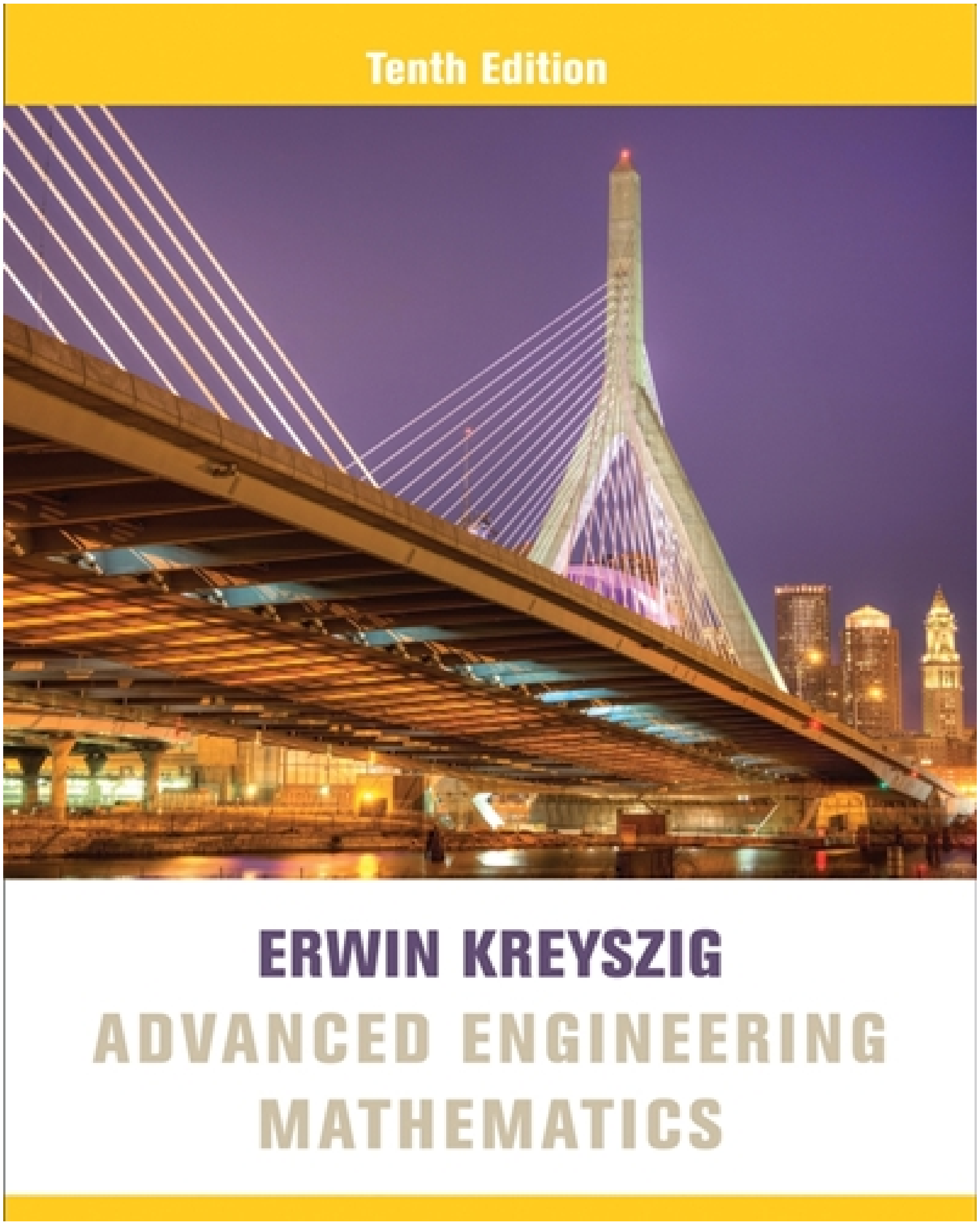
Solutions for Advanced Engineering Mathematics 10th Edition by Kreyszig



Solutions

CHAPTER 2 Second-Order Linear ODEs

Major Changes

Among linear ODEs those of second order are by far the most important ones from the viewpoint of applications, and from a theoretical standpoint they illustrate the theory of linear ODEs of any order (except for the role of the Wronskian). For these reasons we consider linear ODEs of third and higher order in a relatively short separate chapter, Chap. 3.

Section 2.2 combines all three cases of the roots of the characteristic equation governing homogeneous linear ODEs with constant coefficients. (In some of the previous editions the complex case was discussed in a separate section, which seems of no great advantage to the student.)

Section 2.3 is a short introduction to differential operators.

Modeling begins in Sec. 2.4 with the mass–spring system, which is now derived more simply than before and in a better logical order.

After a discussion of the Euler–Cauchy equation and its application to electric fields between concentric spheres in Sec. 2.6, we discuss in Sec. 2.7 the existence and uniqueness of the solution of IVPs involving the homogeneous linear ODE of second order.

This is the end of discussing homogeneous ODEs. It is followed in Sec. 2.7 by the method of undetermined coefficients for nonhomogeneous ODEs, which is basic in applications since it is simpler than the general method (variation of parameters, Sec. 2.10) and covers many, if not most of the standard engineering applications.

Modeling of forced mechanical oscillations is discussed in Sec. 2.8, and electric *RLC*-circuits in Sec. 2.9. Note that we have placed the *RL*-circuit, governed by a first-order ODE into Sec. 1.5, which the student may perhaps wish to review. This was a request by various users of the book, as a stepping stone that may lessen difficulties and simplify the derivation of the model from physics.

SECTION 2.1. Homogeneous Linear ODEs of Second-Order, page 46

Purpose. To extend the basic concepts from first-order to second-order ODEs and to present the basic properties of linear ODEs.

Comment on the Standard Form (1)

The form (1), with 1 as the coefficient of y'', is practical, because if one starts from

$$f(x)y'' + g(x)y' + h(x)y = \tilde{r}(x),$$

one usually considers the equation in an interval I in which $f(x)$ is nowhere zero, so that in I one can divide by $f(x)$ and obtain an equation of the form (1). Points at which $f(x) = 0$ require a special study, which we present in Chap. 5.

Main Content, Important Concepts

Linear and nonlinear ODEs

Homogeneous linear ODEs (to be discussed in Secs. 2.1–2.6)

Superposition principle for homogeneous ODEs

General solution, basis, linear independence

Initial value problem (2), (4), particular solution

Reduction to first order (text and Probs. 3–10)

Comment on the Three ODEs after (2)

These are for illustration, not for solution, but should a student ask, answers are that the first will be solved by methods in Secs. 2.7 and 2.10, the second is a Bessel equation (Sec. 5.5) and the third has the solutions $\pm\sqrt{c_1 x + c_2}$ with any c_1 and c_2.

Comment on Footnote 1

In 1760, Lagrange gave the first methodical treatment of the calculus of variations. The book mentioned in the footnote includes all major contributions of others in the field and made him the founder of analytical mechanics.

Examples in the Text. The examples show the following.

Example 1 shows the superposition of solutions of the homogeneous linear ODE.

Examples 2 and 3 are counter-examples to the superposition for a nonhomogeneous linear ODE and a nonlinear ODE.

Example 4 is an initial value problem, suggesting the concepts of a general solution, a particular solution, and a basis.

Examples 5 and 6 give further illustrations of those concepts.

Example 7 shows the reduction of order of

$$y'' + p(x)y' + q(x)y = 0$$

using a known solution y_1, followed by the derivation of a general formula for a second solution

$$y_2 = y_1 \int \frac{1}{{y_1}^2} e^{-\int p\,dx}\,dx.$$

Hence solving the ODE for finding a second solution is reduced to two integrations, and the student should understand that this is a simpler task.

Comment on Terminology

p and q are called the **coefficients** of (1) and (2). The function r on the right is *not* called a coefficient, to avoid the misunderstanding that r must be constant when we talk about an ODE *with constant coefficients*.

SOLUTIONS TO PROBLEM SET 2.1, page 53

2. $y'' = \dfrac{dy'}{dx} = \dfrac{dy'}{dy}\dfrac{dy}{dx} = \dfrac{dz}{dy} z$

4. $z = y'$, $2xz' = 3z$. Separation of variables and integration gives

$$\frac{dz}{z} = \frac{3}{2x}\,dx, \qquad \ln|z| = \tfrac{3}{2}\ln|x| + \tilde{c}, \qquad z = cx^{3/2}.$$

Integrating once more, we have

$$y = \int z\,dx = c_1 x^{5/2} + c_2.$$

6. The formula in the text was derived under the assumption that the ODE is in standard form; in the present case,

$$y'' + \frac{2}{x}y' + y = 0.$$

Hence $p = 2/x$, so that $e^{-\int p\,dx} = x^{-2}$. It follows from (9) in the text that

$$U = \frac{x^2}{\cos^2 x} \cdot \frac{1}{x^2} = \frac{1}{\cos^2 x}.$$

The integral of U is $\tan x$; we need no constants of integration because we merely want to obtain a particular solution. The *answer* is

$$y_2 = y_1 \tan x = \frac{\sin x}{x}.$$

8. $z' = 1 + z^2$, $dz/(1 + z^2) = dx$, $\arctan z = x + c_1$, $z = \tan(x + c_1)$, $y = -\ln|\cos(x + c_1)| + c_2$

This is an obvious use of problems from Chap. 1 in setting up problems for this section. The only difficulty may be an unpleasant additional integration.

10. $z = \dfrac{dy}{dx}$, $\dfrac{dz}{dy}z + \left(1 + \dfrac{1}{y}\right)z^2 = 0$, divide by z, separate variables, and integrate:

$$\frac{dz}{z} = -\left(1 + \frac{1}{y}\right)dy, \qquad \ln|z| = -y - \ln|y| + \tilde{c}.$$

Take exponentials, separate again, and integrate:

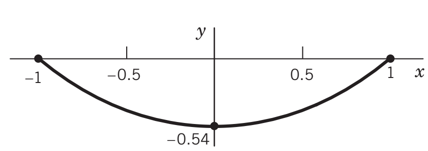
$$\frac{dy}{dx} = z = \frac{c}{y}e^{-y}, \qquad ye^y\,dy = c\,dx, \qquad \int ye^y\,dy = cx + c_2.$$

Evaluation of the integral gives the *answer* $(y - 1)e^y = c_1x + c_2$.

