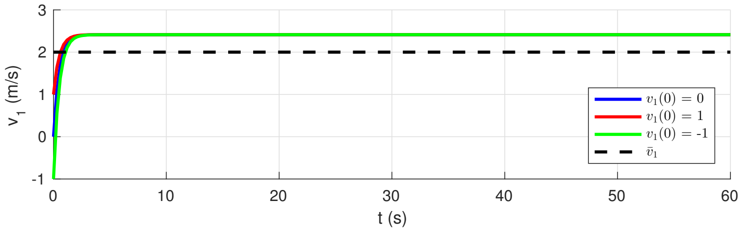


P2.26. The solution is as in P2.25 with $\tau = 0.5$s. The relevant quantities are:

$$K \approx 3440,$$
$$\tau \approx 8.5\text{s} \qquad \text{(open-loop)},$$
$$\bar{v}_1 - \tilde{v}_1 \approx -0.41\text{m/s}$$

Note how high the gain is. This high gain will produce high closed-loop torques that the motors might have trouble delivering.

The closed-responses when $v_1(0) = 0$, $v_1(0) = 1$m/s, and $v_1(0) = -1$m/s should be as in the following plot:



P2.27. With Newton's second law help we write

$$m\ddot{x} = f - b\dot{x} - k\Delta\ell = f - b\dot{x} - k(x + x_0 - \ell_0)$$

which becomes $m\ddot{x} + b\dot{x} + kx = f$ with the choice $x_0 = \ell_0$. A different choice of x would make the equation depend on ℓ_0.

P2.28. With Newton's second law help we write

$$m\ddot{x} = mg\sin\theta - b\dot{x} - k(x + x_0 - \ell_0)$$

which is equal to the equation in the statement after the choice of $x_0 = \ell_0$.

P2.29. Let

$$x = y + k^{-1} m g \sin\theta, \qquad \dot{x} = \dot{y}, \qquad \ddot{x} = \ddot{y}$$

and substitute into the equation obtained in P2.28 to obtain

$$m\ddot{y} + b\dot{y} + ky + mg\sin\theta = mg\sin\theta$$

or

$$m\ddot{y} + b\dot{y} + ky = 0.$$

This means that the constant force provided by gravity does not affect the dynamic behavior of this system. As expected it shifts the static equilibrium from $x_0 = \ell_0$ to $x_0 = \ell_0 + k^{-1} m g \sin\theta$.

P2.30. With Newton's second law help we write

$$\begin{aligned} m\ddot{x} &= -b\dot{x} - k_1(x + x_0 - \ell_{0,1}) + k_2(d - x - x_0 - w - \ell_{0,2}) \\ &= -b\dot{x} - (k_1 + k_2)x - k_1(x_0 - \ell_{0,1}) + k_2(d - x_0 - w - \ell_{0,2}) \\ &= -b\dot{x} - (k_1 + k_2)x \end{aligned}$$

after the choice

$$x_0 = \frac{k_1 \ell_{0,1} + k_2(d - w - \ell_{0,2})}{k_1 + k_2}.$$

When $d \geq w + \ell_{0,1} + \ell_{0,2}$ both springs are stretched so that the mass experiences tensile forces. When $d \leq w + \ell_{0,1} + \ell_{0,2}$ both springs are compressed so that the mass experiences compressive forces.

P2.31. Yes, a spring with stiffness $k = k_1 + k_2$.

P2.32. With Newton's second law help we write

$$\begin{aligned} m_1\ddot{x}_1 &= -b_1\dot{x}_1 + b_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 + k_2(x_2 - x_1) \\ &= -(b_1 + b_2)\dot{x}_1 - (k_1 + k_2)x_1 - b_2\dot{x}_2 - k_2x_2 \end{aligned}$$

and

$$m_2\ddot{x}_2 = -b_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) + f_2$$

after choosing x_1 and x_2 as displacements from equilibrium.

P2.33. With Newton's second law help we write

$$\begin{aligned} m_1\ddot{x}_1 &= -b(\dot{x}_1 - \dot{x}_2) - k(x_1 - x_2), \\ m_2\ddot{x}_2 &= -b(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1). \end{aligned}$$

Because

$$x_1 = y_1 + \frac{m_2}{M} y_2, \qquad x_2 = y_1 - \frac{m_1}{M} y_2, \qquad M = \frac{m_1 + m_2}{2},$$

we have

$$\begin{aligned} m_1\ddot{y}_1 + \frac{m_1 m_2}{M}\ddot{y}_2 &= -b\dot{y}_2 - ky_2, \\ m_2\ddot{y}_1 - \frac{m_1 m_2}{M}\ddot{y}_2 &= b\dot{y}_2 + ky_2. \end{aligned}$$

Adding the two equations:

$$M\ddot{y}_1 = 0,$$

and subtracting after multiplying the first equation by m_2 and the second by m_1:

$$m_1 m_2 \ddot{y}_2 = -(m_1 + m_2) b\dot{y}_2 - (m_1 + m_2) k y_2$$

which are "decoupled". The first equation is the dynamic of the *center of mass* $(m_1x_1 + m_2x_2)/M$ and the second is the dynamics of the *body*, which we expect to be decoupled from physics.

