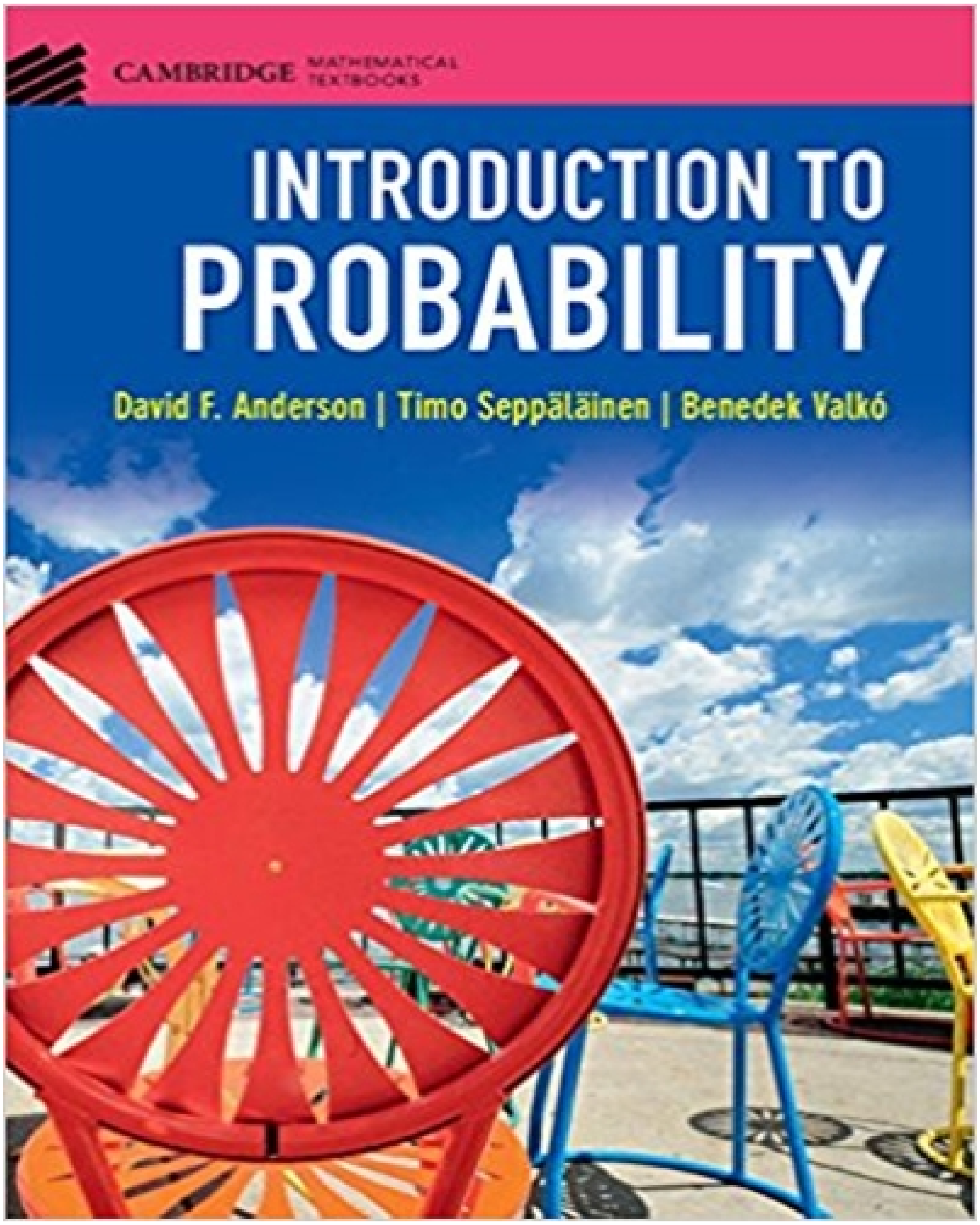
Solutions for Introduction to Probability 1st Edition by Anderson

CLICK HERE TO ACCESS COMPLETE Solutions



Solutions

CLICK HERE TO ACCESS THE COMPLETE Solutions

Solutions to Chapter 2

2.1. We can set our sample space to be $\Omega = \{(a_1, a_2) : 1 \le a_i \le 6\}$. We have $\#\Omega = 36$ and each outcome is equally likely.

Denote by A the event that at least one number is even and by B the event that the sum is 8. Then we need $P(A|B)$ which can be computed from the definition as $P(A|B) = \frac{P(AB)}{P(B)}$.

We have $B = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$, and hence $P(B) = \frac{\#B}{\#\Omega} = \frac{5}{36}$. Moreover, $AB = \{(2,6), (4,4), (6,2)\}$ and hence $P(AB) = \frac{\#AB}{\#\Omega} = \frac{3}{36} = \frac{1}{12}$. Thus $P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{12}}{\frac{5}{36}} = \frac{3}{5}$.

Since the outcomes are equally likely, we can equivalently find the answer from $P(A|B) = \frac{\#AB}{\#B} = \frac{3}{5}$.

2.2. $A = \{\text{second flip is tails}\} = \{(H,T,H), (H,T,T), (T,T,H), (T,T,T)\}$,

$$B = \{\text{at most one tails}\} = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H)\}.$$

Hence $AB = \{(H,T,H)\}$, and since we have equally likely outcomes,

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\#AB}{\#B} = \tfrac{1}{4}.$$

2.3. We set the sample space as $\Omega = \{1, 2, \dots, 100\}$. We have $\#\Omega = 100$ and each outcome is equally likely.

Let A denote the event that the chosen number is divisible by 3 and B denote the event that at least one digit is equal to 5. Then

$$B = \{5, 15, 25, \dots, 95\} \cup \{50, 51, \dots, 59\}$$

and $\#B = 19$. (As there are 10 numbers with 5 as the last digit, 10 numbers with 5 at the tens place, and 55 was counted both times.) We also have

$$AB = \{15, 45, 51, 54, 57, 75\}, \quad \#AB = 6.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

This gives $P(A|B) = \frac{P(AB)}{P(B)} = \frac{6/100}{19/100} = \frac{6}{19}$.

2.4. Let A be the event that we picked the ball labeled 5 and B the event that we picked the first urn. Then we have $P(B) = 1/2$, $P(B^c) = P(\text{we picked the second urn}) = 1/2$. Moreover, from the setup if the problem

$$P(A|B) = P(\text{we chose the number 5} \,|\, \text{we chose from the first urn}) = 0,$$

$$P(A|B^c) = P(\text{we chose the number 5} \,|\, \text{we chose from the second urn}) = \frac{1}{3}.$$

We compute $P(A)$ by conditioning on B and B^c:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

2.5. Let A be the event that we picked the number 2 and B the event that we picked the first urn. Then we have $P(B) = 1/5$, $P(B^c) = P(\text{we picked the second urn}) = 4/5$. Moreover, from the setup if the problem

$$P(A|B) = P(\text{we chose the number 2} \,|\, \text{we chose from the first urn}) = \frac{1}{3},$$

$$P(A|B^c) = P(\text{we chose the number 2} \,|\, \text{we chose from the second urn}) = \frac{1}{4}.$$

Then we can compute $P(A)$ by conditioning on B and B^c:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{4}{5} = \frac{4}{15}.$$

2.6. Define events

$$A = \{\text{Alice watches TV tomorrow}\} \quad \text{and} \quad B = \{\text{Betty watches TV tomorrow}\}.$$

(a) $P(AB) = P(A)P(B|A) = 0.6 \cdot 0.8 = 0.48$.

(b) Intuitively, the answer must be the same 0.48 as in part (a) because Betty cannot watch TV unless Alice is also watching. Mathematically, this says that $P(B|A^c) = 0$. Then by the law of total probability,

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 0.8 \cdot 0.6 + 0 \cdot 0.4 = 0.48.$$

(c) $P(AB^c) = P(A) - P(AB) = 0.6 - 0.48 = 0.12$. Or, by conditioning and using the outcome of Exercise 2.7(a),

$$P(AB^c) = P(A)P(B^c|A) = P(A)\big(1 - P(B|A)\big) = 0.6 \cdot 0.2 = 0.12.$$

2.7. (a) By definition $P(A^c|B) = \frac{P(A^cB)}{P(B)}$. We have $A^cB \cup AB = B$, and the two sets on the left are disjoint, so $P(A^cB) + P(AB) = P(B)$, and $P(A^cB) = P(B) - P(AB)$. This gives

$$P(A^c|B) = \frac{P(A^cB)}{P(B)} = \frac{P(B) - P(AB)}{P(B)} = 1 - \frac{P(AB)}{P(B)} = 1 - P(A|B).$$

(b) From part (a) we have $P(A^c|B) = 1 - P(A|B) = 0.4$. Then $P(A^cB) = P(A^c|B)P(B) = 0.4 \cdot 0.5 = 0.2$.

2.8. Let A_1, A_2, A_3 denote the events that the first, second and third cards are queen, king and ace, respectively. We need to compute $P(A_1A_2A_3)$. One could do this by counting favorable outcomes. But conditional probabilities provide an

CLICK HERE TO ACCESS THE COMPLETE Solutions

easier way because then we can focus on picking one card at a time. We just have to keep track of how earlier picks influence the probabilities of the later picks.

We have $P(A_1) = \frac{4}{52} = \frac{1}{13}$ since there are 52 equally likely choices for the first pick and four of them are queens. The conditional probability $P(A_2 \mid A_1)$ must reflect the fact that one queen has been removed from the deck and is no longer a possible outcome. Since the outcomes are still equally likely, the conditional probability of getting a king for the second pick is $\frac{4}{51}$. Similarly, when we compute $P(A_3 \mid A_1A_2)$ we can assume that we pick a card out of 50 (with one queen and one king removed) and thus the conditional probability of picking an ace will be $\frac{4}{50} = \frac{2}{25}$. Thus the probability of $A_1A_2A_3$ is given by

$$P(A_1A_2A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_2A_1) = \tfrac{1}{13} \cdot \tfrac{4}{51} \cdot \tfrac{2}{25} = \tfrac{8}{16{,}575}.$$

2.9. Let C be the event that we chose the ball 3 and D the event that we chose from the second urn. Then we have

$$P(D) = \frac{4}{5}, \quad P(D^c) = \frac{1}{5}, \quad P(C|D) = \frac{1}{4}, \quad P(C|D^c) = \frac{1}{3}.$$

We need to compute $P(D|C)$, which we can do using Bayes' formula:

$$P(D|C) = \frac{P(C|D)P(D)}{P(C|D)P(D) + P(C|D^c)P(D^c)} = \frac{\frac{1}{4}\cdot\frac{4}{5}}{\frac{1}{4}\cdot\frac{4}{5} + \frac{1}{3}\cdot\frac{1}{5}} = \frac{3}{4}.$$

2.10. Define events:

$$A = \{\text{outcome of the roll is 4}\} \quad \text{and} \quad B_k = \{\text{the } k\text{-sided die is picked}\}.$$

Then

$$P(B_6|A) = \frac{P(A \cap B_6)}{P(A)} = \frac{P(A|B_6)P(B_6)}{P(A|B_4)P(B_4) + P(A|B_6)P(B_6) + P(A|B_{12})P(B_{12})}$$
$$= \frac{\frac{1}{6}\cdot\frac{1}{3}}{\frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{12}\cdot\frac{1}{3}} = \frac{1}{3}.$$

2.11. Let A be the event that the chosen customer is reckless. Let B be the event that the chosen customer has an accident. We know the following:

$$P(A) = 0.2, \quad P(A^c) = 0.8, \quad P(B|A) = 0.04, \quad \text{and } P(B|A^c) = 0.01.$$

The probability asked for is $P(A^c|B)$. Using Bayes' formula we get

$$P(A^c|B) = \frac{P(B|A^c)P(A^c)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.01 \cdot 0.80}{0.04 \cdot 0.2 + 0.01 \cdot 0.80} = \frac{1}{2}.$$

2.12. (a) $A = \{X \text{ is even}\}$, $B = \{X \text{ is divisible by } 5\}$. $\#A = 50$, $\#B = 20$ and $AB = \{10, 20, \dots, 100\}$ so $\#AB = 10$. Thus

$$P(A)P(B) = \tfrac{50}{100} \cdot \tfrac{20}{100} = \tfrac{1}{10} \quad \text{and} \quad P(AB) = \tfrac{10}{100} = \tfrac{1}{10}.$$

This shows $P(A)P(B) = P(AB)$ and verifies the independence of A and B.