12. $z' = (1 + z^2)^{1/2}$, $(1 + z^2)^{-1/2}\,dz = dx$, $\text{arcsinh}\, z = x + c_1$. From this we have $z = \sinh(x + c_1)$, $y = \cosh(x + c_1) + c_2$. From the boundary conditions $y(1) = 0$, $y(-1) = 0$ we get

$$\cosh(1 + c_1) + c_2 = 0 = \cosh(-1 + c_1) + c_2.$$

Hence $c_1 = 0$ and then $c_2 = -\cosh 1$. The *answer* is (see the figure) $y = \cosh x - \cosh 1$.



Section 2.1. Problem 12

14. $y'' = 1/y'$, $y''y' = 1$, $\dfrac{dz}{dy}z^2 = 1$. Integration with respect to y gives

$$\frac{z^3}{3} = y + c.$$

Hence $z^3 = 3y + \tilde{c}$ and thus

$$\frac{dy}{dt} = z = (3y + \tilde{c})^{1/3}.$$

By separation of variables,

$$\frac{dy}{(3y + \tilde{c})^{1/3}} = dt.$$

By integration,

$$\tfrac{1}{2}(3y + \tilde{c})^{2/3} = t + \tilde{\tilde{c}}.$$

Hence

$$(3y + \tilde{c})^{2/3} = 2t + c_2$$

and thus

$$3y + \tilde{c}_1 = (2t + c_2)^{3/2}.$$

The answer is

$$y = \tfrac{1}{3}(2t + c_2)^{3/2} + c_1.$$

16. $y = (2.2 + 0.8x)e^{-0.3x}$

18. $y = 4.3x - 3.8x \ln x$

SECTION 2.2. Homogeneous Linear ODEs with Constant Coefficients, page 53

Purpose. To show that homogeneous linear ODEs with constant coefficients can be solved by algebra, namely, by solving the quadratic characteristics equation (3). The roots may be:

(Case I) Real distinct roots

(Case II) A real double root ("Critical case")

(Case III) Complex conjugate roots

In Case III the roots are conjugate because the coefficients of the ODE, and thus of (3), are real, a fact the student should remember.

To help poorer students, we have shifted the derivation of the real form of the solutions in Case III to the end of the section, but the verification of these real solutions is done immediately when they are introduced. This will also help to a better understanding.

The student should become aware of the fact that Case III includes both undamped (harmonic) oscillations (if $c = 0$) and damped oscillations.

Also it should be emphasized that in the transition from the complex to the real form of the solutions we use the superposition principle.

Furthermore, one should emphasize the general importance of the Euler formula (11), which we shall use on various occasions.

Examples in the Text. The examples show the following.

Examples 1 and 2 concern Case I, the case of distinct real roots. In this case, as well as in the other two cases, an initial value problem requires the solution of a system of two linear equations in two unknowns, whose values are determined by the two initial conditions. A typical solution in Case I is shown in Fig. 30.

Examples 3 and 4 concern Case II, the case of a real double root, which is the limiting case between Cases I and III. Figure 31 shows a typical solution, having a real root at $x = 1.5$, which is the solution of $3 - 2x = 0$, where $3 - 2x$ is a factor in the solution of the IVP in Example 4.

Example 5 concerns Case III, in which one obtains solutions (9), representing oscillations. These may be damped as in Fig. 32, or of increasing maximum amplitude if $a > 0$, or of constant maximum amplitude if $a = 0$, as in Example 6, giving a harmonic oscillations.

Comment on How to Avoid Working in Complex

The average engineering student will profit from working a little with complex numbers. However, if one has reasons for avoiding complex numbers here, one may apply the method of eliminating the first derivative from the equation, that is, substituting $y = uv$ and determining v so that the equation for u does not contain u'. For v this gives

$$2v' + av = 0. \qquad \text{A solution is} \qquad v = e^{-ax/2}.$$

With this v, the equation for u takes the form

$$u'' + (b - \tfrac{1}{4}a^2)u = 0$$

and can be solved by remembering from calculus that $\cos \omega x$ and $\sin \omega x$ reproduce under two differentiations, multiplied by $-\omega^2$. This gives (9), where

$$\omega = \sqrt{b - \tfrac{1}{4}a^2}.$$

Of course, ***the present approach can be used to handle all three cases***. In particular, $u'' = 0$ in Case II gives $u = c_1 + c_2 x$ at once.

SOLUTIONS TO PROBLEM SET 2.2, page 59

2. $y = c_1 \cos 6x + c_2 \sin 6x$

4. $y = e^{-2x}(c_1 \cos \pi x + c_2 \sin \pi x)$

6. $y = (c_1 + c_2 x)e^{1.6x}$

8. $y = e^{-x/2}(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x))$

10. $y = e^{-1.2x}(c_1 \cos(1.4\pi x) + c_2 \sin(1.4\pi x))$

12. $y = c_1 e^{-5x} + c_2 e^{-4x}$

14. $y = (c_1 + c_2 x)e^{-k^2 x}$

16. $y'' + 1.7y' - 11.18y = 0$

18. $y'' + 4\pi^2 y = 0$

20. $y'' + 6.2y' + 14.02y = 0$

22. A general solution is

$$y(x) = e^{-2x}(c_1 \cos \pi x + c_2 \sin \pi x).$$

The first initial condition gives $y(\frac{1}{2}) = e^{-1}(0 + c_2) = 1$, hence $c_2 = e$. The derivative is

$$y'(x) = e^{-2x}(-2c_1 \cos \pi x - 2c_2 \sin \pi x - c_1\pi \sin \pi x + c_2\pi \cos \pi x).$$

From this and the second initial condition we obtain

$$y'(\tfrac{1}{2}) = e^{-1}(-2c_2 - c_1\pi) = e^{-1}(-2e - c_1\pi) = -2 - c_1 e^{-1}\pi = -2.$$

Hence $c_1 = 0$. This gives the answer

$$y = e^{-2x+1} \sin \pi x.$$

Notice that it depends on the initial condition whether both solutions of a basis appear in the particular solution or just one; this is worthwhile pointing out to the students.

24. A general solution is

$$y = c_1 e^{-x/2} + c_2 e^{3x/2}.$$

The first initial condition gives

$$y(-2) = c_1 e + c_2 e^{-3} = e.$$

The derivative is

$$y' = -\tfrac{1}{2}c_1 e^{-x/2} + \tfrac{3}{2}c_2 e^{3x/2}.$$

From this and the second initial condition we have

$$y'(-2) = -\tfrac{1}{2}c_1 e + \tfrac{3}{2}c_2 e^{-3} = -e/2.$$

Division by e and by $-\frac{1}{2}$, respectively, gives

$$c_1 + c_2 e^{-4} = 1$$
$$c_1 - 3c_2 e^{-4} = 1.$$

The solution of this system is $c_2 = 0$, $c_1 = 1$. Hence the answer is

$$y = e^{-x/2}.$$

26. A general solution is

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

From this and the first initial condition we have

$$c_1 + c_2 = 1.$$

The derivative is

$$y' = c_1 k e^{kx} - c_2 k e^{-kx}.$$

From this and the second initial condition we obtain

$$c_1 - c_2 = 1/k.$$

The solution of this system is

$$c_1 = \frac{k+1}{2k}, \qquad c_2 = \frac{k-1}{2k}.$$

Hence the answer (the particular solution of the IVP) is

$$y = [(k+1)e^{kx} + (k-1)e^{-kx}]/(2k).$$

28. A general solution is

$$y = c_1 e^{-x/4} + c_2 e^{x/2}.$$

From this and the first initial condition we have

$$y(0) = c_1 + c_2 = -0.2.$$

The derivative is

$$y' = \tfrac{1}{4} c_1 e^{-x/4} + \tfrac{1}{2} c_2 e^{x/2}.$$

From this and the second initial condition we have

$$y'(0) = -\tfrac{1}{4} c_1 + \tfrac{1}{2} c_2 = -0.325.$$

The solution of this system is $c_1 = 0.3$, $c_2 = -0.5$. Hence the particular solution satisfying the initial condition is

$$y = 0.3e^{-x/4} - 0.5e^{x/2}.$$

30. A general solution is

$$y = (c_1 + c_2 x)e^{5x/3}.$$

This is a case of a double root of the characteristic equation. The first initial condition yields $y(0) = c_1 = 3.3$. By differentiation,

$$y' = [c_2 + \tfrac{5}{3}(c_1 + c_2 x)]e^{5x/3}.$$

From this and the second initial condition we obtain

$$y'(0) = c_2 + \tfrac{5}{3}c_1 = c_2 + 5.5 = 10.$$

Hence $c_2 = 4.5$, so that the solution of the IVP is

$$y = (3.3 + 4.5x)e^{5x/3}.$$

32. Independent if $a \neq 0$

34. Dependent since $\ln(x^3) = 3 \ln x$

36. If one of the functions is identically zero, the set is linearly dependent because $c_1 f_1(x) + c_2 \cdot 0 = 0$ holds with any $c_2 \neq 0$ (and $c_1 = 0$).

The intervals given in the problems are just a reminder that linear independence or independence always refers to some interval. In the present case we could choose as the interval the real axis, or the positive half-axis if a logarithm is involved.