P2.34. Using Kirchoff's voltage and current law:

$$\begin{aligned} v &= v_C + v_R \\ &= v_C + R i_R \\ &= v_C + R i_C \\ &= v_C + RC\dot{v}_C \end{aligned}$$

which is the desired differential equation.

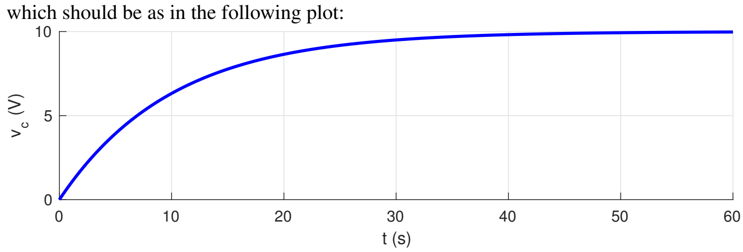
P2.35. The solution to the differential equation from P2.34 when $v(t) = \bar{v}$ is constant and under the assumption of zero initial conditions is

$$v_C(t) = \tilde{v}(t)(1 - e^{\lambda t})$$

where

$$\lambda = -\frac{1}{RC} = -\frac{1}{1 \times 10^6 \times 10 \times 10^{-6}} = -0.1\mathrm{s}^{-1}, \qquad \tilde{v} = \bar{v} = 10\mathrm{V}$$

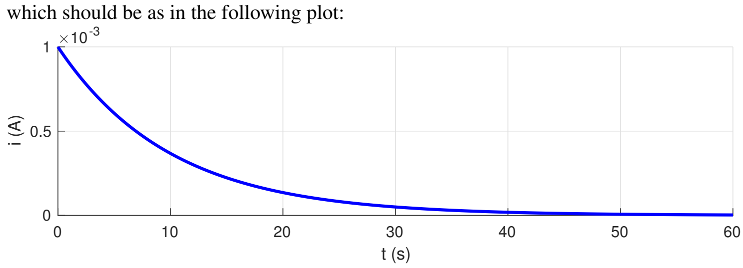
which should be as in the following plot:



The current

$$i(t) = i_C(t) = C\dot{v}_C(t) = -\frac{C\tilde{v}}{\lambda}e^{\lambda t}, \qquad -\frac{C\tilde{v}}{\lambda} = \frac{10 \times 10^{-6} \times 10}{0.1} = 1\mathrm{mA}.$$

which should be as in the following plot:



P2.36. Using Kirchoff's voltage and current law:

$$\begin{aligned} v &= v_C + v_R + v_L \\ &= v_C + R i_R + L\dot{i}_L \\ &= v_C + R i_C + L\dot{i}_C \\ &= v_C + RC\dot{v}_C + LC\ddot{v}_C, \end{aligned}$$

which is the desired equation.

P2.37. Comparing

$$LC\ddot{v}_C + RC\dot{v}_C + v_C = v$$

to

$$m\ddot{x} + b\dot{x} + kx = f$$

we first normalize

$$\ddot{v}_C + \frac{R}{L}\dot{v}_C + \frac{1}{LC}v_C = \frac{1}{LC}v,$$
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}\frac{f}{k},$$

from which one can simulate the response of the mass-spring-damper system to a scaled force f/k by setting R, L and C so that

$$\frac{R}{L} = \frac{b}{m}, \qquad \frac{1}{LC} = \frac{k}{m}.$$

P2.38. Use the given equations to write:

$$\begin{aligned} v &= R_1 i_{R_1} \\ &= R_1(i_{C_2} - i_{C_1}) \\ &= -R_1 C_2 \dot{v}_o - R_1 C_1 \dot{v} \end{aligned}$$

which is the desired equation.

The solution to the auxiliary equation is

$$z(t) = z(0) - \frac{1}{R_1 C_2}\int_0^t v(\tau)\,d\tau$$

and

$$\begin{aligned} v_o(t) &= R_1 C_1 \dot{z}(t) + z(t), \\ &= z(0) - \frac{C_1}{C_2}v(t) - \frac{1}{R_1 C_2}\int_0^t v(\tau)\,d\tau. \end{aligned}$$

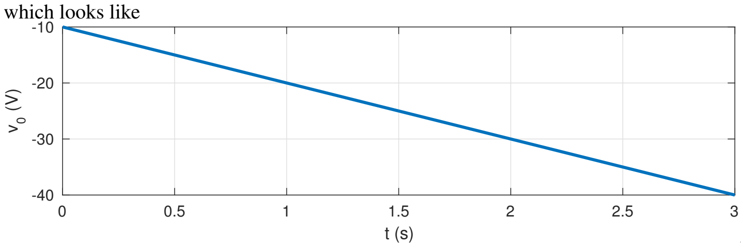
Multiplying by $R_1 C_2$ and differentiating under the integral

$$R_1 C_2 \dot{v}_o(t) = -R_1 C_1 \dot{v} - v(t).$$

P2.39. From P2.38 With a constant voltage $v(t) = 10$ V and zero initial conditions the response looks like

$$\begin{aligned} v_o(t) &= -\frac{C_1}{C_2}v(t) - \frac{1}{R_1 C_2}\int_0^t v(\tau)\,d\tau \\ &= -10 - 10\int_0^t d\tau = -10(1+t), \end{aligned}$$

which looks like



.