(b) $C = \{X \text{ has two digits}\} = \{10, 11, 12, \dots, 99\}$ and $\#C = 90$.

$$D = \{X \text{ is divisible by } 3\} = \{3, 6, 9, 12, \dots, 99\} \text{ and } \#D = 33.$$

$CD = \{12, 15, \dots, 99\}$ and $\#C = 30$. Thus

$$P(C)P(D) = \tfrac{90}{100} \cdot \tfrac{33}{100} \approx 0.297 \quad \text{and} \quad P(CD) = \tfrac{30}{100} = \tfrac{3}{10}.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

This shows $P(C)P(D) \neq P(CD)$ and verifies that C and D are not independent.

(c) $E = \{X \text{ is a prime}\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$,

and $\#E = 25$.

$$F = \{X \text{ has a digit } 5\} = \{5, 15, 25, \dots, 95\} \cup \{50, 51, \dots, 59\}$$

and $\#F = 19$. $EF = \{5, 53, 59\}$ and $\#EF = 3$. We have

$$P(E)P(F) = \tfrac{25}{100} \cdot \tfrac{19}{100} = 0.0475 \quad \text{and} \quad P(EF) = \tfrac{3}{100}.$$

This shows $P(E)P(F) \neq P(EF)$ and verifies that E and F are not independent.

2.13. We need to check whether or not we have

$$P(AB) = P(A)P(B).$$

We know that $P(A)P(B) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. We also know that $A = AB \cup AB^c$ and that the events AB and AB^c are disjoint. Thus,

$$\frac{1}{3} = P(A) = P(AB) + P(AB^c) = P(AB) + \frac{2}{9}.$$

Thus,

$$P(AB) = \frac{1}{3} - \frac{2}{9} = \frac{1}{9} = P(A)P(B),$$

so A and B are independent.

2.14. Since $P(AB) = P(\varnothing) = 0$ and independence requires $P(A)P(B) = P(AB)$, disjoint events A and B are independent if and only if at least one of them has probability zero.

2.15. Number the days by 1,2,3,4,5 starting from Monday. Let $X_i = 1$ if Ramona catches her bus on day i and $X_i = 0$ if she misses it. Then we need to compute $P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 0)$. By assumption, the events $\{X_1 = 1\}$, $\{X_2 = 1\}$, $\{X_3 = 0\}$, $\{X_4 = 1\}$, $\{X_5 = 0\}$ are independent from each other, and $P(X_i = 1) = \frac{9}{10}$ and $P(X_i = 0) = \frac{1}{10}$. Thus

$$\begin{aligned} &P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 1, X_5 = 0) \\ &\quad = P(X_1 = 1)P(X_2 = 1)P(X_3 = 0)P(X_4 = 1)P(X_5 = 0) \\ &\quad = \frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{10} \cdot \frac{9}{10} \cdot \frac{1}{10} = \frac{729}{10000}. \end{aligned}$$

2.16. Let us label heads as 0 and tails as 1. The sample space is

$$\Omega = \{(s_1, s_2, s_3) : \text{each } s_i \in \{0, 1\}\},$$

the set of ordered triples of zeros and ones. $\#\Omega = 8$ and so for equally likely outcomes we have $P(\omega) = 1/8$ for each $\omega \in \Omega$. The events and their probabilities

CLICK HERE TO ACCESS THE COMPLETE Solutions

we need for answering the question of independence are

$$\begin{aligned}
P(A_1) &= P\{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\} = \tfrac{4}{8} = \tfrac{1}{2},\\
P(A_2) &= P\{(0,1,0),(0,1,1),(1,0,0),(1,0,1)\} = \tfrac{4}{8} = \tfrac{1}{2},\\
P(A_3) &= P\{(0,1,1),(1,0,1),(1,1,0),(0,0,0)\} = \tfrac{4}{8} = \tfrac{1}{2},\\
P(A_1A_2) &= \{(0,1,0),(0,1,1)\} = \tfrac{2}{8} = \tfrac{1}{4} = \tfrac{1}{2}\cdot\tfrac{1}{2} = P(A_1)P(A_2),\\
P(A_1A_3) &= \{(0,1,1),(0,0,0)\} = \tfrac{2}{8} = \tfrac{1}{4} = \tfrac{1}{2}\cdot\tfrac{1}{2} = P(A_1)P(A_3),\\
P(A_2A_3) &= \{(0,1,1),(1,0,1)\} = \tfrac{2}{8} = \tfrac{1}{4} = \tfrac{1}{2}\cdot\tfrac{1}{2} = P(A_1)P(A_3),\\
P(A_1A_2A_3) &= \{(0,1,1)\} = \tfrac{1}{8} = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = P(A_1)P(A_2)P(A_3).
\end{aligned}$$

All the four possible combinations of more than two events from A_1, A_2, A_3 satisfy the product identity. Hence independence of A_1, A_2, A_3 has been verified.

2.17. We have $AB \cup C = ABC^c \cup C$, and the events ABC^c and C are disjoint. Thus $P(AB \cup C) = P(ABC^c) + P(C)$. Since A, B, C are mutually independent, this is also true for A, B, C^c. Thus

$$P(ABC^c) = P(A)P(B)P(C^c) = \frac{1}{2}\cdot\frac{1}{3}\cdot\Big(1-\frac{1}{4}\Big) = \frac{1}{8},$$

From this we get

$$P(AB \cup C) = P(ABC^c) + P(C) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

Here is another solution: by inclusion-exclusion $P(AB\cup C) = P(AB)+P(C)-P(ABC)$. Because of independence

$$P(AB) = P(A)P(B) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}, \qquad P(ABC) = P(A)P(B)P(C) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{24}.$$

Thus

$$P(AB \cup C) = P(AB) + P(C) - P(ABC) = \frac{1}{6} + \frac{1}{4} - \frac{1}{24} = \frac{3}{8}.$$

2.18. There are 90 numbers to choose from and so each outcome has probability $\frac{1}{90}$.

(a) From enumerating the possible values of X, we see that $P(X = k) = \frac{1}{9}$ for each $k \in \{1, 2, \ldots, 9\}$. (For example, the event $\{X = 3\} = \{30, 31, \ldots, 39\}$ has 10 outcomes from the 90 total.) For Y we have $P(Y = \ell) = \frac{1}{10}$ for each $\ell \in \{0, 1, 2, \ldots, 9\}$. (For example, the event $\{Y = 3\} = \{13, 23, 33, \ldots, 93\}$ has 9 outcomes from the 90 total.)

The intersection $\{X = k, Y = \ell\}$ contains exactly one number from the 90 outcomes, namely $10k + \ell$. (For example $\{X = 3, Y = 5\} = \{35\}$). Thus for each pair (k, ℓ) of possible values,

$$P(X = k, Y = \ell) = P\{10k + \ell\} = \tfrac{1}{90} = \tfrac{1}{9}\cdot\tfrac{1}{10} = P(X = k)P(Y = \ell).$$

Thus we have checked that X and Y are independent.

(b) To show that independence fails, we need to find only one case where the product property $P(X = k, Z = m) = P(X = k)P(Z = m)$ fails. Let's take an extreme case. The smallest possible value for Z is 1 that comes only from the outcome 10, since the sum of the digits is $1 + 0 = 1$. (Formally, since Z is

CLICK HERE TO ACCESS THE COMPLETE Solutions

a function on Ω, $Z(10) = 1 + 0 = 1$.) And so $P(Z = 1) = P\{10\} = \frac{1}{90}$. If we take $X = 2$, we cannot get $Z = 1$. Here is the precise derivation:

$$P(X = 2, Z = 1) = P(\{20, 21, \dots, 29\} \cap \{10\}) = P(\varnothing) = 0.$$

Since $P(X = 2)P(Z = 1) = \frac{1}{9} \cdot \frac{1}{90} = \frac{1}{810} \neq 0$, we have shown that X and Z are not independent.

2.19. (a) If we draw with replacement then we have 7^2 equally likely outcomes for the two picks. Counting the favorable outcomes gives

$$\begin{aligned} P(X_1 = 4) &= \frac{1 \cdot 7}{7 \cdot 7} = \frac{1}{7} \\ P(X_2 = 5) &= \frac{7 \cdot 1}{7 \cdot 7} = \frac{1}{7} \\ P(X_1 = 4, X_2 = 5) &= \frac{1}{7 \cdot 7} = \frac{1}{49}. \end{aligned}$$

(b) If we draw without replacement then we have $7 \cdot 6$ equally likely outcomes for the two picks. Counting the favorable outcomes gives

$$\begin{aligned} P(X_1 = 4) &= \frac{1 \cdot 6}{7 \cdot 6} = \frac{1}{7} \\ P(X_2 = 5) &= \frac{6 \cdot 1}{7 \cdot 6} = \frac{1}{7} \\ P(X_1 = 4, X_2 = 5) &= \frac{1}{7 \cdot 6} = \frac{1}{42}. \end{aligned}$$

(c) The answer to part (b) showed that $P(X_1 = 4)P(X_2 = 5) \neq P(X_1 = 4, X_2 = 5)$. This proves that X_1 and X_2 are not independent when drawing without replacement.

Part (a) showed that the *events* $\{X_1 = 4\}$ and $\{X_2 = 5\}$ are independent when drawing with replacement, but this is not enough for proving that the *random variables* X_1 and X_2 are independent. Independence of random variables requires checking $P(X_1 = a)P(X_2 = b) = P(X_1 = a, X_2 = b)$ for all possible choices of a and b. (This can be done and so independence of X_1 and X_2 does actually hold here.)