38. **Team Project. (a)** We obtain

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 + a\lambda + b = 0.$$

Comparison of coefficients gives $a = -(\lambda_1 + \lambda_2)$, $b = \lambda_1\lambda_2$.
(b) $y'' + ay' = 0$. (i) $y = c_1 e^{-ax} + c_2 e^{0x} = c_1 e^{-ax} + c_2$. (ii) $z' + az = 0$, where $z = y'$, $z = ce^{-ax}$ and the second term comes in by integration:

$$y = \int z\, dx = \tilde{c}_1 e^{-ax} + \tilde{c}_2.$$

(d) $e^{(k+m)x}$ and e^{kx} satisfy $y'' - (2k + m)y' + k(k + m)y = 0$, by the coefficient formulas in part (a). By the superposition principle, another solution is

$$\frac{e^{(k+m)x} - e^{kx}}{m}.$$

We now let $m \to 0$. This becomes $0/0$, and by l'Hôpital's rule (differentiation of numerator and denominator separately with respect to m, not x!) we obtain

$$xe^{kx}/1 = xe^{kx}.$$

The ODE becomes $y'' - 2ky' + k^2 y = 0$. The characteristic equation is

$$\lambda^2 - 2k\lambda + k^2 = (\lambda - k)^2 = 0$$

and has a double root. Since $a = -2k$, we get $k = -a/2$, as expected.

SECTION 2.3. Differential Operators. *Optional,* page 60

Purpose. To take a short look at the operational calculus of second-order differential operators with constant coefficients. This parallels and confirms our discussion of ODEs with constant coefficients.

A discussion of the case of *variable* coefficients would exceed the level and the area of interest of the book, sidetrack the attention of the student, and give no substantial additional insights that might be helpful to our further work.

SOLUTIONS TO PROBLEM SET 2.3, page 61

2. The first function gives

$$6x + 3 - 9x^2 - 9x = -9x^2 - 3x + 3.$$

The second function gives

$$9e^{3x} - 9e^{3x} = 0.$$

The third function gives

$$-4 \sin 4x - 4 \cos 4x - 3 \cos 4x + 3 \sin 4x = -7 \cos 4x - \sin 4x.$$

4. For the first function,

$$\begin{aligned}&(D + 6I)(6 + 6 \cos 6x + 36x + 6 \sin 6x)\\ &\quad = -36 \sin x + 36 + 36 \cos 6x + 36 + 36 \cos 6x + 216x + 36 \sin 6x\\ &\quad = 72 + 72 \cos 6x + 216x.\end{aligned}$$

For the second function,

$$(D + 6I)(e^{-6x} - 6xe^{-6x} + 6xe^{-6x}) = -6e^{-6x} + 6e^{-6x} = 0.$$

6. $(D + 2.8I)(D + 1.2I), \quad y = c_1e^{-2.8x} + c_2e^{-1.2x}$

8. $(D - \sqrt{3}iI)(D + \sqrt{3}iI), \quad y = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x$

10. $(D + 2.4I)^2, \quad y = (c_1 + c_2x)e^{-2.4x}$

12. $[D + (1.5 + 0.5i)I][D + (1.5 - 0.5i)I], \quad y = e^{-1.5x}(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x)$

14. y is a solution, as follows from the superposition principle in Sec. 2.1 since the ODE is homogeneous and linear. In applying l'Hopital's rule, regard y as a function of μ, the variable that approaches the limit, whereas λ is fixed. Differentiation of the numerator with respect to μ gives $xe^{\mu x} - 0$ and differentiation of the denominator gives 1. The limit of this is $xe^{\lambda x}$.

SECTION 2.4. Modeling of Free Oscillations of a Mass–Spring System, page 62

Purpose. To present a main application of second-order constant-coefficient ODEs

$$my'' + cy' + ky = 0$$

resulting as models of motions of a mass m on an elastic spring of modulus k (>0) under linear damping c ($\geqq 0$) applying Newton's second law and Hooke's law. These are free motions (no driving force). Forced motions follow in Sec. 2.8.

This system should be regarded as a basic building block of more complicated systems, a prototype of a vibrating system that shows the essential features of more sophisticated systems as they occur in various forms and for various purposes in engineering.

The quantitative agreement between experiments of the physical system and its mathematical model is surprising. Indeed, the student should not miss performing experiments if there is an opportunity, as I had as a student of Prof. Blaess, the inventor of a (now obscure) graphical method for solving ODEs.

Main Content, Important Concepts

Restoring force ky, damping force cy', force of inertia my''

No damping, harmonic oscillations (4), natural frequency $\omega_0/(2\pi)$

Overdamping, critical damping, nonoscillatory motions (7), (8)

Underdamping, damped oscillations (10)

In the text, the derivation of the model has been simplified by clarifying the role of the force $-F_0$, which has no effect on the motion.

The model, like many others, is obtained from Newton's second law.

We discuss the undamped case $c = 0$ and the damped case $c > 0$ separately because the types of motion are basically different, as follows. The undamped case $c = 0$ gives a harmonic motion (4) for an infinite time interval (practically: for a long time). The damped case $c > 0$ gives a damped motion, which is either oscillatory or, if c is large enough, is a nonoscillatory approach to zero.

Hence it is interesting that the formal distinction of Cases I–III mechanically corresponds to quite different types of motion.

No damping ($c = 0$) means no loss of the energy corresponding to the initial displacement and initial velocity.

Make sure that the student understands the physics behind (4*), that shows the phase shift.

Examples in the Text. The examples illustrate the following.

Example 1 discusses the undamped case $c = 0$.

Example 2 compares the three cases, the three types of motion, graphically shown in Fig. 40, namely, Case I giving a rapid approach to zero, Case II looking almost the same, also showing a rapid and monotone approach to zero, and, finally, Case III a damped oscillation, of a frequency smaller than that of the harmonic oscillation when $c = 0$.

Problem Set 2.4. Problems 1–6 enhance the physical understanding and insight into the basic properties of the undamped model.

Team Project 10 shows that the model in the text is in fact the prototype of various physical systems governed by the same mathematical formulas.

Problems 11–19 play a role for the damped case similar to that of Probs. 1–9 for the undamped model.

CAS Project 20 shows the "continuity" in the transitions between Cases I–III, that is also illustrated in Fig. 47.

SOLUTIONS TO PROBLEM SET 2.4, page 69

2. $W = 20$ and $s_0 = 2$ gives $k = W/s_0 = 10$ by Hooke's law. Thus

$$f = \frac{\omega_0}{2\pi} = \frac{\sqrt{k/m}}{2\pi} = \frac{\sqrt{k/(W/g)}}{2\pi} = \frac{\sqrt{10/(20/980)}}{2\pi} = 3.52 \text{ [Hz]}.$$

From this we get the period $1/f = 0.284$ [sec].

4. No because the frequency depends only on k/m, not on initial conditions.

6. By Hooke's law, $F_1 = k_1 = 8$ stretches spring S_1 by 8, and $F_2 = k_2 = 12$ stretches spring S_2 by 12. Hence the unknown k of the combination of the springs stretches S_1 by $k/k_1 = k/8$ and S_2 by $k/k_2 = k/12$. And k is such that the sum of these stretches equals 1, because k is the force that corresponds to the stretch 1 of the combination. Thus

$$\frac{k}{k_1} + \frac{k}{k_2} = 1, \qquad \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k}. \qquad \textit{Answer: } k = 4.8.$$

8. $my'' = -\pi \cdot 0.3^2 y\gamma$, where $\pi \cdot 0.3^2 y$ is the volume of water displaced when the buoy is depressed y meters from its equilibrium position, and $\gamma = 9800$ nt is the weight of water per cubic meter. Thus $y'' + \omega_0{}^2 y = 0$, where $\omega_0{}^2 = \pi \cdot 0.3^2\gamma/m$ and the period is $2\pi/\omega_0 = 2$; hence

$$m = \pi \cdot 0.3^2\gamma/\omega_0^2 = 0.3^2\gamma/\pi = 280.7$$

$$W = mg = 280.7 \cdot 9.80 = 2750.86 \text{ [nt]} \quad \text{(about 620 lb)}.$$

10. Team Project. (a) By Prob. 7 the frequency is

$$\frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.80}{1}} = 0.498,$$

so it takes about 2 sec to complete 1 cycle. *Answer:* It ticks about 30 times per minute.

(b) $W = ks_0 = 8$. Now $s_0 = 1$ because the system has its equilibrium position 1 cm below the horizontal line. Also, $m = W/g$, so that

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{W/s_0}{W/g}} = \sqrt{g} = \sqrt{980} = 31.3,$$

and we get the general solution

$$y = A \cos 31.3t + B \sin 31.3t.$$

The initial conditions give $y(0) = A = 0$ and $y'(0) = 31.3B = 10$. Hence $B = 0.319$ and the *answer* is

$$y = 0.319 \sin 31.3t \text{ [cm]}.$$

(c) $\theta(t) = 0.5235 \cos 3.7t + 0.0943 \sin 3.7t$ [rad]

12. $y = 0$ gives $c_1 = -c_2 e^{-2\beta t}$, which has at most one solution because the exponential function is monotone.

14. Case (II) of (5) with $c = \sqrt{4mk} = \sqrt{4 \cdot 500 \cdot 4500} = 3000$ [kg/sec], where 500 kg is the mass per wheel.

16. $2\pi/\omega^*$ since Eq. (10) and $y' = 0$ give $\tan(\omega^* t - \delta) = -\alpha/\omega^*$; tan is periodic with period π/ω^*.

18. If an extremum is at t_0, the next one is at $t_1 = t_0 + \pi/\omega^*$, by Prob. 16. Since the cosine and sine in (10) have period $2\pi/\omega^*$, the amplitude ratio is

$$\exp(-\alpha t_0)/\exp(-\alpha t_1) = \exp(-\alpha(t_0 - t_1)) = \exp(\alpha\pi/\omega^*).$$

The natural logarithm is $\alpha\pi/\omega^*$, and maxima alternate with minima. Hence $\Delta = 2\pi\alpha/\omega^*$ follows.

For the ODE, $\Delta = 2\pi \cdot 1/(\frac{1}{2}\sqrt{4 \cdot 5 - 2^2}) = \pi$.