P2.40. From the solution to P2.38 when $C_1 = 0$ we obtain

$$v_o(t) = z(t) = z(0) - \frac{1}{R_1 C_2}\int_0^t v(\tau)\,d\tau.$$

This circuit is an *integrator*. It can be used as a building block to realize linear systems.

P2.41. Solving for the current

$$i_a = \frac{1}{R_a}v_a - \frac{K_e}{R_a}\omega$$

so that

$$J\,\dot\omega + b\,\omega = \tau$$
$$= K_t\, i_a,$$
$$= \frac{K_t}{R_a} v_a - \frac{K_t K_e}{R_a}\omega$$

which is equal to the expression sought after rearranging.

P2.42. The mechanical power is $\tau\omega = K_t\, i_a\, \omega$ and the electrical power is $v_e i_a = K_e\, \omega\, i_a$, therefore $K_t = K_e$.

P2.43. The differential equation can be rewritten as

$$\dot\omega + \alpha\,\omega = \beta\, v_a$$

where

$$\alpha = \frac{b}{J} + \frac{K_t K_e}{J R_a}, \qquad \beta = \frac{K_t}{J R_a}.$$

The solution to the differential equation when $v_a = \bar v_a$ is constant is

$$\omega(t) = \tilde\omega\,(1 - e^{\lambda t}) + \omega(0)\, e^{-\lambda t}, \qquad \lambda = -\alpha, \qquad \tilde\omega = \frac{\beta}{\alpha}\bar v_a.$$

Knowing that when $\bar v_a = 12$V the time-constant is equal to 0.1s and the terminal velocity is 5000RPM means that

$$\lambda = -\alpha = -1/0.1 = -10\text{s}^{-1}, \qquad \tilde\omega = \frac{\beta\,\bar v_a}{\alpha} = 5000\text{RPM},$$

from which it is possible to estimate

$$\alpha = 10\text{s}^{-1}, \qquad \beta = \frac{\tilde\omega\alpha}{\bar v_a} = \frac{5000\times 10}{12} \approx 4166\frac{\text{RPM}}{\text{V s}}.$$

In SI units

$$\beta = \frac{2\pi}{60}4166 \approx 436.3\frac{\text{rad}}{\text{V s}^2}.$$

There is not enough information to estimate all physical parameters.

P2.44. With knowledge of the *stall torque*, τ, and the motor resistance, R_a, we determine

$$K_t = \frac{R_a\tau}{v_a} = \frac{0.2\times 1.2}{12} \approx 0.02\frac{\text{N m}}{\text{A}}$$

and from P2.42 $K_e = K_t = 0.02$V s/rad.

Since β is already known we calculate

$$J = \frac{K_t}{\beta R_a} = \frac{0.02}{436.3\times 0.2} \approx 229.1\times 10^{-6}\text{kg m}^2$$

and

$$b = \alpha J - \frac{K_t K_e}{R_a} = 10\times 229.1\times 10^{-6} - \frac{0.02^2}{0.2} \approx 291.8\times 10^{-6}\text{kg m}^2/\text{s}.$$

P2.45. Because

$$\alpha = \frac{1}{J}\left(b + \frac{K_t K_e}{R_a}\right), \qquad \alpha' = \frac{1}{J+J'}\left(b + \frac{K_t K_e}{R_a}\right),$$

$$\frac{J+J'}{J} = \frac{1+\frac{J}{J'}}{\frac{J}{J'}} = \frac{\alpha}{\alpha'}$$

we calculate with $J' = 0.001\text{kg m}^2$ $\alpha = 1/0.1 = 10\text{s}^{-1}$ and $\alpha' = 1/0.54 \approx 1.86\text{s}^{-1}$

$$J = \frac{J'}{\frac{\alpha}{\alpha'} - 1} = \frac{J'\alpha'}{\alpha - \alpha'} \approx 227.2 \times 10^{-6}\text{kg m}^2$$

After determining J we can calculate

$$K_t = \frac{\alpha J R_a \bar{\omega}}{\bar{v}_a} = \frac{10 \times 227.2 \times 10^{-6} \times 0.2 \times 2\pi 5000}{60 \times 12} \approx 0.02\frac{\text{N m}}{\text{A}}$$

and $K_e = K_t = 0.02$V s/rad. The last unknown quantity is

$$b = \alpha J - \frac{K_t K_e}{R_a} = 10 \times 227.2 \times 10^{-6} - \frac{0.02^2}{0.2} \approx 289.4 \times 10^{-6}\text{kg m}^2/\text{s}.$$