2.20. (a) Let S_5 denote the number of threes in the first five rolls. Then

$$P(S_5 \leq 2) = \sum_{k=0}^{2} \binom{5}{k} \left(\tfrac{1}{6}\right)^k \left(\tfrac{5}{6}\right)^{5-k}.$$

(b) Let N be the number of rolls needed to see the first three. Then from the p.m.f. of a geometric random variable,

$$P(N > 4) = \sum_{k=5}^{\infty} \left(\tfrac{5}{6}\right)^{k-1} \left(\tfrac{1}{6}\right) = \left(\tfrac{5}{6}\right)^4.$$

Equivalently,

$$P(N > 4) = P(\text{no three in the first four rolls}) = \left(\tfrac{5}{6}\right)^4.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(c) We can approach this in a couple different ways. By using the independence of the rolls,

$$\begin{aligned}&P(5 \le N \le 20)\\ &= P(\text{no three in the first four rolls, at least one three in rolls 5–20})\\ &= \left(\tfrac{5}{6}\right)^4\left(1 - \left(\tfrac{5}{6}\right)^{16}\right) = \left(\tfrac{5}{6}\right)^4 - \left(\tfrac{5}{6}\right)^{20}.\end{aligned}$$

Equivalently, thinking of the roll at which the first three comes,

$$\begin{aligned}P(5 \le N \le 20) &= P(N \ge 5) - P(N \ge 21)\\ &= \sum_{k=5}^{\infty}\left(\tfrac{5}{6}\right)^{k-1}\left(\tfrac{1}{6}\right) - \sum_{k=21}^{\infty}\left(\tfrac{5}{6}\right)^{k-1}\left(\tfrac{1}{6}\right)\\ &= \left(\tfrac{5}{6}\right)^4 - \left(\tfrac{5}{6}\right)^{20}.\end{aligned}$$

2.21. (a) Let S be the number of problems she gets correct. Then $S \sim \text{Bin}(4, 0.8)$ and

$$\begin{aligned}P(\text{Jane gets an A}) = P(S \ge 3) &= P(S = 3) + P(S = 4)\\ &= \binom{4}{3}(0.8)^3(0.2) + (0.8)^4\\ &= 0.8192.\end{aligned}$$

(b) Let S_2 be the number of problems Jane gets correct out of the last three. Then $S_2 \sim \text{Bin}(3, 0.8)$. Let $X_1 \sim \text{Bern}(0.8)$ model whether or not she gets the first problem correct. By assumption, S_2 and X_1 are independent. We have

$$\begin{aligned}P(S \ge 3 \,|\, X_1 = 1) &= \frac{P(S \ge 3, X_1 = 1)}{P(X_1 = 1)}\\ &= \frac{P(S_2 \ge 2, X_1 = 1)}{P(X_1 = 1)} = \frac{P(S_2 \ge 2)P(X_1 = 1)}{P(X_1 = 1)}.\end{aligned}$$

The last equality followed by the independence of S_2 and X_1. Hence,

$$P(S \ge 3|X_1 = 1) = P(S_2 \ge 2) = \binom{3}{2}(0.8)^2(0.2) + (0.8)^3 = 0.896.$$

2.22. (a) Let us encode the possible events in a single round as

$$A_R = \{\text{Annie chooses rock}\}, \quad A_P = \{\text{Annie chooses paper}\}$$
$$\text{and} \quad A_S = \{\text{Annie chooses scissors}\}$$

and similarly B_R, B_P and B_S for Bill. Then, using the independence of the players' choices,

$$\begin{aligned}&P(\text{Ann wins the round}) = P(A_R B_S) + P(A_P B_R) + P(A_S B_P)\\ &= P(A_R)P(B_S) + P(A_P)P(B_R) + P(A_S)P(B_P)\\ &= \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{3} = \tfrac{1}{3}.\end{aligned}$$

Conceptually quicker than enumerating cases would be to notice that no matter what Ann chooses, the probability that Bill makes a losing choice is $\frac{1}{3}$.

CLICK HERE TO ACCESS THE COMPLETE Solutions

Hence by the law of total probability, Ann's probability of winning must be $\frac{1}{3}$. Here is the calculation:

$$\begin{aligned}P(\text{Ann wins the round}) &= P(\text{Ann wins the round}\,|\,A_R)P(A_R)\\&\quad + P(\text{Ann wins the round}\,|\,A_P)P(A_P)\\&\quad + P(\text{Ann wins the round}\,|\,A_S)P(A_S)\\&= \tfrac{1}{3}\cdot P(A_R) + \tfrac{1}{3}\cdot P(A_P) + \tfrac{1}{3}\cdot P(A_S)\\&= \tfrac{1}{3}\cdot \big(P(A_R)+P(A_P)+P(A_S)\big) = \tfrac{1}{3}.\end{aligned}$$

(b) By the independence of the outcomes of different rounds,

$$\begin{aligned}&P(\text{Ann's first win happens in the fourth round})\\&= P(\text{Ann does not win any of the first three rounds,}\\&\qquad\qquad \text{Ann wins the fourth round})\\&= \tfrac{2}{3}\cdot\tfrac{2}{3}\cdot\tfrac{2}{3}\cdot\tfrac{1}{3} = \tfrac{8}{81}.\end{aligned}$$

(c) Again by the independence of the outcomes of different rounds,

$$P(\text{Ann does not win any of the first four rounds}) = \left(\tfrac{2}{3}\right)^4 = \tfrac{16}{81}.$$

2.23. Whether there is an accident on a given day can be treated as the outcome of a trial (where success means that there is at least one accident). The success probability is $p = 1 - 0.95 = 0.05$ and the failure probability is 0.95.

(a) The probability of no accidents at this intersection during the next 7 days is the probability that the first seven trials failed, which is $(1-p)^7 = 0.95^7 \approx 0.6983$.

(b) There are 30 days in September. Let X be the number of days that have at least one accident. X counts the number of 'successes' among 30 trials, so $X \sim \text{Bin}(30, 0.05)$. Using the probability mass function of the binomial we get

$$P(X=2) = \binom{30}{2}0.05^2 0.95^{28} \approx 0.2586.$$

(c) Let N denote the number of days we have to wait for the next accident, or equivalently, the number of trials needed for the first success. N has geometric distribution with parameter $p = 0.05$. We need to compute $P(4 < N \le 10)$. The event $\{4 < N \le 10\}$ is the same as $\{N \in \{5,6,7,8,9,10\}\}$. Using the probability mass function of the geometric distribution,

$$\begin{aligned}P(4 < N \le 10) &= \sum_{k=5}^{10} P(N=k) = \sum_{k=5}^{10}(1-p)^{k-1}p = \sum_{k=5}^{10} 0.95^{k-1}0.05\\&\approx 0.2158.\end{aligned}$$

Here is an alternative solution. Note that

$$\begin{aligned}P(4 < N \le 10) &= P(N \le 10) - P(N \le 4)\\&= (1 - P(N > 10)) - (1 - P(N > 4))\\&= P(N > 4) - P(N > 10).\end{aligned}$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

For any positive integer k the event $\{N > k\}$ is the same as having k failures in the first k trials. By part (a) the probability of this is $(1-p)^k$, which gives $P(N > k) = (1-p)^k = 0.95^k$ and then

$$\begin{aligned} P(4 < N \leq 10) &= P(N > 4) - P(N > 10) = (1-p)^4 - (1-p)^{10} \\ &= 0.95^4 - 0.95^{10} \approx 0.2158. \end{aligned}$$

2.24. (a) X is hypergeometric with parameters $(6, 4, 3)$.

(b) The probability mass function of X is

$$P(X = k) = \frac{\binom{4}{k}\binom{2}{3-k}}{\binom{6}{3}} \qquad \text{for } k \in \{0, 1, 2, 3\},$$

with the convention that $\binom{a}{k} = 0$ for integers $k > a \geq 0$. In particular, $P(X = 0) = 0$ because with only 2 men available, a team of 3 cannot consist of men alone.

2.25. Define events: $A = \{\text{first roll is a three}\}$, $B = \{\text{second roll is a four}\}$, $D_i = \{\text{the die has } i \text{ sides}\}$. Assume that A and B are independent, given D_i, for each $i = 4, 6, 12$.

$$\begin{aligned} P(AB) &= \sum_{i=4,6,12} P(AB|D_i)P(D_i) = \sum_{i=4,6,12} P(A|D_i)P(B|D_i)P(D_i) \\ &= \big((\tfrac{1}{4})^2 + (\tfrac{1}{6})^2 + (\tfrac{1}{12})^2\big) \cdot \tfrac{1}{3}. \end{aligned}$$

$$P(D_6|AB) = \frac{P(AB|D_6)P(D_6)}{P(AB)} = \frac{(\frac{1}{6})^2 \cdot \frac{1}{3}}{\big((\frac{1}{4})^2 + (\frac{1}{6})^2 + (\frac{1}{12})^2\big) \cdot \frac{1}{3}} = \tfrac{2}{7}.$$

2.26.

$$P((AB) \cap (CD)) = P(ABCD) = P(A)P(B)P(C)P(D) = P(AB)P(CD).$$

The very first equality is set algebra, namely, the associativity of intersection. This can be taken as intuitively obvious, or verified from the definition of intersection and common sense logic:

$$\begin{aligned} \omega \in (AB) \cap (CD) &\iff \omega \in AB \;\text{ and }\; \omega \in CD \\ &\iff \big(\omega \in A \;\text{ and }\; \omega \in B\big) \;\text{ and }\; \big(\omega \in C \;\text{ and }\; \omega \in D\big) \\ &\iff \omega \in A \;\text{ and }\; \omega \in B \;\text{ and }\; \omega \in C \;\text{ and }\; \omega \in D \\ &\iff \omega \in ABCD. \end{aligned}$$

Then we used the product rule first for all four events A, B, C, D, and then separately for the pairs A, B and C, D.

2.27. (a) First introduce the necessary events. Let A be the event that we picked Urn I. Then A^c is the event that we picked Urn II. Let B_1 the event that we picked a green ball. Then

$$P(A) = P(A^c) = \frac{1}{2}, \qquad P(B_1|A) = \frac{1}{3}, \qquad P(B_1|A^c) = \frac{2}{3}.$$

$P(B_1)$ is computed from the law of total probability:

$$P(B_1) = P(B_1|A)P(A) + P(B_1|A^c)P(A^c) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(b) The two experiments are identical and independent. Thus the probability of picking green both times is the square of the probability from part (a): $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$.

(c) Let B_2 be the event that we picked a green ball in the second draw. The events B_1, B_2 are conditionally independent given A (and given A^c), since we are sampling with replacement from the same urn. Thus

$$P(B_2|A) = \frac{1}{3}, \quad P(B_2|A^c) = \frac{2}{3},$$
$$P(B_1B_2|A) = P(B_1|A)P(B_2|A), \quad P(B_1B_2|A^c) = P(B_1|A^c)P(B_2|A^c).$$

From this we get

$$\begin{aligned} P(B_1B_2) &= P(B_1B_2|A)P(A) + P(B_1B_2|A^c)P(A^c) \\ &= P(B_1|A)P(B_2|A)P(A) + P(B_1|A^c)P(B_2|A^c)P(A^c) \\ &= (\tfrac{1}{3})^2\tfrac{1}{2} + (\tfrac{2}{3})^2\tfrac{1}{2} = \tfrac{5}{18}. \end{aligned}$$

(d) The probability of getting a green from the first urn is $\frac{1}{3}$ and the probability of getting a green from the second urn is $\frac{2}{3}$. Since the picks are independent, the probability of both picks being green is $\frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$.