20. CAS Project. (a) The three cases appear, along with their typical solution curves, regardless of the numeric values of k/m, $y(0)$, etc.

(b) The first step is to see that Case II corresponds to $c = 2$. Then we can choose other values of c by experimentation. In Fig. 47 the values of c (omitted on purpose; the student should choose!) are 0 and 0.1 for the oscillating curves, 1, 1.5, 2, 3 for the others (from below to above).

(c) This addresses a general issue arising in various problems involving heating, cooling, mixing, electrical vibrations, and the like. One is generally surprised how quickly certain states are reached whereas the theoretical time is infinite.

(d) General **sol**ution $y(t) = e^{-ct/2}(A \cos \omega^* t + B \sin \omega^* t)$, where $\omega^* = \frac{1}{2}\sqrt{4 - c^2}$. The first initial condition $y(0) = 1$ gives $A = 1$. For the second initial condition we need the derivative (we can set $A = 1$)

$$y'(t) = e^{-ct/2}\left(-\frac{c}{2}\cos \omega^* t - \frac{c}{2} B \sin \omega^* t - \omega^* \sin \omega^* t + \omega^* B \cos \omega^* t\right).$$

From this we obtain $y'(0) = -c/2 + \omega^* B = 0$, $B = c/(2\omega^*) = c/\sqrt{4 - c^2}$. Hence the particular solution (with c still arbitrary, $0 < c < 2$) is

$$y(t) = e^{-ct/2}\left(\cos \omega^* t + \frac{c}{\sqrt{4 - c^2}} \sin \omega^* t\right).$$

It derivative is, since the cosine terms drop out,

$$y'(t) = e^{-ct/2}(-\sin \omega^* t)\left(\frac{c^2}{2\sqrt{4 - c^2}} + \frac{1}{2}\sqrt{4 - c^2}\right)$$

$$= \frac{-2}{\sqrt{4 - c^2}} e^{-ct/2} \sin \omega^* t.$$

The tangent of the y-curve is horizontal when $y' = 0$, for the first positive time when $\omega^* t = \pi$, thus $t = t_2 = \pi/\omega^* = 2\pi/\sqrt{4 - c^2}$. Now the y-curve oscillates between $\pm e^{-ct/2}$, and (11) is satisfied if $e^{-ct/2}$ does not exceed 0.001. Thus $ct = 2 \ln 1000$, and $t = t_2$ gives the best c satisfying (11). Hence

$$c = \frac{2 \ln 1000}{t_2}, \qquad c^2 = \frac{(\ln 1000)^2}{\pi^2}(4 - c^2).$$

The solution of this is $c = 1.821$, approximately. For this c we get by substitution $\omega^* = 0.4141$, $t_2 = 7.587$, and the particular solution

$$y(t) = e^{-0.9103t}(\cos 0.4139t \; + \; 2.199 \sin 0.4139t)$$

The graph shows a positive maximum near 15, a negative minimum near 23, a positive maximum near 30, and another negative minimum at 38.

(e) The main difference is that Case II gives

$$y = (1 - t)e^{-t}$$

which is negative for $t > 1$. The experiments with the curves are as before in this project.

SECTION 2.5. Euler–Cauchy Equations, page 71

Purpose. Algebraic solution of the Euler–Cauchy equation, which appears in certain applications (see our Example 4) and which we shall need again in Sec. 5.4 as the simplest equation to which the Frobenius method applies. We have three cases; this is similar to the situation for constant-coefficient equations, to which the Euler–Cauchy equation can be transformed (Team Project 20(d)); however, this fact is of theoretical rather than of practical interest.

Comment on Footnote 4

Euler worked in St. Petersburg 1727–1741 and 1766–1783 and in Berlin 1741–1766. He investigated Euler's constant (See. 5.6) first in 1734, used Euler's formula (Secs. 2.2, 13.5, 13.6) beginning in 1740, introduced integrating factors (Sec. 1.4) in 1764, and studied conformal mappings (Chap. 17) starting in 1770. His main influence on the development of mathematics and mathematical physics resulted from his textbooks, in particular from his famous *Introductio in analysin infinitorum* (1748), in which he also introduced many of the modern notations (for trigonometric functions, etc.). Euler was the central figure of the mathematical activity of the 18th century. His Collected Works are still incomplete, although some seventy volumes have already been published.

Cauchy worked in Paris, except during 1830–1838, when he was in Turin and Prague. In his two fundamental works, *Cours d'Analyse* (1821) and *Résumé des leçons données à l'École royale polytechnique* (vol. 1, 1823), he introduced more rigorous methods in calculus, based on an exactly defined limit concept; this also includes his convergence principle (Sec. 15.1). Cauchy also was the first to give existence proofs in ODEs. He initiated complex analysis; we discuss his main contributions to this field in Secs. 13.4, 14.2–14.4, and 15.2, His famous integral theorem (Sec. 14.2) was published in 1825 and his paper on complex power series and their radius of convergence (Sec. 15.2), in 1831.

Examples in the Text. The examples illustrate the following.

Examples 1–3 and Fig. 48 illustrate Cases I–III, respectively. In particular, Example 3 shows the derivation of real solutions from complex ones.

Example 4 shows the occurrence of an Euler–Cauchy equation in connection with the electric potential field between concentric spheres kept at different constant potentials. Here, the student may wish to find a solution formula for arbitrary r_1, r_2 and potentials v_1, v_2.

SOLUTIONS TO PROBLEM SET 2.5, page 73

2. $y = c_1x^5 + c_2x^{-4}$

4. $y = c_1 + c_2/x$

6. $y = c_1x^{0.5} + c_2x^{-0.2}$

8. $y = (c_1 + c_2 \ln x)x^2$

10. $y = x(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$

12. $y = 1.2x^2 - 0.8x^3$

14. $y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)$ is a general solution, and from the initial conditions we obtain the answer

$$y = \tfrac{5}{6} \sin(3 \ln x)$$

because $y(1) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$ and

$$\begin{aligned} y'(x) &= c_1(-\sin(3 \ln x) \cdot \frac{3}{x} + c_2 \cos(3 \ln x) \cdot \frac{3}{x} \\ &= c_1 \cdot 0 \; + \; c_2 \cdot 3 = 2.5, \end{aligned}$$

so that $c_2 = \tfrac{5}{6}$.

16. A general solution is

$$y(x) = (c_1 + c_2 \ln x)x^2,$$

so that $y(1) = c_1 = -\pi$. The derivative is

$$y'(x) = \frac{c_2}{x} x^2 + (c_1 + c_2 \ln x) \cdot 2x,$$

so that $y'(1) = c_2 + 2c_1 = c_2 - 2\pi = 2\pi$, hence $c_2 = 4\pi$. This gives the answer

$$y = (-\pi + 4\pi \ln x)x^2.$$

18. The auxiliary equation is

$$\begin{aligned} &9(m(m-1) + \tfrac{1}{3}m + \tfrac{1}{9}) \\ &= 9(m^2 - \tfrac{2}{3}m + \tfrac{1}{9}) \\ &= 9(m - \tfrac{1}{3})^2 \\ &= 0 \end{aligned}$$

has the double root $\tfrac{1}{3}$. Hence a general solution is

$$y(x) = (c_1 + c_2 \ln x)x^{1/3}.$$

By the first initial condition, $y(1) = c_1 = 1$. Differentiation gives

$$y'(x) = \frac{c_2}{x} x^{1/3} + (c_1 + c_2 \ln x) \cdot \tfrac{1}{3} x^{-2/3}.$$

The second initial condition thus gives

$$y'(1) = c_2 + c_1 \cdot \tfrac{1}{3} = 0.$$

Hence $c_2 = -\frac{1}{3}$. This yields the answer (the solution of the initial value problem)

$$y = (1 - \tfrac{1}{3}\ln x)\, x^{1/3}.$$

20. Team Project. (a) The student should realize that the present steps are the same as in the general derivation of the method in Sec. 2.1. An advantage of such specific derivations may be that the student gets a somewhat better understanding of the method and feels more comfortable with it. Of course, once a general formula is available, there is no objection to applying it to specific cases, but often a direct derivation may be simpler. In that respect the present situation resembles, for instance, that of the integral solution formula for first-order linear ODEs in Sec. 1.5.
(b) The Euler–Cauchy equation to start from is

$$x^2y'' + (1 - 2m - s)xy' + m(m + s)y = 0$$

where $m = (1 - a)/2$, the exponent of the one solution we first have in the critical case. For $s \to 0$ the ODE becomes

$$x^2y'' + (1 - 2m)xy' + m^2y = 0.$$

Here $1 - 2m = 1 - (1 - a) = a$, and $m^2 = (1 - a)^2/4$, so that this is the Euler–Cauchy equation in the critical case. Now the ODE is homogeneous and linear; hence another solution is

$$Y = (x^{m+s} - x^m)/s.$$

L'Hôpital's rule, applied to Y as a function of s (not x, because the limit process is with respect to s, not x), gives

$$(x^{m+s}\ln|x|)/1 \to x^m \ln|x| \qquad \text{as } s \to 0.$$

This is the expected result.
(c) This is less work than perhaps expected, an exercise in the technique of differentiation (also necessary in other cases). We have $y = x^m \ln x$, and with $(\ln x)' = 1/x$ we get

$$\begin{aligned} y' &= mx^{m-1}\ln|x| + x^{m-1} \\ y'' &= m(m-1)x^{m-2}\ln|x| + mx^{m-2} + (m-1)x^{m-2}. \end{aligned}$$

Since $x^m = x^{(1-\alpha)/2}$ is a solution, in the substitution into the ODE the ln-terms drop out. Two terms from y'' and one from y' remain and give

$$x^2(mx^{m-2} + (m-1)x^{m-2}) + ax^m = x^m(2m - 1 + a) = 0$$

because $2m = 1 - a$.
(d) $t = \ln x$, $dt/dx = 1/x$, $y' = \dot{y}t' = \dot{y}/x$, where the dot denotes the derivative with respect to t. By another differentiation,

$$y'' = (\dot{y}/x)' = \ddot{y}/x^2 + \dot{y}/(-x^2).$$

Substitution of y' and y'' into (1) gives the constant-coefficient ODE

$$\ddot{y} - \dot{y} + a\dot{y} + by = \ddot{y} + (a-1)\dot{y} + by = 0.$$

The corresponding characteristic equation has the roots

$$\lambda = \tfrac{1}{2}(1-a) \pm \sqrt{\tfrac{1}{4}(1-a)^2 - b}.$$

With these λ, solutions are $e^{\lambda t} = (e^t)^\lambda = (e^{\ln|x|})^\lambda = x^\lambda$.
(e) $te^{\lambda t} = (\ln|x|)e^{\lambda \ln|x|} = (\ln|x|)(e^{\ln|x|})^\lambda = x^\lambda \ln|x|$.