P2.46. Kirchoff's voltage law for the circuit is:

$$v_a = R_a i_a + L_a \dot{i}_a + K_e \omega$$

The mechanical equation of motion is

$$J\,\dot{\omega} + b\,\omega = \tau = K_t i_a$$

Multiplying by R_a

$$JR_a\,\dot{\omega} + bR_a\,\omega = \tau = K_t R_a i_a$$

and then by L_a and differentiating

$$JL_a\,\ddot{\omega} + bL_a\,\dot{\omega} = K_t L_a \dot{i}_a$$

so that

$$\begin{aligned} JL_a\,\ddot{\omega} + (bL_a + JR_a)\,\dot{\omega} + bR_a\,\omega &= K_t(L_a\dot{i}_a + R_a i_a) \\ &= K_t(v_a - K_e\omega) \end{aligned}$$

which is equal to the expression sought after rearranging.

When $L_a = 0$

$$JR_a\,\dot{\omega} + (bR_a - K_t K_e)\,\omega = K_t v_a$$

which is the equation obtained before multiplied by R_a.

P2.47. In P2.43 you calculated the model

$$\dot{\omega} + \alpha\omega = \beta v_a, \qquad \alpha = 10\text{s}^{-1}, \qquad \beta \approx 436.3\ \text{rad}/(\text{V s}^2)$$

The closed-loop connection of this model with the controller produces the differential equation:

$$\dot{\omega} + (\alpha + K\beta)\omega = K\beta\bar{\omega}$$

Its solution is

$$\omega(t) = \tilde{\omega}\,(1 - e^{\lambda t}) + \omega(0)e^{\lambda t}$$

where

$$\lambda = -(\alpha + K\beta), \qquad \tilde{\omega} = \frac{K\beta\,\bar{\omega}}{\alpha + K\beta}.$$

For $\bar{\omega} = (2\pi/60)4000 \approx 418.9$rad/s the steady-state error is

$$\bar{\omega} - \tilde{\omega} = \bar{\omega} - \frac{K\beta\,\bar{\omega}}{\alpha + K\beta} = \frac{\alpha\bar{\omega}}{\alpha + K\beta}$$

In order to obtain

$$\frac{|\bar{\omega}-\tilde{\omega}|}{|\bar{\omega}|} = \frac{|\alpha|}{|\alpha+K\beta|} \leq 0.1$$

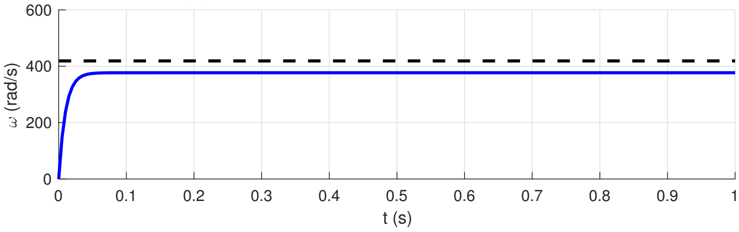
we must select

$$K \geq \frac{9\alpha}{\beta} \approx 0.2.$$

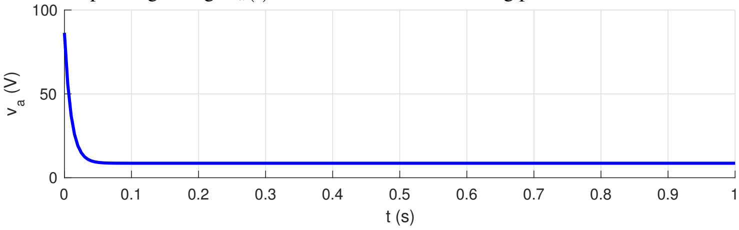
The closed-loop time-constant corresponding to $K = 0.2$ is

$$\tau = \frac{1}{\alpha+K\beta} \approx 0.01\text{s}$$

The response when $\omega(0) = 0$ should be as in the following plot:



The corresponding voltage $v_a(t)$ should be as in the following plot:



The maximum value of $v_a(t)$ is at $t = 0$ which is

$$v_a(0) = K(\bar{\omega} - \omega(0)) = K\bar{\omega} \approx 86.4\text{V}.$$

P2.48. We proceed as in P2.47 but this time we select K such that

$$v_a(0) = K\bar{\omega} \leq 12$$

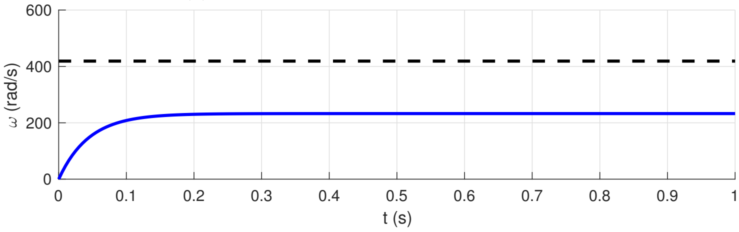
when $\bar{\omega} = (2\pi/60)4000 \approx 418.9\text{rad/s}$