2.28. (a) The number of aces I get in the first game is hypergeometric with parameters $(52, 4, 13)$.

(b) The number of games in which I receive at least one ace during the evening is binomial with parameters $(50, 1 - (\binom{48}{13}/\binom{52}{13}))$.

(c) The number of games in which all my cards are from the same suit is binomial with parameters $(50, \binom{52}{13}^{-1})$.

(d) The number of spades I receive in the 5th game is hypergeometric with parameters $(52, 13, 13)$.

2.29. Let E_1, E_2, E_3, N be the events that Uncle Bob hits a single, double, triple, or not making it on base, respectively. These events form a partition of our sample space. We also define S as the event Uncle Bob scores in this turn at bat. By the law of total probability we have

$$\begin{aligned} P(S) &= P(SE_1) + P(SE_2) + P(SE_3) + P(SN) \\ &= P(S|E_1)P(E_1) + P(S|E_2)P(E_2) + P(S|E_3)P(E_3) + P(S|N)P(N) \\ &= 0.2\cdot 0.35 + 0.3\cdot 0.25 + 0.4\cdot 0.1 + 0\cdot 0.3 \\ &= 0.185. \end{aligned}$$

2.30. Identical twins have the same gender. We assume that identical twins are equally likely to be boys or girls. Fraternal twins are also equally likely to be boys or girls, but independently of each other. Thus fraternal twins are two girls with probability $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Let I be the event that the twins are identical, F the event that the twins are fraternal.

(a) $P(\text{two girls}) = P(\text{two girls}\,|\,I)P(I) + P(\text{two girls}\,|\,F)P(F) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{3}$.

(b) $P(I\,|\,\text{two girls}) = \dfrac{P(\text{two girls}\,|\,I)P(I)}{P(\text{two girls})} = \dfrac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{3}} = \dfrac{1}{2}$.

CLICK HERE TO ACCESS THE COMPLETE Solutions

2.31. (a) The sample space is

$$\Omega = \{(g, b), (b, g), (b, b), (g, g)\},$$

and the probability measure is simply

$$P(g, b) = P(b, g) = P(b, b) = P(g, g) = \frac{1}{4},$$

since we assume that each outcome is equally likely.

(b) Let A be the event that there is a girl in the family. Let B be the event that there is a boy in the family. Note that the question is asking for $P(B|A)$. Begin to solve by noting that

$$A = \{(g, b), (b, g), (g, g)\} \text{ and } P(A) = \frac{3}{4}.$$

Similarly,

$$B = \{(g, b), (b, g), (b, b)\} \text{ and } P(B) = \frac{3}{4}.$$

Finally, we have

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(\{(g, b), (b, g)\})}{3/4} = \frac{2/4}{3/4} = \frac{2}{3}.$$

(c) Let $C = \{(g, b), (g, g)\}$ be the event that the first child is a girl. B is as above. We want $P(B|C)$. Since $P(C) = 1/2$ we have

$$P(B|C) = \frac{P(BC)}{P(C)} = \frac{P\{(g, b)\}}{1/2} = \frac{1/4}{1/2} = \frac{1}{2}.$$

2.32. (a) The sample space is

$$\Omega = \{(b, b, b), (b, b, g), (b, g, b), (b, g, g), (g, b, b), (g, b, g), (g, g, b), (g, g, g)\},$$

and each sample point has probability $\frac{1}{8}$ since we assume all outcomes equally likely.

(b) Let $A = \{(b, g, g), (g, b, g), (g, g, b), (g, g, g)\}$ be the event that there are at least two girls in the family. Let

$$B = \{(b, b, b), (b, b, g), (b, g, b), (b, g, g), (g, b, b), (g, b, g), (g, g, b)\}$$

be the event that there is a boy in the family.

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(\{(b, g, g), (g, b, g), (g, g, b)\})}{P\{(b, g, g), (g, b, g), (g, g, b), (g, g, g)\}} = \frac{3/8}{4/8} = \frac{3}{4}.$$

(c) Let $C = \{(g, g, b), (g, g, g)\}$ be the event that the first two children are girls. B is as above. We want $P(B|C)$. We have

$$P(B|C) = \frac{P(BC)}{P(C)} = \frac{P\{(g, g, b)\}}{P\{(g, g, b), (g, g, g)\}} = \frac{1}{2}.$$

2.33. (a) Let B_k be the event that we choose urn k and let A be the event that we chose a red ball. Then

$$P(B_k) = \tfrac{1}{5}, \qquad P(A|B_k) = \tfrac{k}{10}, \qquad \text{for } 1 \le k \le 5.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

By conditioning on the urn we chose and using (2.7) we get

$$P(A) = \sum_{k=1}^{5} P(A \,|\, B_k)P(B_k) = \sum_{k=1}^{5} \tfrac{k}{10} \cdot \tfrac{1}{5} = \tfrac{1+2+3+4+5}{50} = \tfrac{3}{10}.$$

(b)

$$P(B_k \,|\, A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{5} P(A \,|\, B_k)P(B_k)} = \frac{\frac{k}{10} \cdot \frac{1}{5}}{\frac{3}{10}} = \frac{k}{15}.$$

2.34. Since the urns are interchangeable, we can put the marked ball in urn 1. There are three ways to arrange the two unmarked balls. Let case i for $i \in \{0, 1, 2\}$ denote the situation where we put i unmarked balls together with the marked ball, and the remaining $2 - i$ unmarked balls in the other urn. Let M denote the event that your friend draws the marked ball, and A_j the event that she chooses urn j, $j = 1, 2$. Since $P(M|A_2) = 0$, we get the following probabilities.

Case 0: $P(M) = P(M|A_1)P(A_1) = 1 \cdot \frac{1}{2} = \frac{1}{2}$.

Case 1: $P(M) = P(M|A_1)P(A_1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Case 2: $P(M) = P(M|A_1)P(A_1) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

So (a) you would put all the balls in one urn (Case 2) while (b) she would put the marked ball in one urn and the other balls in the other urn.

(c) The situation is analogous. If we put k unmarked balls together with the marked ball in urn 1, then

$$P(M) = P(M|A_1)P(A_1) = \tfrac{1}{k+1} \cdot \tfrac{1}{2} = \tfrac{1}{2(k+1)}.$$

Hence to minimize the chances of drawing the marked ball, put all the balls in one urn, and to maximize the chances of drawing the marked ball, put the marked ball in one urn and all the unmarked balls in the other.

2.35. Let A be the event that the first card is a queen and B the event that the second card is a spade. Note that A and B are not independent, and there is no immediate way to compute $P(B|A)$. We can compute $P(AB)$ by counting favorable outcomes. Let Ω be the collection of all ordered pairs drawn without replacement from 52 cards. $\#\Omega = 52 \cdot 51$ and all outcomes are equally likely. We can break up AB into the union of the following two disjoint events:

$$C = \{\text{first card is queen of spades, second is a spade}\},$$

$$D = \{\text{first card is a queen but not a spade, the second card is a spade}\}.$$

We have $\#C = 12$, as we can choose the second card 12 different ways. We have $\#D = 3 \cdot 13 = 39$ as the first card can be any of the three non-spade queens, and the second card can be any of the 13 spades. Thus $\#AB = \#C + \#D = 12 + 39 = 51$ and we get $P(AB) = \frac{\#AB}{\#\Omega} = \frac{51}{52 \cdot 51} = \frac{1}{52}$.

2.36. Let A_j be the event that a j-sided die was chosen and B the event that a six was rolled.

CLICK HERE TO ACCESS THE COMPLETE Solutions

(a) By the law of total probability,

$$\begin{aligned} P(B) &= P(B|A_4)P(A_4) + P(B|A_6)P(A_6) + P(B|A_{12})P(A_{12}) \\ &= 0 \cdot \tfrac{7}{12} + \tfrac{1}{6} \cdot \tfrac{3}{12} + \tfrac{1}{12} \cdot \tfrac{2}{12} = \tfrac{1}{18}. \end{aligned}$$

(b)

$$P(A_6|B) = \frac{P(B|A_6)P(A_6)}{P(B)} = \frac{\frac{1}{6} \cdot \frac{3}{12}}{\frac{1}{18}} = \tfrac{3}{4}.$$

2.37. (a) Let S, E, T, and W be the events that the six, eight, ten, and twenty sided die is chosen. Let X be the outcome of the roll. Then

$$\begin{aligned} P(X = 6) &= P(X = 6|S)P(S) + P(X = 6|E)P(E) \\ &\quad + P(X = 6|T)P(T) + P(X = 6|W)P(W) \\ &= \frac{1}{6} \cdot \frac{1}{10} + \frac{1}{8} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{3}{10} + \frac{1}{20} \cdot \frac{4}{10} \\ &= \frac{11}{120}. \end{aligned}$$

(b) We want

$$P(W|X = 7) = \frac{P(W, X = 7)}{P(X = 7)} = \frac{P(X = 7|W)P(W)}{P(X = 7)}.$$

Following part (a), we have

$$\begin{aligned} P(X = 7) &= P(X = 7|S)P(S) + P(X = 7|E)P(E) \\ &\quad + P(X = 7|T)P(T) + P(X = 7|W)P(W) \\ &= 0 \cdot \frac{1}{10} + \frac{1}{8} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{3}{10} + \frac{1}{20} \cdot \frac{4}{10} = \frac{3}{40}. \end{aligned}$$

Thus,

$$P(W|X = 7) = \frac{(1/20) \cdot (4/10)}{(3/40)} = \frac{4}{15}.$$

2.38. Let R denote the event that the chosen letter is R and let A_i be the event that the ith word of the sentence is chosen.

(a) $P(R) = \sum_{i=1}^{4} P(R|A_i)P(A_i) = 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4} = \frac{2}{15}$.

(b) $P(X = 3) = \frac{1}{4}$, $P(X = 4) = \frac{1}{2}$, $P(X = 5) = \frac{1}{4}$.