SECTION 2.6. Existence and Uniqueness of Solutions. Wronskian, page 74

Purpose. To explain the theory of existence of solutions of ODEs with variable coefficients in standard form (that is, with y'' as the first term, not, say, $f(x)y''$)

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

and of their uniqueness if initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \qquad (2)$$

are imposed. Of course, no such theory was needed in the last sections on ODEs for which we were able to write all solutions explicitly.

Main Content. The theorems show the following.

Theorem 1 shows that the continuity of the coefficients suffices for the existence and uniqueness of a solution of the initial values problem (1), (2).

Theorem 2 gives a criterion for linear dependence and independence involving the Wronskian. Simple basic applications are shown in Examples 1 and 2.

Theorem 3 on the existence of a general solution follows from Theorems 1 and 2 by the trick of using two special initial value problems; this idea is worth remembering.

For Theorem 4 see below.

Comment on Wronskian

For $n = 2$, where linear independence and dependence can be seen immediately, the Wronskian serves primarily as a tool in our proofs; the practical value of the independence criterion will appear for higher n in Chap. 3.

Comment on General Solution

Theorem 4 shows that linear ODEs (actually, of any order) have no singular solutions. This also justifies the term "general solution," on which we commented earlier. We did not pay much attention to singular solutions, which sometimes occur in geometry as envelopes of one-parameter families of straight lines or curves.

Altogether, this provides a general theory that is useful in practice.

SOLUTIONS TO PROBLEM SET 2.6 page 79

2. $W = \begin{vmatrix} e^{4x} & e^{-1.5x} \\ 4e^{4x} & -1.5e^{-1.5x} \end{vmatrix} = e^{2.5x}\begin{vmatrix} 1 & 1 \\ 4 & -1.5 \end{vmatrix} = -5.5e^{2.5x}$

4. $W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\dfrac{2}{x}$

6. We use the abbreviations $c = \cos \omega x$, $s = \sin \omega x$. Then

$$W = \begin{vmatrix} e^{-x}c & e^{-x}s \\ e^{-x}(-c - \omega s) & e^{-x}(-s + \omega c) \end{vmatrix} = e^{-2x} \begin{vmatrix} c & s \\ -c - \omega s & -s + \omega c \end{vmatrix} = \omega e^{-2x}.$$

8. Equation (6*) saves much work and avoids sources of errors. We obtain

$$\left(\frac{y_2}{y_1}\right)' = (\tan(\ln x))' = \frac{1}{\cos^2(\ln x)} \cdot \frac{1}{x},$$

Multiplication by $y_1^2 = x^{2k} \cos^2(\ln x)$ gives $W = x^{2k-1}$.

10. $x^2y'' - (m_1 + m_2 - 1)xy' + m_1m_2y = 0$. For the Wronskian we obtain from (6*)

$$W = -\left(\frac{x^{m_1}}{x^{m_2}}\right)' x^{2m_2} = -(m_1 - m_2)x^{m_1+m_2-1}.$$

From the initial conditions we obtain the particular solution

$$y = 2x^{m_1} - 4x^{m_2}.$$

12. $x^2y'' - 3xy' + 4y = 0$. The Wronskian is

$$W = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3.$$

The solution of the initial value problem is

$$y = (4 - 2 \ln x)x^2.$$

14. $y_1 = e^{-kx} \cos \pi x$, $y_2 = e^{-kx} \sin \pi x$. By (6*),

$$\begin{aligned} W = -(y_2/y_1)'y_1^2 &= -(\tan \pi x)' e^{-2kx} \cos^2 \pi x \\ &= -\pi e^{-2kx}. \end{aligned}$$

The characteristics equation is

$$(\lambda + k)^2 + \pi^2 = 0.$$

This gives the ODE

$$y'' + 2ky' + (k^2 + \pi^2)y = 0.$$

From the initial conditions we obtain the particular solution

$$y = e^{-kx}(\cos \pi x - \sin \pi x).$$

16. Team Project. (a) $c_1e^x + c_2e^{-x} = c_1^* \cosh x + c_2^* \sinh x$. Expressing cosh and sinh in terms of exponential functions [see (17) in App. 3.1], we have

$$\tfrac{1}{2}(c_1^* + c_2^*)e^x + \tfrac{1}{2}(c_1^* - c_2^*)e^{-x};$$

hence $c_1 = \frac{1}{2}(\overset{*}{c}_1 + \overset{*}{c}_2)$, $c_2 = \frac{1}{2}(\overset{*}{c}_1 - \overset{*}{c}_2)$. The student should become aware that for second-order ODEs there are several possibilities for choosing a basis and making up a general solution. For this reason we say "*a* general solution," whereas for first-order ODEs we said "*the* general solution."

(b) If two solutions are 0 at the same point x_0, their Wronskian is 0 at x_0, so that these solutions are linearly dependent by Theorem 2.

(c) The derivatives would be 0 at that point. Hence so would be their Wronskian W, implying linear dependence.

(d) y_2/y_1 is constant in the case of linear dependence; hence the derivative of this quotient is 0, whereas in the case of linear independence this is not the case. This makes it likely that such a formula should exist.

(e) The first two derivatives of y_1 and y_2 are continuous at $x = 0$ (the only x at which something could happen). Hence these functions qualify as solutions of a second-order ODE. y_1 and y_2 are linearly dependent for $x \geqq 0$ as well as for $x < 0$ because, in each of these two intervals, one of the functions is identically 0. On $-1 < x < 1$ they are linearly independent because $c_1y_1 + c_2y_2 = 0$ gives $c_1 = 0$ when $x \geqq 0$, and $c_2 = 0$ when $x < 0$. The Wronskian is

$$W = y_1y_2' - y_2y_1' = \left\{ \begin{matrix} 0 \cdot 3x^2 - x^3 \cdot 0 \\ x^3 \cdot 0 - 0 \cdot 3x^2 \end{matrix} \right\} = 0 \qquad \text{if } \begin{cases} x < 0 \\ x \geqq 0 \end{cases}.$$

The Euler–Cauchy equation satisfied by these functions has the auxiliary equation

$$(m - 3)m = m(m - 1) - 2m = 0.$$

Hence the ODE is

$$xy'' - 2y' = 0.$$

Indeed, $xy_1'' - 2y_1' = x \cdot 6x - 2 \cdot 3x^2 = 0$ if $x \geqq 0$, and $0 = 0$ for $x < 0$. Similarly for y_2. Now comes the point. In the present case the standard form, as we use it in all our present theorems, is

$$y'' - \frac{2}{x}y' = 0$$

and shows that $p(x)$ is not continuous at 0, as required in Theorem 2. Thus there is no contradiction.

This illustrates why the continuity assumption for the two coefficients is quite important.

(f) According to the hint given in the enunciation, the first step is to write the ODE (1) for y_1 and then again for y_2. That is,

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0 \\ y_2'' + py_2' + qy_2 &= 0 \end{aligned}$$

where p and q are variable. The hint then suggests eliminating q from these two ODEs. Multiply the first equation by $-y_2$, the second by y_1, and add:

$$(y_1 y_2'' - y_1'' y_2) + p(y_1 y_2' - y_1' y_2) = W' + pW = 0$$

where the expression for W' results from the fact that $y_1' y_2'$ appears twice and drops out. Now solve this by separating variables or as a homogeneous linear ODE.

In Prob. 6 we have $p = 2$, hence $W = ce^{-2x}$ by integration from 0 to x, where $c = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1 \cdot \omega - 0 \cdot (-1) = \omega$.

SECTION 2.7. Nonhomogeneous ODEs, page 79

Purpose. We show that for getting a general solution y of a nonhomogeneous linear ODE we must find a general solution y_h of the corresponding homogeneous ODE and then—this is our new task—any particular solution y_p of the nonhomogeneous ODE,

$$y = y_h + y_p.$$

Main Content, Important Concepts

General solution, particular solution

Continuity of p, q, r suffices for existence and uniqueness.

A general solution exists and includes all solutions.

Comment on Methods for Particular Solutions

The method of undetermined coefficients is simpler than that of variation of parameters (Sec. 2.10), as is mentioned in the text, and it is sufficient for many applications, of which Secs. 2.8 and 2.9 show standard examples.

Comment on General Solution

Theorem 2 shows that the situation with respect to general solutions is practically the same for homogeneous and nonhomogeneous linear ODEs.

Comment on Table 2.1

It is clear that the table could be extended by the inclusion of products of polynomials times cosine or sine and other cases of limited practical value. Also, $\alpha = 0$ in the last pair of lines gives the previous two lines, which are listed separately because of their practical importance.

General Comments on Text. Determination of Constants. For a good understanding, it is important to realize that a general solution of a nonhomogeneous linear ODE contains two kinds of constants, namely,

(I) the constants in Table 2.1, which depend on the right side of the ODE, but not on the initial conditions, and must be determined *first*,

(II) the two arbitrary constants in a general solution y_h of the homogeneous ODE (and thus in a general solution $y = y_h + y_p$ of the given ODE) and must be determined *after* the constants in (I) have been determined, and are to be determined by using the initial conditions. Examples 1–3 illustrate this important fact, which, for weaker students, sometimes causes difficulties in understanding.