$$K \leq \frac{12}{\bar{\omega}} \approx 0.029$$

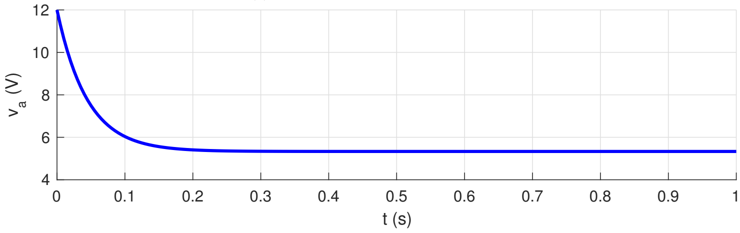
The time-constant corresponding to $K = 0.029$ is

$$\tau = \frac{1}{\alpha+K\beta} \approx 0.044\text{s}$$

The response when $\omega(0) = 0$ should be as in the following plot:



The corresponding voltage $v_a(t)$ should be as in the following plot:



which, as expected has a maximum voltage $v_a(0) = 12$V. Note that the closed-loop response is now slower.

P2.49. When q, w, T_o, and T_i are constants the solution to the differential equation is

$$mc\dot{T} + \left(wc + \frac{1}{R}\right) T = q + wcT_i + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}(1 - e^{\lambda t}) + T(0)e^{\lambda t}$$

where

$$\lambda = -\frac{Rwc+1}{Rmc}, \qquad \tilde{T} = \frac{Rq + RwcT_i + T_o}{Rwc+1}.$$

P2.50. With $q = 0$ and $w = 0$

$$mc\dot{T} + \frac{1}{R}T = \frac{1}{R}T_o$$

which has as solution

$$T(t) = \tilde{T}(1 - e^{\lambda t}) + T(0)e^{\lambda t}$$

where

$$\lambda = -\frac{1}{Rmc}, \qquad \tilde{T} = T_o.$$

With $T(0) = 60$°C and $T_o = 25$°C, after 7 days $t = 7 \times 24 \times 3600 = 604800$s and

$$25 + (60 - 25)e^{\lambda t} = T(t) = 27$$

or

$$\lambda = \frac{1}{604800} \log \frac{27-25}{60-25} \approx -4.73 \times 10^{-6}\text{s}^{-1},$$

from which $m = 997.1 \times 0.19 \approx 189$kg, $c = 4186$J/kg K and

$$R = -\frac{1}{mc\lambda} \approx 0.27\text{K/W}$$

P2.51. In the same conditions as in P2.50, if $w = 21 \times 10^{-6}\text{m}^3/\text{s} \neq 0$ and $T_i = 25$°C

$$mc\dot{T} + \left(wc + \frac{1}{R}\right) T = wcT_i + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}(1 - e^{\lambda t}) + T(0)e^{\lambda t}$$

where $R = 0.27$K/W,

$$\lambda = -\frac{Rwc+1}{Rmc} \approx -115.3 \times 10^{-6}\text{s}^{-1}$$

and

$$\tilde{T} = \frac{RwcT_i + T_o}{Rwc + 1}.$$

If $T_i = T_o = 25°C$ and $T(t) = 27°C$ then $\tilde{T} = T_i = T_o = 25°C$ and

$$t = \frac{1}{\lambda} \log \frac{T(t) - \tilde{T}}{T(0) - \tilde{T}} \approx 24833\text{s}$$

or about 6.9hours. Compare this number with the case when no water was flowing through the heater: from 7 days to 7 hours!

P2.52. In the same conditions as in P2.50, if $q = 12\text{kW}$ and $w = 0$

$$mc\dot{T} + \frac{1}{R}T = q + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}(1 - e^{\lambda t}) + T(0)e^{\lambda t}$$

where

$$\lambda = -\frac{1}{Rmc} \approx -4.73 \times 10^{-6}\text{s}^{-1}$$

and

$$\tilde{T} = T_o + Rq \approx 3222°\text{C}.$$

If $T(0) = 25°C$ and $T(t) = 60°C$ then

$$t = \frac{1}{\lambda} \log \frac{T(t) - \tilde{T}}{T(0) - \tilde{T}} \approx 2326\text{s}$$

or about 39 minutes.

P2.53. With $q = 12\text{kW}$, $w = 21 \times 10^{-6}\text{m}^3/\text{s} \neq 0$ and $T_i = 25°C$

$$mc\dot{T} + \left(wc + \frac{1}{R}\right)T = q + wcT_i + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}(1 - e^{\lambda t}) + T(0)e^{\lambda t}$$

where $R = 0.27\text{K/W}$,

$$\lambda = -\frac{Rwc + 1}{Rmc} \approx -115.3 \times 10^{-6}\text{s}^{-1}$$

and

$$\tilde{T} = \frac{Rq}{Rwc + 1} + \frac{RwcT_i + T_o}{Rwc + 1} \approx 156.3°\text{C}$$

so that for $T(0) = 25°C$ and $T(t) = 60°C$

$$t = \frac{1}{\lambda} \log \frac{T(t) - \tilde{T}}{T(0) - \tilde{T}} \approx 2690\text{s}$$

or about 45 minutes. This is a 15% increase when compared to the case without flow.