(c) $P(X = 3 \,|\, X > 3) = 0$.

$$P(X = 4 \,|\, X > 3) = \frac{P(\{X = 4\} \cap \{X > 3\})}{P(X > 3)} = \frac{P(X = 4)}{P(X = 4) + P(X = 5)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)} = \frac{2}{3}.$$

$$P(X = 5 \,|\, X > 3) = \frac{P(\{X = 5\} \cap \{X > 3\})}{P(X > 3)} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(d) Use below that $R \cap A_1 = R \cap A_2 = A_3 \cap \{X > 3\} = \varnothing$.

$$\begin{aligned}
P(R \,|\, X > 3) &= \sum_{i=1}^{4} P(RA_i|X > 3) = P(RA_3|X > 3) + P(RA_4|X > 3) \\
&= \frac{P(R \cap A_3 \cap \{X > 3\})}{P(X > 3)} + \frac{P(R \cap A_4 \cap \{X > 3\})}{P(X > 3)} \\
&= \frac{P(R \cap A_4)}{P(X = 4) + P(X = 5)} = \frac{P(R \,|\, A_4)P(A_4)}{P(X = 4) + P(X = 5)} \\
&= \frac{\frac{1}{5} \cdot \frac{1}{4}}{\frac{1}{2} + \frac{1}{4}} = \frac{1}{15}.
\end{aligned}$$

(e)

$$P(A_4 \,|\, R) = \frac{P(R \,|\, A_4)P(A_4)}{P(R)} = \frac{\frac{1}{5} \cdot \frac{1}{4}}{\frac{2}{15}} = \frac{3}{8}.$$

2.39. (a) Let B_i the event that we chose the ith word ($i = 1, \dots, 8$). Events $B_1, \dots, B_8$ form a partition of the sample space and $P(B_i) = \frac{1}{8}$ for each i. Let A be the event that we chose the letter `O`. Then $P(A|B_3) = \frac{1}{5}$, $P(A|B_4) = \frac{1}{3}$, $P(A|B_6) = \frac{1}{4}$ with all other $P(A|B_i) = 0$. This gives

$$P(A) = \sum_{i=1}^{8} P(A|B_i)P(B_i) = \frac{1}{8}\left(\frac{1}{5} + \frac{1}{3} + \frac{1}{4}\right) = \frac{47}{480}.$$

(b) The length of the chosen word can be 3, 4, 5 or 6, so the range of X is the set $\{3, 4, 5, 6\}$. For each of the possible value x we have to find the probability $P(X = x)$.

$$\begin{aligned}
p_X(3) &= P(X = 3) = P(\text{we chose the 1st, the 4th or the 7th word}) \\
&= P(B_1 \cup B_4 \cup B_7) = \frac{3}{8}, \\
p_X(4) &= P(X = 4) = P(\text{we chose the 6th or the 8th word}) = P(B_6 \cup B_8) = \frac{2}{8}, \\
p_X(5) &= P(X = 5) = P(\text{we chose the 2nd or the 3rd word}) = P(B_2 \cup B_3) = \frac{2}{8}, \\
p_X(6) &= P(X = 6) = P(\text{we chose the 5th word}) = P(B_5) = \frac{1}{8}.
\end{aligned}$$

Note that the probabilities add up to 1, as they should.

2.40. (a) For $i \in \{1, 2, 3, 4\}$ let A_i be the event that the student scores i on the test. Let M be the event that the student becomes a math major.

$$P(M) = \sum_{i=1}^{4} P(M|A_i)P(A_i) = 0 \cdot 0.1 + \tfrac{1}{5} \cdot 0.2 + \tfrac{1}{3} \cdot 0.6 + \tfrac{3}{7} \cdot 0.1 \approx 0.2829.$$

(b)

$$P(A_4|M) = \frac{P(M|A_4)P(A_4)}{P(M)} = \frac{\frac{3}{7} \cdot 0.1}{\frac{1}{5} \cdot 0.2 + \frac{1}{3} \cdot 0.6 + \frac{3}{7} \cdot 0.1} \approx 0.1515.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

2.41. Introduce the following events:

$$B = \{\text{the phone is not defective}\}, \quad A = \{\text{the phone comes from factory II}\}.$$

Then A^c is the event that the phone is from factory I. We know that

$$P(A) = 0.4 = \frac{2}{5}, \quad P(A^c) = 0.6 = \frac{3}{5}, \quad P(B^c|A) = 0.2 = \frac{1}{5}, \quad P(B^c|A^c) = 0.1 = \frac{1}{10}.$$

Note that this also gives

$$P(B|A) = 1 - P(B^c|A) = \frac{4}{5}, \qquad P(B|A^c) = 1 - P(B^c|A^c) = \frac{9}{10}.$$

We need to compute $P(A|B)$. By Bayes' formula,

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{\frac{4}{5} \cdot \frac{2}{5}}{\frac{4}{5} \cdot \frac{2}{5} + \frac{9}{10} \cdot \frac{3}{5}} = \frac{16}{16 + 27}$$
$$= \frac{16}{43} \approx 0.3721.$$

2.42. Let R be the event that the transferred ball was red, and W the event that the transferred ball was white. Let V be the event that a white ball was drawn from urn B. Then $P(R) = \frac{1}{3}$ and $P(W) = \frac{2}{3}$. If a red ball was transferred, then the new composition of urn B is 2 red and 1 white, while if a white ball was transferred, then the new composition of urn B is 1 red and 2 white. Putting all this together gives the following calculation.

$$P(W|V) = \frac{P(WV)}{P(V)} = \frac{P(V|W)P(W)}{P(V|W)P(W) + P(V|R)P(R)}$$
$$= \frac{\frac{2}{3} \cdot \frac{2}{3}}{\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3}} = \frac{4}{5}.$$

2.43. (a) Let A_1 be the event that the first sample had two balls of the same color. If we imagine that the draws are done one at a time in order then there are $5 \cdot 4$ possible outcomes. Counting the green-green and yellow-yellow cases separately we get that $3 \cdot 2 + 2 \cdot 1$ of those outcomes have two balls of the same color. Thus

$$P(A_1) = \frac{3 \cdot 2 + 2 \cdot 1}{5 \cdot 4} = \frac{2}{5}.$$

(b) Let A_2 be the event that the second sample had two balls of the same color. We have $P(A_2|A_1) = 1$, since if the first sample had two balls of the same color then this must be true for the second one. Furthermore, $P(A_2|A_1^c) = \frac{1}{2}$, because if we sample twice with replacement from an urn containing one yellow and one green ball, then $1/2$ is the probability that the second draw has the same color as the first one. (Or, dividing the number of favorable outcomes by the total, $\frac{1 \cdot 1 + 1 \cdot 1}{2 \cdot 2} = \frac{1}{2}$.) From part (a) we know that $P(A_1) = \frac{2}{5}$ and $P(A_1^c) = \frac{3}{5}$. Altogether this gives

$$P(A_2) = P(A_2|A_1)P(A_1) + P(A_2|A_1^c)P(A_1^c) = 1 \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{3}{5} = \frac{7}{10}.$$

(c) Using the already computed probabilities:

$$P(A_1|A_2) = \frac{P(A_2|A_1)P(A_1)}{P(A_2)} = \frac{1 \cdot \frac{2}{5}}{\frac{7}{10}} = \frac{4}{7}.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

2.44. Let A_i be the event that bin i was chosen $(i = 1, 2)$ and Y_j the event that draw j $(j = 1, 2)$ is yellow.

(a)

$$P(A_1|Y_1) = \frac{P(Y_1|A_1)P(A_1)}{P(Y_1|A_1)P(A_1) + P(Y_1|A_2)P(A_2)}$$
$$= \frac{\frac{4}{10} \cdot \frac{1}{2}}{\frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}} = \frac{14}{34} \approx 0.4118.$$

(b) This question asks for the conditional probability of A_1, given that two draws with replacement from the chosen urn yield yellow. We assume that draws with replacement from the same urn are independent. This translates into conditional independence of Y_1 and Y_2, given A_i.

$$P(A_1|Y_1Y_2) = \frac{P(Y_1Y_2|A_1)P(A_1)}{P(Y_1Y_2|A_1)P(A_1) + P(Y_1Y_2|A_2)P(A_2)}$$
$$= \frac{P(Y_1|A_1)P(Y_1|A_1)P(A_1)}{P(Y_1|A_1)P(Y_1|A_1)P(A_1) + P(Y_1|A_2)P(Y_1|A_2)P(A_2)}$$
$$= \frac{\frac{4}{10} \cdot \frac{4}{10} \cdot \frac{1}{2}}{\frac{4}{10} \cdot \frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{1}{2}} = \frac{196}{596} \approx 0.3289.$$

2.45. (a) Let B, G, and O be the events that a 7-year-old like the Bears, Packers, and some other team, respectively. We are given the following:

$$P(B) = 0.10, \quad P(G) = 0.75, \quad P(O) = 0.15.$$

Let A be the event that the 7-year-old goes to a game. Then we have

$$P(A|B) = 0.01, \quad P(A|G) = 0.05, \quad P(A|O) = 0.005.$$

$P(A)$ is computed from the law of total probability:

$$P(A) = P(A|B)P(B) + P(A|G)P(G) + P(A|O)P(O)$$
$$= 0.01 \cdot 0.1 + 0.05 \cdot 0.75 + 0.005 \cdot 0.15 = 0.03925.$$

(b) Using the result of (a) (or Bayes' formula directly):

$$P(G|A) = \frac{P(AG)}{P(A)} = \frac{P(A|G)P(G)}{P(A)} = \frac{0.05 \cdot 0.75}{0.03925} = \frac{0.0375}{0.03925} \approx 0.9554.$$

2.46. A sample point is an ordered triple (x, y, z) where x is the number drawn from box A, y is the number drawn from box B, and z the number drawn from box C. All $6 \cdot 12 \cdot 4 = 288$ outcomes are equally likely, so we can solve these problems by counting.

(a) The number of outcomes with exactly two 1s is

$$1 \cdot 1 \cdot 3 + 1 \cdot 11 \cdot 1 + 5 \cdot 1 \cdot 1 = 19.$$

The number of outcomes with a 1 from box A and exactly two 1s is

$$1 \cdot 1 \cdot 3 + 1 \cdot 11 \cdot 1 = 14.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

Thus

$$P(\text{ball 1 from } A \,|\, \text{exactly two 1s}) = \frac{P(\text{ball 1 from } A \text{ and exactly two 1s})}{P(\text{exactly two 1s})} = \frac{14/288}{19/288} = \frac{14}{19}.$$

(b) There are three sample points whose sum is 21: $(6, 12, 3), (6, 11, 4), (5, 12, 4)$. Two of these have 12 drawn from B. Hence the answer is $2/3$. Here is the formal calculation.

$$P(\text{ball 12 from } B \,|\, \text{sum of balls 21}) = \frac{P(\text{ball 12 from } B \text{ and sum of balls 21})}{P(\text{sum of balls 21})}$$
$$= \frac{P\{(6,12,3),(5,12,4)\}}{P\{(6,12,3),(6,11,4),(5,12,4)\}} = \frac{2/288}{3/288} = \frac{2}{3}.$$

2.47. Define random variables X and Y and event S:

$X =$ total number of patients for whom the drug is effective

$Y =$ number of patients for whom the drug is effective, excluding your friends

$S =$ trial is a success for your two friends.