Examples in the Text. Examples 1–3 are very similar, illustrating Rules (**a**), (**b**), (**c**). In particular, the student should realize that Example 3 on the sum rule is not more difficult than the other two examples. The only additional idea in the case, say, $r = r_1 + r_2$ on the right, is to split y_p into a sum, $y_p = y_{p1} + y_{p2}$, whose coefficients, according to Table 2.1, can be determined for y_{p1} and y_{p2} separately. Hence we have to determine two sets of constants from two systems of algebraic equations, each of which is not larger than it would be had we only y_{p1} or only y_{p2} on the right side of the given ODE.

Problem Set 2.7

This problem set begins with the determination of general solutions of nonhomogeneous linear ODEs, Probs. 1–6 with a single term on the right, Probs. 7–9 with a sum of two terms each, and Prob. 10 showing a simple extension of the method of undetermined coefficients beyond functions r shown in Table 2.1.

Problems 11–18 concern IVPs for nonhomogeneous linear ODEs with one term on the right (Probs. 11–14 and 17) and with two terms on the right (Probs. 15, 16, and 18).

CAS Project 19 should make the student aware that, depending on the initial conditions and on the kind of the homogeneous ODE, the solution $y = y_h + y_p$ may approach y_p as: $x \to \infty$, or may contain an increasing $|y_h|$, or may be of the form $y = y_p$ with y_h absent.

Team Project 20 is an invitation to explore more general functions on the right, and to what Euler–Cauchy equation the present method can be extended.

SOLUTIONS TO PROBLEM SET 2.7, page 84

2. $y = c_1e^{-3.2x} + c_2e^{-1.8x} + 0.00999 \cos x + 0.0105 \sin x$. This is a typical solution of a forced oscillation problem in the overdamped case. The general solution of the homogeneous ODE dies out, practically after some short time (theoretically never), and the transient solution goes over into a harmonic oscillation whose frequency is equal to that of the driving force (or electromotive force).

Note that the input (the driving force) is a cosine, whereas the output (the response) is a cosine and sine; this means a phase shift. It is due to the presence of a y'-term, mechanically a linear damping force, as we shall see in the next section.

4. $y = c_1e^{3x} + c_2e^{-3x} - \dfrac{2}{1 + (\pi/3)^2} \cos \pi x$. Observe that the ODE has no y'-term, so we have no phase shift, and the output is a pure cosine term, just like the input. Compare this with Prob. 2.

6. A general solution of the homogeneous ODE is

$$y_h = e^{-x/2}(c_1 \cos \pi x + c_2 \sin \pi x).$$

We see that the function on the right side of the ODE is a solution of the homogeneous ODE. Hence we have to apply the Modification Rule, starting from

$$y_p = xe^{-x/2}(K \cos \pi x + M \sin \pi x).$$

Substitution gives $K = -1/(2\pi)$; $M = 0$. Hence the answer is

$$y = y_h + y_p = e^{-x/2}\left(c_1 \cos \pi x + c_2 \sin \pi x - \frac{x}{2\pi} \cos \pi x\right).$$

Note that the output involves cosine, whereas the input involves sine; and, although we have a y'-term, the output is a single term. Compare this with Probs. 2 and 4, which differ from the present situation.

8. $y = A \cos 3x + B \sin 3x + \frac{1}{8} \cos x + \frac{1}{18} x \sin 3x$. An important point is that the Modification Rule applies to the second term on the right. Hence the best way seems to split y_p additively, $y_p = y_{p1} + y_{p2}$, where

$$y_{p1} = K_1 \cos x + M_1 \sin x, \qquad y_{p2} = K_2 x \cos 3x + M_2 x \sin 3x.$$

In Prob. 9 the situation is similar.

10. $2x \sin x$ is not listed in the table because it is of minor practical importance. However, by looking at its derivatives, we see that

$$y_p = Kx \cos x + Mx \sin x + N \cos x + P \sin x$$

should be general enough. Indeed, by substitution and collecting cosine and sine terms separately we obtain

$$(1) \qquad (2K + 2Mx + 2P + 2M) \cos x = 0$$

$$(2) \qquad (-2Kx + 2M - 2N - 2K) \sin x = 2x \sin x.$$

In (1) we must have $2Mx = 0$; hence $M = 0$ and then $P = -K$. In (2) we must have $-2Kx = 2x$; hence $K = -1$, so that $P = 1$ and from (2), finally, $-2N - 2K = 0$, hence $N = 1$. *Answer:*

$$y = (c_1 + c_2 x)e^{-x} + (1 - x) \cos x + \sin x.$$

12. The Modification Rule is needed. The *answer* is $y = 1.8 \cos 2x + \sin 2x + 3x \cos 2x$.

14. The characteristic equation of the homogeneous ODE has a double root -2. The function on the right is such that the Modification Rule does not apply. A general solution of the nonhomogeneous ODE is

$$y = (c_1 + c_2 x)e^{-2x} - \tfrac{1}{4} e^{-2x} \sin 2x.$$

From this and the initial conditions we obtain the *answer*

$$y = (1 + x)e^{-2x} - \tfrac{1}{4} e^{-2x} \sin 2x.$$

16. $y_h = c_1 e^{2x} + c_2$, $y_p = C_1 x e^{2x} + C_2 e^{-2x}$ by the Modification Rule for a simple root.

$$y = e^{2x} - \tfrac{3}{2} + 3xe^{2x} - \tfrac{1}{2} e^{-2x}.$$

18. The Basic Rule and the Sum Rule are needed. We obtain

$$y_h = e^{-x}(A \cos 3x + B \sin 3x)$$

$$y = e^{-x} \cos 3x - 0.4 \cos x + 1.8 \sin x + 6 \cos 3x - \sin 3x.$$

20. Team Project. (b) Perhaps the simplest way is to take a specific ODE, e.g.,

$$x^2y'' + 6xy' + 6y = r(x)$$

and then experiment by taking various $r(x)$ to find the form of choice functions. The simplest case is a single power of x. However, almost all the functions that work as $r(x)$ in the case of an ODE with constant coefficients can also be used here.

SECTION 2.8. Modeling: Forced Oscillations. Resonance, page 85

Purpose. To extend Sec. 2.4 from free to forced vibrations by adding an input (a driving force, here assumed to be sinusoidal). Mathematically, we go from a homogeneous to a nonhomogeneous ODE which we solve by undetermined coefficients.

New Features

Undamped Forced Oscillations. Resonance (Fig. 55)

Resonance appears if the physical system is (theoretically) undamped (in practice, if it has small enough damping that the damping effect can be neglected), and if the input frequency is *exactly equal* to the natural frequency of the system. Then a solution

$$y = At \sin \omega_0 t \tag{11}$$

has a factor t which makes it increase to ∞ (Fig. 55).

The approach to resonance as $\omega \to \omega_0$ $(= \sqrt{k/m})$ is also characterized by the resonance factor in Fig. 54.

Undamped Forced Oscillations. Beats (Fig. 56)

Beats occur if the input frequently is *approximately equal* to the natural frequency of the physical system. Then

$$y = K(\cos \omega t - \cos \omega_0 t) \tag{12}$$

with $K = F_0/[m(\omega_0^2 - \omega^2)]$.

Damped Forced Oscillations. For these the transient solution approaches the steady-state solution as $t \to \infty$, practically after some time which may often be rather short. If $c > 0$, there is no more true resonance, but the maximum amplitude (16) may still be large, as Fig. 57 illustrates. Also, there is a phase lag η, discontinuous and equal to 0 or π when $c = 0$, and continuous and monotone when $c > 0$ (Fig. 58).

Problem Set 2.8

Problems 3–7 concern damped systems. Hence a general solution of such a physical system is that of a homogeneous linear ODE and approaches 0 as $t \to \infty$, so that solving these problems amounts to determining a particular solution of the corresponding ODE.

Problems 8–15 amount to finding a general solution of the nonhomogeneous ODE.

Problems 16–20 are IVPs for nonhomogeneous linear ODEs. Problem 17 is of the kind that will occur in connection with partial sums of Fourier series in Chap. 11. Problem 18 is a typical example illustrating the rapidity of approach to the steady-state solution.

Beats are considered in Probs. 21, 22, and 25 with $\omega = 0.9$, whereas for ω farther away from $\omega_0 = 1$ (corresponding to the natural frequency of the physical system) the form of vibrations cannot be guessed immediately (see Fig. 60).

Practical resonance is considered experimentally in Team Experiment 23.

Continuity Conditions

Nonhomogeneous linear ODEs with a driving force acting for some finite interval of time only will require the idea of continuity conditions of y and y' at the instant of time when the driving force becomes identically zero. This makes such problems more involved, as Prob. 24 illustrates, and motivates the application of an "operational method," such as the Laplace transform (Chap. 6).

SOLUTIONS TO PROBLEM SET 2.8, page 91

2. Problems 2, 8, 18. Note that the damping and restoring terms must have positive coefficients, and that Prob. 12 shows resonance; hence it is not a candidate.

4. $y_p = \cos 4t + 0.6 \sin 4t$

6. $y_p = \frac{1}{10} \cos t - \frac{1}{90} \cos 3t + \frac{1}{5} \sin t + \frac{1}{45} \sin 3t$

8. $y = e^{-t}(A \cos 1.5t + B \sin 1.5t) - 0.6 \cos 1.5t + 0.2 \sin 1.5t$

10. $y = A \cos 4t + B \sin 4t + 7t \sin 4t$

12. $y = e^{-t}(A \cos 2t + B \sin 2t) + 2 \sin t$. Note that, whereas a single term on the right side of the ODE will usually produce two terms in the solution (the response), the present problem shows that sometimes the opposite will also occur.

14. $y = A \cos t + B \sin t + e^{-t}(\cos t - 2 \sin t)$. Note that this does not give resonance, but, on the contrary, $y_p \to 0$, which is understandable because the driving force on the right approaches 0 as t approaches ∞.