P2.54. We solve the problem by considering two phases: a) after the heater is turned on at $T = \underline{T} = 50°$C until it is turned off at $T = \overline{T} = 60°$C; b) after the heater is turned off at $T = \overline{T} = 60°$C until it is turned on at $T = \underline{T} = 50°$C.

In the first phase, $q = 12$kW, $w = 0$, $T(0) = \underline{T}$ and

$$mc\dot{T} + \frac{1}{R}T = q + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}_1(1 - e^{\lambda t}) + \underline{T}e^{\lambda t}$$

where

$$\lambda = -\frac{1}{Rmc} \approx -4.67 \times 10^{-6}\text{s}^{-1}$$

and

$$\tilde{T}_1 = T_o + Rq \approx 3265°\text{C}.$$

The heater stays in this phase for

$$t_1 = \frac{1}{\lambda}\log\frac{\overline{T} - \tilde{T}_1}{\underline{T} - \tilde{T}_1} \approx 667\text{s}$$

or about 11 minutes. The average temperature in this phase is

$$T_1 = \frac{1}{t_1}\int_0^{t_1} T(\tau)d\tau = \int_0^{t_1} \tilde{T}_1 + (\underline{T} - \tilde{T}_1)e^{\lambda\tau}d\tau = \tilde{T}_1 + \frac{\underline{T} - \tilde{T}_1}{\lambda t_1}(e^{\lambda t_1} - 1) \approx 55.0°\text{C}.$$

In the second phase, $q = 0$, $w = 0$, $T(0) = \overline{T}$ and

$$mc\dot{T} + \frac{1}{R}T = \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}_2(1 - e^{\lambda t}) + \overline{T}e^{\lambda t}$$

where λ is as before and

$$\tilde{T}_2 = T_o \approx 25°\text{C}.$$

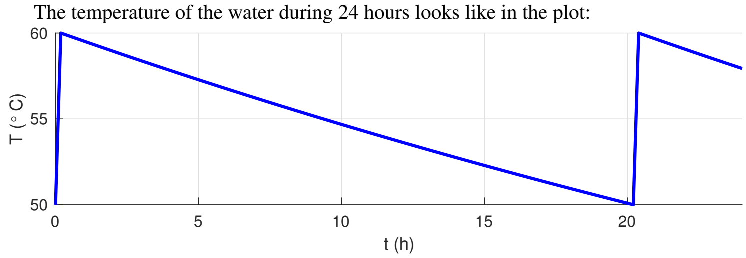
The heater stays in this phase for

$$t_2 = \frac{1}{\lambda}\log\frac{\underline{T} - \tilde{T}_2}{\overline{T} - \tilde{T}_2} \approx 72045\text{s}$$

or about 1201 minutes or 20 hours. The average temperature in this phase is

$$T_2 = \frac{1}{t_2}\int_0^{t_1} T(\tau)d\tau = \int_0^{t_2} \tilde{T}_2 + (\overline{T} - \tilde{T}_2)e^{\lambda\tau}d\tau = \tilde{T}_2 + \frac{\overline{T} - \tilde{T}_2}{\lambda t_2}(e^{\lambda t_1} - 1) \approx 54.7°\text{C}.$$

The temperature of the water during 24 hours looks like in the plot:



During one complete on/off cycle the average temperature is

$$T = \frac{t_1 T_1 + t_2 T_2}{t_1 + t_2} \approx 54.7°\text{C}.$$

The average power consumption was

$$P = \frac{t_1 q}{t_1 + t_2} \approx 110\text{W}$$

since power is only consumed in phase 1.

P2.55. We solve the problem by considering two phases: a) after the heater is turned on at $T = \underline{T} = 50°$C until it is turned off at $T = \overline{T} = 60°$C; b) after the heater is turned off at $T = \overline{T} = 60°$C until it is turned on at $T = \underline{T} = 50°$C.

In the first phase, $q = 12$kW, $w = 21 \times 10^{-6}$m^3/s, $T(0) = \underline{T}$ and

$$mc\dot{T} + \left(wc + \frac{1}{R}\right)T = q + wcT_i + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}_1(1 - e^{\lambda t}) + \underline{T}e^{\lambda t}$$

where

$$\lambda = -\frac{Rwc + 1}{Rmc} \approx -115.3 \times 10^{-6}\text{s}^{-1}$$

and

$$\tilde{T}_1 = \frac{Rq}{Rwc + 1} + \frac{RwcT_i + T_o}{Rwc + 1} \approx 156.4°\text{C}$$

The heater stays in this phase for

$$t_1 = \frac{1}{\lambda}\log\frac{\overline{T} - \tilde{T}_1}{\underline{T} - \tilde{T}_1} \approx 857\text{s}$$

or about 14 minutes. The average temperature in this phase is

$$T_1 = \frac{1}{t_1}\int_0^{t_1} T(\tau)\,d\tau = \int_0^{t_1} \tilde{T}_1 + (\underline{T} - \tilde{T}_1)e^{\lambda\tau}\,d\tau = \tilde{T}_1 + \frac{\underline{T} - \tilde{T}_1}{\lambda t_1}(e^{\lambda t_1} - 1) \approx 55.1°\text{C}.$$