We need to find

$$P(S|X = 55) = \frac{P(S \cap \{X = 55\})}{P(X = 55)}.$$

Note that $X \sim \text{Bin}(80, p)$, and thus $P(X = 55) = \binom{80}{55} p^{55}(1-p)^{25}$. Moreover, $S \cap \{X = 55\} = S \cap \{Y = 53\}$. The events S and $\{Y = 53\}$ are independent, as S depends on the trial outcomes for your friends, and Y on the trial outcomes of the other patients. Thus

$$P(S \cap \{X = 55\}) = P(S \cap \{Y = 53\}) = P(S)P(Y = 53).$$

We have $P(S) = p^2$ and $P(Y = 53) = \binom{78}{53} p^{53}(1-p)^{25}$, as $Y \sim \text{Bin}(78, p)$. Collecting everything:

$$P(S|X = 55) = \frac{P(S \cap \{X = 55\})}{P(X = 55)} = \frac{p^2 \cdot \binom{78}{53} p^{53}(1-p)^{25}}{\binom{80}{55} p^{55}(1-p)^{25}} = \frac{\binom{78}{53}}{\binom{80}{55}}$$
$$= \frac{297}{632} \approx 0.4699.$$

2.48. Define events $G = \{\text{Kevin is guilty}\}$, $A = \{\text{DNA match}\}$. Before the DNA evidence $P(G) = 1/100,000$. After the DNA match

$$P(G|A) = \frac{P(A|G)P(G)}{P(A|G)P(G) + P(A|G^c)P(G^c)} = \frac{1 \cdot \frac{1}{100{,}000}}{1 \cdot \frac{1}{100{,}000} + \frac{1}{10{,}000} \cdot \frac{99{,}999}{100{,}000}}$$
$$= \frac{1}{1 + 10 - 10^{-4}} \approx \frac{1}{11}.$$

2.49. (a) The given numbers are nonnegative, so we just need to check that $\sum_{k=0}^{\infty} P(X = k) = 1$:

$$\sum_{k=0}^{\infty} P(X = k) = \frac{4}{5} + \sum_{k=1}^{\infty} \tfrac{1}{10} \cdot \left(\tfrac{2}{3}\right)^k = \frac{4}{5} + \frac{\frac{1}{10} \cdot \frac{2}{3}}{1 - \frac{2}{3}} = 1.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(b) For $k \geq 1$, by changing the summation index from j to $i = j - k$:

$$P(X \geq k) = \sum_{j=k}^{\infty} \tfrac{1}{10} \cdot \left(\tfrac{2}{3}\right)^j = \tfrac{1}{10} \cdot \left(\tfrac{2}{3}\right)^k \sum_{i=0}^{\infty} \left(\tfrac{2}{3}\right)^i = \tfrac{1}{10} \cdot \left(\tfrac{2}{3}\right)^k \frac{1}{1 - \frac{2}{3}} = \tfrac{1}{5} \left(\tfrac{2}{3}\right)^{k-1}.$$

Thus again for $k \geq 1$,

$$P(X \geq k \,|\, X \geq 1) = \frac{P(\{X \geq k\} \cap \{X \geq 1\})}{P(X \geq 1)} = \frac{P(X \geq k)}{P(X \geq 1)}$$
$$= \frac{\frac{1}{5}\left(\frac{2}{3}\right)^{k-1}}{\frac{1}{5}} = \left(\tfrac{2}{3}\right)^{k-1}.$$

The numerator simplified because $\{X \geq k\} \subset \{X \geq 1\}$. The answer shows that conditional on $X \geq 1$, X has Geom($\frac{1}{3}$) distribution.

2.50. (a)

$$P(A|D) = \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)}$$
$$= \frac{p \cdot \frac{1}{3}}{p \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{p}{1+p}.$$

(b)

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{1 \cdot \frac{1}{3}}{(p+1) \cdot \frac{1}{3}} = \frac{1}{1+p}.$$

If the guard is equally likely to name either B or C when both of them are slated to die, then A has not gained anything (his probability of pardon is still $\frac{1}{3}$) but C's chances of pardon have increased to $\frac{2}{3}$. In the extreme case where the guard would never name B unless he had to ($p = 0$), C is now sure to be pardoned.

2.51. Since $C \subset B$ we have $B \cup C = B$ and thus $A \cup B \cup C = A \cup B$. Then

$$P(A \cup B \cup C) = P(A \cup B) = P(A) + P(B) - P(AB).$$

Since A and B are independent we have $P(AB) = P(A)P(B)$. This gives

$$P(A \cup B \cup C) = P(A) + P(B) - P(A)P(B) = 1/2 + 1/4 - 1/8 = 5/8.$$

2.52. Yes, A, B, and C are mutually independent. There are four equations to check:

(i) $P(AB) = P(A)P(B)$
(ii) $P(AC) = P(A)P(C)$
(iii) $P(BC) = P(B)P(C)$
(iv) $P(ABC) = P(A)P(B)P(C)$.

(i) comes from inclusion-exclusion:

$$P(AB) = P(A) + P(B) - P(A \cup B) = 0.06 = P(A)P(B).$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(ii) comes from $P(AC) = P(C) - P(A^cC) = 0.03 = P(A)P(C)$. (iii) is given. Finally, (iv) comes from using inclusion-exclusion once more and the previous computations:

$$\begin{aligned} P(ABC) &= P(A \cup B \cup C) - P(A) - P(B) - P(C) \\ &\qquad + P(AB) + P(AC) + P(BC) \\ &= 0.006 = P(A)P(B)P(C). \end{aligned}$$

2.53. (a) If the events are disjoint then

$$P(A \cup B) = P(A) + P(B) = 0.3 + 0.6 = 0.9.$$

(b) If the events are independent then

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) = P(A) + P(B) - P(A)P(B) \\ &= 0.3 + 0.6 - 0.3 \cdot 0.6 = 0.72. \end{aligned}$$

2.54. (a) It is possible. We use the fact that $A = AB \cup AB^c$ and that these are mutually exclusive:

$$\begin{aligned} P(A) &= P(AB) + P(AB^c) = P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= \frac{1}{3}P(B) + \frac{1}{3}P(B^c) = \frac{1}{3}(P(B) + P(B^c)) = \frac{1}{3}. \end{aligned}$$

(b) A and B are independent. By part (a) and the given information,

$$P(A) = P(A|B) = \frac{P(AB)}{P(B)}$$

from which $P(AB) = P(A)P(B)$ and independence has been verified. (Note that the value $\frac{1}{3}$ was not needed for this conclusion.)

2.55. (a) Since Peter throws the first dart, in order for Mary to win Peter must fail once more than she does.

$$\begin{aligned} P(\text{Mary wins}) &= \sum_{k=1}^{\infty} P(\text{Mary wins on her } k\text{th throw}) \\ &= \sum_{k=1}^{\infty} ((1-p)(1-r))^{k-1}(1-p)r = \frac{(1-p)r}{1-(1-p)(1-r)} \\ &= \frac{(1-p)r}{p+r-pr}. \end{aligned}$$

(b) The possible values of X are the nonnegative integers.

$$P(X = 0) = P(\text{Peter wins on his first throw}) = p.$$

For $k \geq 1$,

$$\begin{aligned} P(X = k) &= P(\text{Mary wins on her } k\text{th throw}) \\ &\qquad + P(\text{Peter wins on his } (k+1)\text{st throw}) \\ &= ((1-p)(1-r))^{k-1}(1-p)r + ((1-p)(1-r))^k p \\ &= ((1-p)(1-r))^{k-1}(1-p)(p+r-pr). \end{aligned}$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

We check that the values for $k \geq 1$ add up to $1-$(the value at $k=0$):

$$\sum_{k=1}^{\infty}((1-p)(1-r))^{k-1}(1-p)(p+r-pr) = \frac{(1-p)(p+r-pr)}{1-(1-p)(1-r)} = 1-p.$$

This is not one of our named distributions.

(c) For $k \geq 1$,

$$\begin{aligned} P(X=k \mid \text{Mary wins}) &= \frac{P(\text{Mary wins on her } k\text{th throw})}{P(\text{Mary wins})} \\ &= \frac{((1-p)(1-r))^{k-1}(1-p)r}{\frac{(1-p)r}{p+r-pr}} \\ &= ((1-p)(1-r))^{k-1}(p+r-pr). \end{aligned}$$

Thus given that Mary wins, $X \sim \text{Geom}(p+r-pr)$.

2.56. Suppose $P(A)=0$. Then for any B, $AB \subset A$ implies $P(AB)=0$. We also have $P(A)P(B) = 0 \cdot P(B) = 0$. Thus $P(AB) = 0 = P(A)P(B)$ and independence of A and B has been verified.

Suppose $P(A)=1$. Then $P(A^c)=0$ and the previous case gives the independence of A^c and B, from which follows the independence of A and B. Alternatively, we can prove this case by first observing that $P(AB) = P(B) - P(A^cB) = P(B) - 0 = P(B)$ and then $P(A)P(B) = 1 \cdot P(B) = P(B)$. Again $P(AB) = P(A)P(B)$ has been verified.

2.57. (a) Let E_1 be the event that the first component functions. Let E_2 be the event that the second component functions. Let S be the event that the entire system functions. $S = E_1 \cap E_2$ since both components must function in order for the whole system to be operational. By the assumption that each component acts independently, we have

$$P(S) = P(E_1 \cap E_2) = P(E_1)P(E_2).$$

Next we find the probabilities $P(E_1)$ and $P(E_2)$.

Let X_i be a Bernoulli random variable taking the value 1 if the ith element of the first component is working. The information given is that $P(X_i = 1) = 0.95$, $P(X_i = 0) = 0.05$ and $X_1, \dots, X_8$ are mutually independent. Similarly, let Y_i be a Bernoulli random variable taking the value 1 if the ith element of the second component is working. Then $P(Y_i = 1) = 0.90$, $P(Y_i = 0) = 0.1$ and $Y_1, \dots, Y_4$ are mutually independent. Let $X = \sum_{i=1}^{8} X_i$ give the total number of working elements in component number one and $Y = \sum_{i=1}^{4} Y_i$ the total number of working elements in component number 2. Then $X \sim \text{Bin}(8, 0.95)$ and $Y \sim \text{Bin}(4, 0.90)$, and X and Y are independent (by the assumption that

CLICK HERE TO ACCESS THE COMPLETE Solutions

the components behave independently). We have

$$\begin{aligned} P(E_1) &= P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) \\ &= \binom{8}{6}(0.95)^6(0.05)^2 + \binom{8}{7}(0.95)^7(0.05)^1 + \binom{8}{8}(0.95)^8(0.05)^0 \\ &= 0.9942117, \end{aligned}$$

and

$$\begin{aligned} P(E_2) &= P(Y \geq 3) = P(Y = 3) + P(Y = 4) \\ &= \binom{4}{3}(0.9)^3(0.1) + (0.9)^4 \\ &= 0.9477. \end{aligned}$$

Thus,

$$P(S) = P(E_1)P(E_2) = 0.9942117 \cdot 0.9477 \approx 0.9422.$$

(b) We look for $P(E_2^c \,|\, S^c)$. We have

$$P(E_2^c|S^c) = \frac{P(E_2^c S^c)}{P(S^c)} = \frac{P(E_2^c)}{1 - P(S)},$$

where we used that $E_2^c \subset S^c$. (If the first component does not work, then the system does not work; mathematically a consequence of de Morgan's law: $S^c = E_1^c \cup E_2^c$.) Thus,

$$P(E_2^c|S^c) = \frac{1 - P(E_2)}{1 - P(S)} = \frac{1 - 0.9477}{1 - 0.9422} \approx 0.9048.$$