16. A general solution is

$$y = A \cos 5t + B \sin 5t + \sin t.$$

From the initial condition we obtain $A = 1$ and $B = 0$. Hence the *answer* is

$$y = \cos 5t + \sin t.$$

Note that, whereas in general both solutions of a basis occur in the solution of an IVP, here we have only one of them. Of course, this could be changed by changing the initial conditions.

18. $y = e^{-4t} \cos t + 26.8 \sin 0.5t - 6.4 \cos 0.5t$. At $t = 1.2$ the exponential term has decreased to less than 1% of its original value. This marks the end of the transition from a practical point of view. $t = 1.8$ is the time when that term has become less than $1/10$ of a percent in absolute value.

20. A general solution is

$$y = A \cos \sqrt{5}\, t + B \sin \sqrt{5}\, t - (\cos \pi t - \sin \pi t)/(\pi^2 - 5).$$

Using the initial conditions, we obtain the answer

$$y = \frac{1}{\pi^2 - 5}\left(\cos \sqrt{5}\, t - \frac{\pi}{\sqrt{5}} \sin \sqrt{5}\, t - \cos \pi t + \sin \pi t\right).$$

22. $y = 100 \cos 4.9t - 98 \cos 5t$.

(a) changes the coefficient of $\cos 5t$

(b) changes the amplitude

24. If $0 \leqq t \leqq \pi$, then a particular solution

$$y_p = K_0 + K_1 t + K_2 t^2$$

gives $y_p'' = 2k_2$ and

$$y_p'' + y_p = K_0 + 2K_2 + K_1 t + K_2 t^2 = 1 - \frac{1}{\pi^2} t^2;$$

thus,

$$K_2 = -\frac{1}{\pi^2}, \qquad K_1 = 0, \qquad K_0 = 1 - 2K_2 = 1 + \frac{2}{\pi^2}.$$

Hence a general solution is

$$y = A \cos t + B \sin t + 1 + \frac{2}{\pi^2} - \frac{1}{\pi^2} t^2.$$

From this and the first initial conditions,

$$y(0) = A + 1 + \frac{2}{\pi^2} = 0, \qquad A = -\left(1 + \frac{2}{\pi^2}\right).$$

The derivative is

$$y' = -A \sin t + B \cos t - \frac{2}{\pi^2} t$$

and gives $y'(0) = B = 0$. Hence the solution is

(I) $\qquad y(t) = (1 + 2/\pi^2)(1 - \cos t) - t^2/\pi^2 \qquad$ if $0 \leqq t \leqq \pi$,

and if $t > \pi$, then

(II) $\qquad y = y_2 = A_2 \cos t + B_2 \sin t$

with A_2 and B_2 to be determined from the **continuity conditions**

$$y(\pi) = y_2(\pi), \qquad y'(\pi) = y_2'(\pi).$$

So we need from (I) and (II)

$$y(\pi) = 2(1 + 2/\pi^2) - 1 = 1 + 4/\pi^2 = y_2(\pi) = -A_2$$

and

$$y'(t) = (1 + 2/\pi^2) \sin t - 2t/\pi^2$$

and from this and (II),

$$y'(\pi) = -2/\pi = B \cos \pi = -B_2.$$

This gives the solution

$$y = -(1 + 4/\pi^2) \cos t + (2/\pi) \sin t \qquad \text{if} \quad t > \pi.$$

Answer:

$$y = \begin{cases} (1 + 2/\pi^2)(1 - \cos t) - t^2/\pi^2 & \text{if} \quad 0 \leqq t \leqq \pi \\ -(1 + 4/\pi^2) \cos t + (2/\pi) \sin t & \text{if} \quad t > \pi \end{cases}.$$

The function in the second line gives a harmonic oscillation because we disregarded damping.

SECTION 2.9. Modeling: Electric Circuits, page 93

Purpose. To discuss the current in an *RLC*-circuit with sinusoidal input $E_0 \sin \omega t$, as follows.

Modeling the *RLC*-circuit

Solving the model (1) for the current $I(t)$

Discussion of a typical IVP

Discussion of a electrical–mechanical analogy

Modeling the *RLC*-circuit. The student should first review the special case of an *RL*-circuit in Example 2 of Sec. 1.5, which is modeled by a first-order ODE, using Kirchhoff's KVL. The present addition of a capacitor is very simple in terms of setting up the model, resulting in a second-order ODE. Proceed stepwise in this way:

Write the voltage drop across the capacitor in the form $(1/C)Q$ (rather than CQ) is a standard convention to obtain generally more convenient numbers.

Solve the model (1) by the method of undetermined coefficients (see Sec. 2.7).

ATTENTION! The right side in (1) is $E_0\omega \cos \omega t$, because of differentiation.

In solving, two quantities of practical importance are introduced, namely, the *reactance*

$$S = \omega L - 1/(\omega C) \tag{3}$$

and the *impedance* (also called the apparent resistance)

$$\sqrt{R^2 + C^2}.$$

(Its complex analog, the *complex impedance* $Z = R + iS$, is mentioned in the answer to Prob. 20.)

Example 1 shows a typical IVP with I_h rapidly going to 0, as illustrated in Fig. 62, so that the transient current rapidly approaches a harmonic steady-state current.

Table 2.2 shows a strictly quantitative electrical–mechanical analogy, which is used in *transducers*, as explained in the text.

SOLUTIONS TO PROBLEM SET 2.9, page 98

2. This is another special case of a circuit that leads to an ODE of first order,

$$RI' + I/C = E' = \omega E_0 \cos \omega t.$$

Integration by parts gives the solution

$$\begin{aligned} I(t) &= e^{-t/(RC)}\left[\frac{\omega E_0}{R}\int e^{t/(RC)} \cos \omega t \, dt + c\right] \\ &= ce^{-t/(RC)} + \frac{\omega E_0 C}{1 + (\omega RC)^2}(\cos \omega t + \omega RC \sin \omega t) \\ &= ce^{-t/(RC)} + \frac{\omega E_0 C}{\sqrt{1 + (\omega RC)^2}} \sin(\omega t - \delta), \end{aligned}$$

where $\tan \delta = -1/(\omega RC)$. The first term decreases steadily as t increases, and the last term represents the steady-state current, which is sinusoidal. The graph of $I(t)$ is similar to that in Fig. 62.

4. The integral that occurs can be evaluated by integration by parts, as is shown (with other notations) in standard calculus texts. From (4) in Sec. 1.5 we obtain

$$\begin{aligned} I &= e^{-Rt/L}\left[\frac{E_0}{L}\int e^{Rt/L} \sin \omega t \, dt + c\right] \\ &= ce^{-Rt/L} + \frac{E_0}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t) \\ &= ce^{-Rt/L} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \delta), \qquad \delta = \arctan \frac{\omega L}{R}. \end{aligned}$$

6. $E = 2t^2$, $E' = 4t$, $0.5I'' + 200I = 4t$, $I'' + 400I = 8t$, $I(0) = 0$ is given. $I'(0) = 0$ follows from

$$LI'(0) + Q(0)/C = E(0) = 0.$$

Answer:

$$I = 0.02(t - 0.05 \sin 20t).$$

8. $E' = 1000 \cos 2t$, $0.5I'' + 4I' + 10I = 1000 \cos 2t$, so that the steady-state solution is

$$I = 62.5(\cos 2t + \sin 2t) \text{ A}.$$

10. The ODE is

$$I'' + 2I' + 20I = 157 \cdot 3.$$

The steady-state solution is

$$I = 33 \cos 3t + 18 \sin 3t.$$

Note that if you let C decrease, the sine term in the solution will become smaller and smaller, compared with the cosine term.

12. The ODE is

$$0.1I'' + 0.2I' + 0.5I = 220 \cdot 314 \cos 314t.$$

Its characteristic equation is

$$0.1[(\lambda + 1)^2 + 4] = 0.$$

Hence a general solution of the homogenous ODE is

$$e^{-t}(A \cos 2t + B \sin 2t).$$

The transient solution (rounded to 4 decimals) is

$$I = e^{-t}(A \cos 2t + B \sin 2t) - 7.0064 \cos 314t + 0.0446 \sin 314t \text{ A}.$$

14. Write $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, as in the text before Example 1. Here $\alpha = R/(2L) > 0$, and β can be real or imaginary. If β is real, then $\beta \leqq R/(2L)$ because $R^2 - 4L/C \leqq R^2$. Hence $\lambda_1 < 0$ (and $\lambda_2 < 0$, of course). If β is imaginary, then $I_h(t)$ represents a damped oscillation, which certainly goes to zero as $t \to \infty$.