In the second phase, $q = 0$, $w = 21 \times 10^{-6}$m^3/s, $T(0) = \overline{T}$ and

$$mc\dot{T} + \left(wc + \frac{1}{R}\right)T = wcT_i + \frac{1}{R}T_o.$$

which has as solution

$$T(t) = \tilde{T}_2(1 - e^{\lambda t}) + \overline{T}e^{\lambda t}$$

where λ is as before and

$$\tilde{T}_2 = T_i = T_o \approx 25°\text{C}.$$

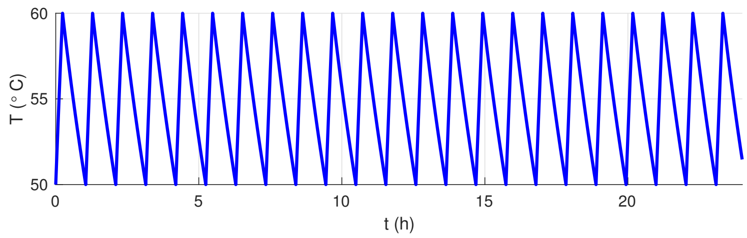
The heater stays in this phase for

$$t_2 = \frac{1}{\lambda}\log\frac{\underline{T} - \tilde{T}_2}{\overline{T} - \tilde{T}_2} \approx 2921\text{s}$$

or about 48 minutes. The average temperature in this phase is

$$T_2 = \frac{1}{t_2}\int_0^{t_1} T(\tau)\,d\tau = \int_0^{t_2} \tilde{T}_2 + (\overline{T} - \tilde{T}_2)e^{\lambda\tau}\,d\tau = \tilde{T}_2 + \frac{\overline{T} - \tilde{T}_2}{\lambda t_2}(e^{\lambda t_1} - 1) \approx 54.7°\text{C}.$$

The temperature of the water during 24 hours looks like in the plot:



During one complete on/off cycle the average temperature is

$$T = \frac{t_1 T_1 + t_2 T_2}{t_1 + t_2} \approx 54.8°\text{C}.$$

The average power consumption was

$$P = \frac{t_1 q}{t_1 + t_2} \approx 2723\text{W}$$

since power is only consumed in phase 1.

This is more than 20 times the amount consumed when there was no flow.

P2.56. Solution is the same as P2.54.

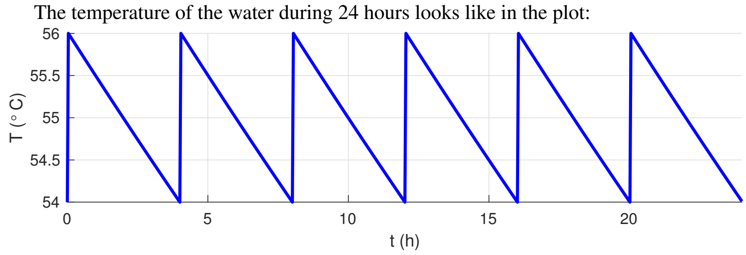
Key quantities in the first phase are:

$$\begin{aligned}
\lambda &= -\frac{1}{Rmc} \approx -4.67 \times 10^{-6}\text{s}^{-1},\\
\tilde{T}_1 &= Rq + \frac{RwcT_i + T_o}{Rwc+1} \approx 3265°\text{C},\\
t_1 &= \frac{1}{\lambda} \log \frac{\overline{T} - \tilde{T}_1}{\underline{T} - \tilde{T}_1} \approx 133\text{s},\\
T_1 &= \tilde{T}_1 + \frac{\underline{T} - \tilde{T}_1}{\lambda t_1}(e^{\lambda t_1} - 1) \approx 55.0°\text{C}.
\end{aligned}$$

In the second phase:

$$\begin{aligned}
\tilde{T}_2 &= T_i = T_o \approx 25°\text{C},\\
t_2 &= \frac{1}{\lambda} \log \frac{\underline{T} - \tilde{T}_2}{\overline{T} - \tilde{T}_2} \approx 14280\text{s},\\
T_2 &= \tilde{T}_2 + \frac{\overline{T} - \tilde{T}_2}{\lambda t_2}(e^{\lambda t_1} - 1) \approx 55.0°\text{C}.
\end{aligned}$$

The temperature of the water during 24 hours looks like in the plot:



During one complete on/off cycle the average temperature is

$$T = \frac{t_1 T_1 + t_2 T_2}{t_1 + t_2} \approx 55.0°\text{C}.$$

The average power consumption is:

$$P = \frac{t_1 q}{t_1 + t_2} \approx 111\text{W}$$

Compared with P2.54, the temperature is better regulated and there is a slight increase in the consumed power. Note how the number of cycles has increased since both t_1 and t_2 got smaller.

P2.57. Solution is the same as P2.55.