2.58. (a) It is enough to show that any two of them are pairwise independent since the argument is the same for any such pair. We show that $P(AB) = P(A)P(B)$. Let

$$\Omega = \{(a, b, c) : a, b, c \in \{1, 2, \dots, 365\}\} \implies \#\Omega = 365^3.$$

We have by counting the possibilities

$$\#AB = \{\text{all three have same birthday}\} = 365 \cdot 1 \cdot 1 \implies P(AB) = \frac{1}{365^2}.$$

Also,

$$\#A = \{\text{Alex and Betty have the same birthday}\} = 365 \cdot 1 \cdot 365,$$

where we counted as follows: there are 365 ways for Alex to have a birthday, then only once choice for Betty, and then another 365 ways for Conlin. Thus,

$$P(A) = \frac{365^2}{365^3} = \frac{1}{365}.$$

Similarly, $P(B) = \frac{1}{365}$ and so,

$$P(AB) = P(A)P(B).$$

(b) The events are not independent. Note that $ABC = AB$ and so,

$$P(ABC) = P(AB) = \frac{1}{365^2} \neq P(A)P(B)P(C) = \frac{1}{365^3}.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

2.59. Define events: $B = \{\text{the bus functions}\}$, $T = \{\text{the train functions}\}$, and $S = \{\text{no storm}\}$. The event that travel is possible is $(B \cup T) \cap S = BS \cup TS$. We calculate the probability with inclusion-exclusion and independence:

$$\begin{aligned} P(BS \cup TS) &= P(BS) + P(TS) - P(BTS) \\ &= P(B)P(S) + P(T)P(S) - P(B)P(T)P(S) \\ &= \tfrac{8}{10} \cdot \tfrac{19}{20} + \tfrac{9}{10} \cdot \tfrac{19}{20} - \tfrac{8}{10} \cdot \tfrac{9}{10} \cdot \tfrac{19}{20} = \tfrac{931}{1000}. \end{aligned}$$

2.60. (a) $P(AB^c) = P(A) - P(AB) = P(A) - P(A)P(B) = P(A)\big(1 - P(B)\big) = P(A)P(B^c)$.

(b) Apply first de Morgan and then inclusion-exclusion:

$$\begin{aligned} P(A^cC^c) &= 1 - P(A \cup C) = 1 - P(A) - P(C) + P(AC) \\ &= 1 - P(A) - P(C) + P(A)P(C) \\ &= \big(1 - P(A)\big)\big(1 - P(C)\big) = P(A^c)P(C^c). \end{aligned}$$

(c) $P(AB^cC) = P(AC) - P(ABC) = P(A)P(C) - P(A)P(B)P(C) = P(A)\big(1 - P(B)\big)P(C = P(A)P(B^c)P(C)$.

(d) Again first de Morgan and then inclusion-exclusion:

$$\begin{aligned} &P(A^cB^cC^c) = 1 - P(A \cup B \cup C) \\ &= 1 - P(A) - P(B) - P(C) + P(AB) + P(AC) + P(BC) - P(ABC) \\ &= 1 - P(A) - P(B) - P(C) + P(A)P(B) + P(A)P(C) + P(B)P(C) \\ &\qquad - P(A)P(B)P(C) \\ &= \big(1 - P(A)\big)\big(1 - P(B)\big)\big(1 - P(C)\big) \\ &= P(A^c)P(B^c)P(C^c). \end{aligned}$$

2.61. (a) Treat each draw as a trial: green is success, red is failure. By counting favorable outcomes, the probability of success is $p = \frac{3}{7}$ for each draw. Because we draw with replacement the outcomes are independent. Thus the number of greens in the 9 picks is the number of successes in 9 trials, hence a $\text{Bin}(9, \frac{3}{7})$ distribution. Using the probability mass function of the binomial distribution gives

$$\begin{aligned} P(X \ge 1) &= 1 - P(X = 0) = 1 - (1 - p)^9 \approx 0.9935, \\ P(X \le 5) &= \sum_{k=0}^{5} P(X = k) = \sum_{k=0}^{5} \binom{9}{k} p^k (1 - p)^{9-k} \approx 0.8653. \end{aligned}$$

(b) N is the number of trials needed for the first success, and so has geometric distribution with parameter $p = \frac{3}{7}$. The probability mass function of the geometric distribution gives

$$P(N \le 9) = \sum_{k=1}^{9} P(N = k) = \sum_{k=1}^{9} p(1 - p)^{k-1} \approx 0.9935.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

(c) We have $P(X \geq 1) = P(N \leq 9)$. We can check this by using the geometric sum formula to get

$$\sum_{k=1}^{9} p(1-p)^{k-1} = p\frac{1-(1-p)^9)}{1-(1-p)} = 1-(1-p)^9.$$

Here is another way to see this, without any algebra. Imagine that we draw balls with replacement infinitely many times. Think of X as the number of green balls in the first 9 draws. N is still the number of draws needed for the first green. Now if $X \geq 1$, then we have at least one green within the first 9 draws, which means that the first green draw happened within the first 9 draws. Thus $X \geq 1$ implies $N \leq 9$. But this works in the opposite direction as well: if $N \leq 9$ then the first green draw happened within the first 9 draws, which means that we must have at least one green within the first 9 picks. Thus $N \leq 9$ implies $X \geq 1$. This gives the equality of event: $\{X \geq 1\} = \{N \leq 9\}$, and hence the probabilities must agree as well.

2.62. Regard the drawing of three marbles as one trial, with success probability p given by

$$p = P(\text{all three marbles blue}) = \frac{\binom{9}{3}}{\binom{13}{3}} = \frac{7 \cdot 8 \cdot 9 \cdot 10}{10 \cdot 11 \cdot 12 \cdot 13} = \frac{42}{143}.$$

$X \sim \text{Bin}(20, \frac{42}{143})$. The probability mass function is

$$P(X = k) = \binom{20}{k}\left(\tfrac{42}{143}\right)^k\left(\tfrac{101}{143}\right)^{20-k} \quad \text{for } k = 0, 1, 2, \dots, 20.$$

2.63. The number of heads in n coin flips has distribution $\text{Bin}(n, 1/2)$. Thus the probability of winning if we choose to flip n times is

$$f_n = P(n \text{ flips yield exactly 2 heads}) = \binom{n}{2}\frac{1}{2^n} = \frac{n(n-1)}{2^{n+1}}.$$

We want to find the n which maximizes f_n. Let us compare f_n and f_{n+1}. We have

$$f_n < f_{n+1} \iff \frac{n(n-1)}{2^{n+1}} < \frac{(n+1)n}{2^{n+2}} \iff 2(n-1) < n+1 \iff n < 3.$$

Similarly, $f_n > f_{n+1}$ if and only if $n > 3$, and $f_3 = f_4$. Thus

$$f_2 < f_3 = f_4 > f_5 > f_6 > \dots.$$

This means that the maximum happens at $n = 3$ and $n = 4$, and the probability of winning at those values is $f_3 = f_4 = \frac{3 \cdot 2}{2^4} = \frac{3}{8}$.

2.64. Let X be the number of correct answers. X is the number of successes in 20 independent trials with success probability $p + \frac{1}{2}r$.

$$P(X \geq 19) = P(X = 19) + P(X = 20) = 20\left(p + \tfrac{1}{2}r\right)^{19}\left(q + \tfrac{1}{2}r\right) + \left(p + \tfrac{1}{2}r\right)^{20}.$$

2.65. Let A be the event that at least one die lands on a 4 and B be the event that all three dice land on different numbers. Our sample space is the set of all triples (a_1, a_2, a_3) with $1 \leq a_i \leq 6$. All outcomes are equally likely and there are 216 outcomes. We need $P(A|B) = \frac{P(AB)}{P(B)}$. There are $6 \cdot 5 \cdot 4 = 120$ elements in B. To count the elements of AB, we first consider A^cB. This is the set of triples where

CLICK HERE TO ACCESS THE COMPLETE Solutions

the three numbers are distinct and none of them is a 4. So $\#A^cB = 5 \cdot 4 \cdot 3 = 60$. Then $\#AB = \#B - \#A^cB = 120 - 60 = 60$ and

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{60}{216}}{\frac{120}{216}} = \frac{1}{2}.$$

2.66. Let

$$f_n = P(n \text{ die rolls give exactly two sixes}) = \binom{n}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^{n-2} = \frac{n(n-1)5^{n-2}}{2 \cdot 6^n}.$$

Next,

$$\begin{aligned} f_n < f_{n+1} \quad &\Longleftrightarrow \quad \frac{n(n-1)5^{n-2}}{2 \cdot 6^n} < \frac{(n+1)n5^{n-1}}{2 \cdot 6^{n+1}} \quad \Longleftrightarrow \quad 6(n-1) < 5(n+1) \\ &\Longleftrightarrow \quad n < 11. \end{aligned}$$

By reversing the inequalities we get the equivalence

$$f_n > f_{n+1} \quad \Longleftrightarrow \quad n > 11.$$

By complementing the two equivalences, we get

$$\begin{aligned} f_n = f_{n+1} \quad &\Longleftrightarrow \quad f_n \ge f_{n+1} \text{ and } f_n \le f_{n+1} \\ &\Longleftrightarrow \quad n \ge 11 \text{ and } n \le 11 \quad \Longleftrightarrow \quad n = 11. \end{aligned}$$

Putting all these facts together we conclude that the probability of two sixes is maximized by $n = 11$ and $n = 12$ and for these two values of n, that probability is

$$\frac{11 \cdot 10 \cdot 5^9}{2 \cdot 6^{11}} \approx 0.2961.$$

2.67. Since $\{X = n + k\} \subset \{X > n\}$ for $k \ge 1$, we have

$$P(X = n + k | X > n) = \frac{P(X = n + k, X > n)}{P(X > n)} = \frac{P(X = n + k)}{P(X > n)} = \frac{(1-p)^{n+k-1}p}{P(X > n)}.$$

Evaluate the denominator:

$$\begin{aligned} P(X > n) &= \sum_{k=n+1}^{\infty} P(X = k) = \sum_{k=n+1}^{\infty} (1-p)^{k-1}p \\ &= p(1-p)^n \sum_{k=0}^{\infty} (1-p)^k = p(1-p)^n \cdot \frac{1}{1-(1-p)} = (1-p)^n. \end{aligned}$$

Thus,

$$\begin{aligned} P(X = n + k | X > n) &= \frac{(1-p)^{n+k-1}p}{P(X > n)} = \frac{(1-p)^{n+k-1}p}{(1-p)^n} \\ &= (1-p)^{k-1}p = P(X = k). \end{aligned}$$

2.68. For $k \ge 1$, the assumed memoryless property gives

$$P(X = k) = P(X = k + 1 \,|\, X > 1) = \frac{P(X = k + 1)}{P(X > 1)}$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

which we convert into $P(X = k+1) = P(X > 1)P(X = k)$. Now let $m \ge 2$, and apply this repeatedly to $k = m-1, m-2, \dots, 2$:

$$\begin{aligned} P(X = m) &= P(X > 1)P(X = m-1) = P(X > 1)^2 P(X = m-2) \\ &= \cdots = P(X > 1)^{m-1} P(X = 1). \end{aligned}$$

Set $p = P(X = 1)$. Then it follows that $P(X = m) = (1-p)^{m-1}p$ for all $m \ge 1$ ($m = 1$ by definition of p, $m \ge 2$ by the calculation above). In other words, $X \sim \text{Geom}(p)$.