Key quantities in the first phase are:

$$\lambda = -\frac{Rwc+1}{Rmc} \approx -115.3 \times 10^{-6}\mathrm{s}^{-1},$$

$$\tilde{T}_1 = \frac{Rq}{Rwc+1} + \frac{RwcT_i + T_o}{Rwc+1} \approx 156.4^\circ\mathrm{C},$$

$$t_1 = \frac{1}{\lambda}\log\frac{\overline{T} - \tilde{T}_1}{\underline{T} - \tilde{T}_1} \approx 171\mathrm{s},$$

$$T_1 = \tilde{T}_1 + \frac{\underline{T} - \tilde{T}_1}{\lambda t_1}(e^{\lambda t_1} - 1) \approx 55.0^\circ\mathrm{C}.$$

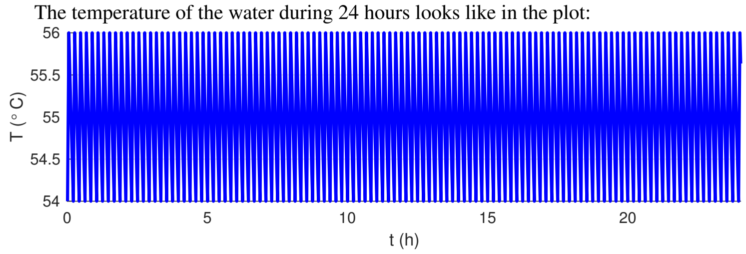
In the second phase:

$$\tilde{T}_2 = T_i = T_o \approx 25^\circ\mathrm{C},$$

$$t_2 = \frac{1}{\lambda}\log\frac{\underline{T} - \tilde{T}_2}{\overline{T} - \tilde{T}_2} \approx 578\mathrm{s}$$

$$T_2 = \tilde{T}_2 + \frac{\overline{T} - \tilde{T}_2}{\lambda t_2}(e^{\lambda t_1} - 1) \approx 55.0^\circ\mathrm{C}.$$

The temperature of the water during 24 hours looks like in the plot:



During one complete on/off cycle the average temperature is

$$T = \frac{t_1 T_1 + t_2 T_2}{t_1 + t_2} \approx 55.0^\circ\mathrm{C}.$$

The average power consumption is:

$$P = \frac{t_1 q}{t_1 + t_2} \approx 2740\mathrm{W}$$

Compared with P2.55, the temperature is better regulated and there is a slight increase in the consumed power. Note how the number of cycles has increased since both t_1 and t_2 got smaller.

Appendix C Chapter 3

P3.1. Follows from the fact that the integral is a linear operator.

P3.2. We seek to evaluate:

$$\mathscr{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \int_{0^-}^{\infty}\int_{0^-}^{t} f(\tau)\,d\tau\, e^{-st}\,dt.$$

Integrating by parts with $u = \int_{0^-}^{t} f(\tau)\,d\tau$ and $dv = e^{-st}dt$:

$$\begin{aligned}\int_{0^-}^{\infty}\int_{0^-}^{t} f(\tau)\,d\tau\, e^{-st}\,dt &= -\frac{e^{-st}}{s}\int_{0^-}^{t} f(\tau)\,d\tau\bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\frac{-e^{-st}}{s}\,dt \\ &= \frac{F(s)}{s} + -\frac{e^{-st}}{s}\int_{0^-}^{t} f(\tau)\,d\tau\bigg|_{0^-}^{\infty}\end{aligned}$$

If $|f(t)| \leq Me^{\alpha t}$ then for any $s = \beta + j\gamma$

$$\left|e^{-st}\int_{0^-}^{t} f(\tau)\,d\tau\right| \leq \left|e^{-st}\right|\int_{0^-}^{t} |f(\tau)|\,d\tau \leq M\left|e^{-st}\right|\int_{0^-}^{t} e^{\alpha\tau}\,d\tau = \frac{M}{\alpha}e^{(\alpha-\beta)t}$$

so that for β large enough

$$-\frac{e^{-st}}{s}\int_{0^-}^{t} f(\tau)\,d\tau\bigg|_{0^-}^{\infty} = 0,$$

which proves the integration property.

P3.3. We seek to evaluate:

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\frac{df(t)}{dt}\,e^{-st}\,dt.$$

Integrating by parts with $u = e^{-st}$ and $dv = \frac{df(t)}{dt}dt$:

$$\begin{aligned}\int_{0^-}^{\infty}\frac{df(t)}{dt}\,e^{-st}\,dt &= f(t)e^{-st}\big|_{0^-}^{\infty} + \int_{0^-}^{\infty} f(t)se^{-st}\,dt \\ &= sF(s) + f(t)e^{-st}\big|_{0^-}^{\infty}\end{aligned}$$

If $|f(t)| \leq Me^{\alpha t}$ then for any $s = \beta + j\gamma$

$$\left|f(t)e^{-st}\right| \leq Me^{(\alpha-\beta)t}$$

so that for β large enough

$$f(t)e^{-st}\big|_{0^-}^{\infty} = f(0^-),$$

which proves the differentiation property.