2.69. We assume that the successive flips of a given coin are independent. This gives us the conditional independence:

$$P(A_1A_2 \,|\, F) = P(A_1 \,|\, F)\, P(A_2 \,|\, F), \quad P(A_1A_2 \,|\, M) = P(A_1 \,|\, M)\, P(A_2 \,|\, M),$$
$$\text{and} \;\; P(A_1A_2 \,|\, H) = P(A_1 \,|\, H)\, P(A_2 \,|\, H).$$

The solution comes by the law of total probability:

$$\begin{aligned} P(A_1A_2) &= P(A_1A_2 \,|\, F)\, P(F) + P(A_1A_2 \,|\, M)\, P(M) + P(A_1A_2 \,|\, H)\, P(H) \\ &= P(A_1 \,|\, F)P(A_2 \,|\, F)P(F) + P(A_1 \,|\, B)P(A_2 \,|\, B)P(B) \\ &= \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{90}{100} + \tfrac{3}{5} \cdot \tfrac{3}{5} \cdot \tfrac{9}{100} + \tfrac{9}{10} \cdot \tfrac{9}{10} \cdot \tfrac{1}{100} = \tfrac{2655}{10{,}000}. \end{aligned}$$

Now $\frac{2655}{10{,}000} \ne (\frac{513}{1000})^2$ which says that $P(A_1A_2) \ne P(A_1)P(A_2)$. In other words, A_1 and A_2 are *not* independent without the conditioning on the type of coin. The intuitive reason is that the first flip gives us information about the coin we hold, and thereby alters our expectations about the second flip.

2.70. The relevant probabilities: $P(A) = P(B) = 2p(1-p)$ and

$$P(AB) = P\{(\mathtt{T},\mathtt{H},\mathtt{T}), (\mathtt{H},\mathtt{T},\mathtt{H})\} = p^2(1-p) + p(1-p)^2 = p(1-p).$$

Thus A and B are independent if and only if

$$\begin{aligned} &\big(2p(1-p)\big)^2 = p(1-p) \iff 4p^2(1-p)^2 - p(1-p) = 0 \\ &\iff p(1-p)\big(4p(1-p) - 1\big) = 0 \\ &\iff p = 0 \;\text{ or }\; 1-p = 0 \;\text{ or }\; 4p(1-p) - 1 = 0 \iff p \in \{0, \tfrac{1}{2}, 1\}. \end{aligned}$$

Note that cancelling $p(1-p)$ from the very first equation misses the solutions $p = 0$ and $p = 1$.

2.71. Let $F = \{\text{coin is fair}\}$, $B = \{\text{coin is biased}\}$ and $A_k = \{k\text{th flip is tails}\}$. We assume that conditionally on F, the events A_k are independent, and similarly conditionally on B. Let $D_n = A_1 \cap A_2 \cap \cdots \cap A_n = \{\text{the first } n \text{ flips are all tails}\}$.

(a)

$$\begin{aligned} P(B|D_n) &= \frac{P(D_n|B)P(B)}{P(D_n|B)P(B) + P(D_n|F)P(F)} = \frac{(\frac{3}{5})^n \frac{1}{10}}{(\frac{3}{5})^n \frac{1}{10} + (\frac{1}{2})^n \frac{9}{10}} \\ &= \frac{(\frac{3}{5})^n}{(\frac{3}{5})^n + 9(\frac{1}{2})^n}. \end{aligned}$$

In particular, $P(B|D_1) = \frac{2}{17}$ and $P(B|D_2) = \frac{4}{29}$.

CLICK HERE TO ACCESS THE COMPLETE Solutions

(b)

$$\frac{(\frac{3}{5})^{24}}{(\frac{3}{5})^{24} + 9(\frac{1}{2})^{24}} \approx 0.898$$

while

$$\frac{(\frac{3}{5})^{25}}{(\frac{3}{5})^{25} + 9(\frac{1}{2})^{25}} \approx 0.914,$$

so 25 flips are needed.

(c)

$$P(A_{n+1}|D_n) = \frac{P(D_{n+1})}{P(D_n)} = \frac{P(D_{n+1}|B)P(B) + P(D_{n+1}|F)P(F)}{P(D_n|B)P(B) + P(D_n|F)P(F)}$$
$$= \frac{(\frac{3}{5})^{n+1}\frac{1}{10} + (\frac{1}{2})^{n+1}\frac{9}{10}}{(\frac{3}{5})^{n}\frac{1}{10} + (\frac{1}{2})^{n}\frac{9}{10}}.$$

(d) Intuitively speaking, an unending sequence of tails would push the probability of a biased coin to 1, and hence the probability of the next tails is 3/5. For a rigorous calculation we take the limit of the previous answer:

$$\lim_{n\to\infty} P(A_{n+1}|D_n) = \lim_{n\to\infty} \frac{(\frac{3}{5})^{n+1}\frac{1}{10} + (\frac{1}{2})^{n+1}\frac{9}{10}}{(\frac{3}{5})^{n}\frac{1}{10} + (\frac{1}{2})^{n}\frac{9}{10}} = \lim_{n\to\infty} \frac{\frac{3}{5} + \frac{9}{2}(\frac{5}{6})^{n+1}}{1 + 9(\frac{5}{6})^{n}} = \frac{3}{5}.$$

2.72. The sample space for n trials is the same, regardless of the probabilities, namely the space of ordered n-tuples of zeros and ones:

$$\Omega = \{\omega = (s_1, \dots, s_n) : \text{each } s_i \text{ equals 0 or 1}\}.$$

By independence, the probability of a sample point $\omega = (s_1, \dots, s_n)$ is obtained by multiplying together a factor p_i for each $s_i = 1$ and $1 - p_i$ for each $s_i = 0$. We can express this in a single formula as follows:

$$P\{(s_1, \dots, s_n)\} = \prod_{i=1}^{n} \big(p_i^{s_i}(1 - p_i)^{1-s_i}\big).$$

2.73. Let X be the number of blond customers at the pancake place. The population of the town is 500, and 100 of them are blond. We may assume that the visitors are chosen randomly from the population, which means that we take a sample of size 14 without replacement from the population. X denotes the number of blonds among this sample. This is exactly the setup for the hypergeometric distribution and $X \sim \text{Hypergeom}(500, 100, 14)$. (Because the total population size is $N = 500$, the number of blonds is $N_A = 100$ and we take a sample of $n = 14$.) We can now use the probability mass function of the hypergeometric distribution to answer the two questions.

(a)

$$P(\text{exactly 10 blonds}) = P(X = 10) = \frac{\binom{100}{10}\binom{400}{4}}{\binom{500}{14}} \approx 0.00003122.$$

(b)

$$P(\text{at most 2 blonds}) = P(X \le 2) = \sum_{k=0}^{2} P(X = k) = \sum_{k=0}^{2} \frac{\binom{100}{k}\binom{400}{14-k}}{\binom{500}{14}}$$
$$\approx 0.4458.$$

CLICK HERE TO ACCESS THE COMPLETE Solutions

2.74. Define events: $D = \{\text{Steve is a drug user}\}$, $A_1 = \{\text{Steve fails the first drug test}\}$ and $A_2 = \{\text{Steve fails the second drug test}\}$. Assume that Steve is no more or less likely to be a drug user than a random person from the company, so $P(D) = 0.01$. The data about the reliability of the tests tells us that $P(A_i|D) = 0.99$ and $P(A_i|D^c) = 0.02$ for $i = 1, 2$, and conditional independence $P(A_1A_2|D) = P(A_1|D)P(A_2|D)$ and also the same under conditioning on D^c.

(a)

$$P(D|A_1) = \frac{P(A_1|D)P(D)}{P(A_1|D)P(D) + P(A_1|D^c)P(D^c)} = \frac{\frac{99}{100}\cdot\frac{1}{100}}{\frac{99}{100}\cdot\frac{1}{100} + \frac{2}{100}\cdot\frac{99}{100}} = \frac{1}{3}$$

(b)

$$\begin{aligned} P(A_2|A_1) &= \frac{P(A_1A_2)}{P(A_1)} = \frac{P(A_1A_2|D)P(D) + P(A_1A_2|D^c)P(D^c)}{P(A_1|D)P(D) + P(A_1|D^c)P(D^c)} \\ &= \frac{\left(\frac{99}{100}\right)^2\cdot\frac{1}{100} + \left(\frac{2}{100}\right)^2\cdot\frac{99}{100}}{\frac{99}{100}\cdot\frac{1}{100} + \frac{2}{100}\cdot\frac{99}{100}} = \frac{103}{300} \approx 0.3433. \end{aligned}$$

(c)

$$\begin{aligned} P(D|A_1A_2) &= \frac{P(A_1A_2|D)P(D)}{P(A_1A_2|D)P(D) + P(A_1A_2|D^c)P(D^c)} \\ &= \frac{\left(\frac{99}{100}\right)^2\cdot\frac{1}{100}}{\left(\frac{99}{100}\right)^2\cdot\frac{1}{100} + \left(\frac{2}{100}\right)^2\cdot\frac{99}{100}} = \frac{99}{103} \approx 0.9612. \end{aligned}$$

2.75. We introduce the following events:

$$\begin{aligned} A &= \{\text{the store gets its phones from factory II}\}, \\ B_i &= \{\text{the } i\text{th phone is defective}\}, \quad i = 1, 2. \end{aligned}$$

Then A^c is the event that the phone is from factory I. We know that

$$P(A) = 0.4 = \frac{2}{5}, \quad P(A^c) = 0.6 = \frac{3}{5}, \quad P(B_i|A) = 0.2 = \frac{1}{5}, \quad P(B_i|A^c) = 0.1 = \frac{1}{10}.$$

We need to compute $P(A|B_1B_2)$. By Bayes' theorem,

$$P(A|B_1B_2) = \frac{P(B_1B_2|A)\cdot P(A)}{P(B_1B_2|A)P(A) + P(B_1B_2|A^c)P(A^c)}.$$

We may assume that conditionally on A the events B_1 and B_2 are independent. This means that given that the store gets its phones from factory II, the defectiveness of the phones stocked there are independent. We may also assume that conditionally on A^c the events B_1 and B_2 are independent. Then

$$P(B_1B_2|A) = P(B_1|A)P(B_2|A) = (\tfrac{1}{5})^2, \quad P(B_1B_2|A^c) = P(B_1|A^c)P(B_2|A^c) = (\tfrac{1}{10})^2$$

and

$$P(A|B_1B_2) = \frac{(\frac{1}{5})^2\cdot\frac{2}{5}}{(\frac{1}{5})^2\cdot\frac{2}{5} + (\frac{1}{10})^2\cdot\frac{3}{5}} = \frac{8}{11} \approx 0.7273.$$