16. The ODE is

$$0.2I'' + 8I' + 80I = 1000 \cos 10t.$$

A general solution is

$$I = (c_1 + c_2 t)e^{-20t} + 6 \cos 10t + 8 \sin 10t.$$

The initial conditions are $I(0) = 0$, $Q(0) = 0$, which because of (1′), that is,

$$LI'(0) + RI(0) + \frac{Q(0)}{C} = E(0) = 0,$$

leads to $I'(0) = 0$. This gives

$$I(0) = c_1 + 6 = 0, \qquad c_1 = -6$$

$$I'(0) = -20c_1 + c_2 + 80 = 0, \qquad c_2 = -200.$$

Hence the answer is

$$I = -(6 + 200t)e^{-20t} + 6 \cos 10t + 8 \sin 10t.$$

18. The characteristic equation of the homogenous ODE is

$$0.2(\lambda + 8)(\lambda + 10) = 0.$$

The initial conditions are $I(0) = 0$ as given, $I'(0) = E(0)/L = 820$ by formula 1$''$ in the text and $Q(0) = 0$. Also, $E' = -8200 \sin 10t$. The ODE is

$$I'' + 8I' + 80I = -8200 \sin 10t.$$

The *answer* is

$$I = 160\, e^{-8t} - 205e^{-10t} + 45 \cos 10t + 5 \sin 10t.$$

20. $\tilde{I}_p = Ke^{i\omega t}, \tilde{I}_p' = i\omega Ke^{i\omega t}, \tilde{I}_p'' = -\omega^2 Ke^{i\omega t}$. Substitution gives

$$\left(-\omega^2 L + i\omega R + \frac{1}{C}\right) Ke^{i\omega t} = E_0 \omega e^{i\omega t}.$$

Divide this by $\omega e^{i\omega t}$ on both sides and solve the resulting equation algebraically for K, obtaining

(A) $$K = \frac{E_0}{-\left(\omega L - \dfrac{1}{\omega C}\right) + iR} = \frac{E_0}{-S + iR}$$

where S is the reactance given by (3). To make the denominator real, multiply the numerator and the denominator of the last expression by $-S - iR$. This gives

$$K = \frac{-E_0(S + iR)}{S^2 + R^2}.$$

The real part of $Ke^{i\omega t}$ is

$$\begin{aligned}(\mathrm{Re}\, K)(\mathrm{Re}\, e^{i\omega t}) - (\mathrm{Im}\, K)(\mathrm{Im}\, e^{i\omega t}) &= \frac{-E_0 S}{S^2 + R^2} \cos \omega t + \frac{E_0 R}{S^2 + R^2} \sin \omega t \\ &= \frac{-E_0}{S^2 + R^2} (S \cos \omega t - R \sin \omega t),\end{aligned}$$

in agreement with (2) and (4).

We mention that (A) can be written

$$K = \frac{E_0}{iZ}$$

where

$$Z = R + iS = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

is called the **complex impedance**. Note that its absolute value $|Z| = \sqrt{R^2 + S^2}$ is the **impedance**, as defined in the text.

SECTION 2.10. Solution by Variation of Parameters, page 99

Purpose. To discuss the general method for particular solutions, which applies in any case but may often lead to difficulties in integration (which we, by and large, have avoided in our problems, as the subsequent solutions show).

Comments

The ODE must be in ***standard form***, with 1 as the coefficient of y''—students tend to forget that.

Here we do need the Wronskian, in contrast with Sec. 2.6 where we could get away without it.

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2. $y_1 = \cos 3x, \quad y_2 = \sin 3x, \quad W = 3, \quad r = \csc 3x$. Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \frac{1}{3}\int \frac{\sin 3x}{\sin 3x}\, dx = \frac{x}{3}$$

$$\int \frac{y_1 r}{W}\, dx = \frac{1}{3}\int \frac{\cos 3x}{\sin 3x}\, dx = \frac{1}{9}\ln|\sin 3x|.$$

Answer:

$$y = A\cos 3x + B\sin 3x - \frac{x}{3}\cos 3x + \frac{1}{9}(\sin 3x)\ln|\sin 3x|$$

4. $y_1 = e^{2x}\cos x, \quad y_2 = e^{2x}\sin x, \quad W = e^{4x}$. Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \int \frac{(e^{2x}\sin x)e^{2x}/\sin x}{e^{4x}}\, dx = x$$

$$\int \frac{y_1 r}{W}\, dx = \int \frac{(e^{2x}\cos x)e^{2x}/\sin x}{e^{4x}}\, dx = \ln|\sin x|.$$

Answer:

$$y = [A\cos x + B\sin x - x\cos x + (\sin x)\ln|\sin x|]e^{2x}.$$

6. $y_1 = e^{-3x}, \quad y_2 = xe^{-3x}, \quad W = e^{6x}$. Hence in (2),

$$\int \frac{y_2 r}{W}\, dx = \int \frac{(xe^{-3x})16e^{-3x}/(x^2+1)}{e^{-6x}}\, dx = \int \frac{16x}{x^2+1}\, dx = 8\ln(x^2+1)$$

$$\int \frac{y_1 r}{W}\, dx = \int \frac{(e^{-3x})16e^{-3x}/(x^2+1)}{e^{-6x}}\, dx = \int \frac{16}{x^2+1}\, dx = 16\arctan x.$$

Answer:

$$y = (c_1 + c_2 x)e^{-3x} + 8[-\ln(x^2+1) + 2x\arctan x]\, e^{-3x}.$$

8. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}\cosh 2x$

10. $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$, $W = e^{-2x}$. Hence in (2),

$$\int \frac{y_2 r}{W}\,dx = \int \frac{(e^{-x}\sin x)4e^{-x}/\cos^3 x}{e^{-2x}}\,dx = \frac{2}{\cos^2 x}$$

$$\int \frac{y_1 r}{W}\,dx = \int \frac{(e^{-x}\cos x)4e^{-x}/\cos^3 x}{e^{-2x}}\,dx = 4\tan x.$$

This gives the particular solution

$$e^{-x}\left[-(\cos x)\frac{2}{\cos^2 x} + 4(\sin x)\tan x\right] = e^{-x}\left(\frac{-2 + 4\sin^2 x}{\cos x}\right)$$
$$= e^{-x}[-2(\cos 2x)/\cos x].$$

Answer:

$$y = e^{-x}[A\cos x + B\sin x - 2(\cos 2x)/\cos x].$$

12. The right side suggests the following choice of a basis of solutions:

$$y_1 = \cosh x, \qquad y_2 = \sinh x.$$

Then $W = 1$, and

$$y_p = -\cosh x \int (\sinh x)/\cosh x\,dx + \sinh x \int (\cosh x)/\cosh x\,dx$$
$$= -(\cosh x)\ln|\cosh x| + x\sinh x.$$

14. TEAM PROJECT. (a) $y_1 = e^{-3x}$, $y_2 = e^{-x}$, $W = 2e^{-4x}$, $r = 65\cos 2x$. From (2),

$$y_p = -e^{-3x}\int \frac{e^{-x}65\cos 2x}{2e^{-4x}}\,dx + e^{-x}\int \frac{e^{-3x}65\cos 2x}{2e^{-4x}}\,dx$$
$$= \frac{65}{2}\left(-e^{-3x}\int e^{3x}\cos 2x\,dx + e^{-x}\int e^{x}\cos 2x\,dx\right)$$
$$= \tfrac{65}{2}\left(-e^{-3x}\tfrac{1}{13}e^{3x}(3\cos 2x + 2\sin 2x) + e^{-x}\tfrac{1}{5}e^{x}(\cos 2x + 2\sin 2x)\right)$$
$$= -\cos 2x + 8\sin 2x.$$

Answer:

$$y = c_1 e^{-3x} + c_2 e^{-x} - \cos 2x + 8\sin 2x.$$

This was much more work than that for undetermined coefficients.

(b) We can treat x^2 on the right by undetermined coefficients, obtaining the contribution $x^2 + 4x + 6$ to the solution. We could treat it by the other method, but we would have to evaluate additional integrals of an exponential function times a power of x. We treat the other part, $35x^{3/2}e^x$, by the method of this section, calling the resulting function y_{p1}. We need $y_1 = e^x$, $y_2 = xe^x$, $W = e^{2x}$. From this and (2),

$$\begin{aligned} y_{p1} &= -e^x \int \frac{xe^x}{e^{2x}} 35x^{3/2}e^x\,dx + xe^x \int \frac{e^x}{e^{2x}} 35x^{3/2}e^x\,dx \\ &= 35\left(-e^x \int x^{5/2}\,dx + xe^x \int x^{3/2}\,dx\right) = 4e^x x^{7/2}. \end{aligned}$$

Complete answer:

$$y = (c_1 + c_2x)e^x + 4e^x x^{7/2} + x^2 + 4x + 6.$$

(c) If the right side is a power of x, say, $r = r_0x^k$, then substitution of $y_p = Cx^k$ gives

$$x^2y'' + axy' + by = (k(k-1) + ak + b)Cx^k = r_0\,x^k.$$

This can be solved for C. To explore further possibilities, one may work "backwards"; that is, assume a solution, substitute it on the left, and see what from one gets as a right side.

SOLUTIONS TO CHAPTER 2 REVIEW QUESTIONS AND PROBLEMS, page 102

8. $y = c_1e^{-4x} + c_2e^{3x}$

10. $y = e^{-0.1x}(A \cos 0.4x + B \sin 0.4x)$

12. $y = (c_1 + c_2x)e^{-2\pi x}$

14. $y = c_1x^3 + c_2x^{-3}$

16. $y = e^{-x}(A \cos x + B \sin x) - e^{-x} \cos 2x$

18. $y' = z$, $y'' = (dz/dy)z$ by the chain rule, $yz\,dz/dy = 2z^2$, $dz/z = 2dy/y$,

$$\ln|z| = 2\ln|y| + c^*,\ z = c_1y^2 = y',\ dy/y^2 = c_1\,dx,\ -1/y = c_1x + c_2;$$

hence

$$y = 1/(\tilde{c}_1x + \tilde{c}_2).$$

Also, $y = 0$ is a solution.

20. Obtain the particular solution by undetermined coefficients.
Answer: $y = 3e^x - 5e^{2x} + 3\cos x + \sin x$

22. The auxiliary equation is

$$m(m-1) + 15m + 49 = (m+7)^2 = 0.$$

Hence a general solution is

$$y = (c_1 + c_2 \ln|x|)x^{-7}.$$

From the initial conditions, $c_1 = 2$, $c_2 = 3$.

24. $I = c_1 e^{-1999.87t} + c_2 e^{-0.125008t}$ A

26. $E' = 220 \cdot 314 \cos 314t$, $I = e^{-50t}(A \cos 150t + B \sin 150t) + 0.847001 \sin 314t - 1.985219 \cos 314t$ A

28. The equation is

$$0.125y'' + 1.125y = \cos t - 4 \sin t;$$

thus,

$$y'' + 9y = 8 \cos t - 32 \sin t.$$

The solution satisfying the initial conditions is

$$y = -\cos 3t + \tfrac{4}{3} \sin 3t + \cos t - 4 \sin t,$$

as obtained by the method of undetermined coefficients.

The last two terms result from the driving force. In the first two terms, $\omega_0 = \sqrt{k/m} = 3$. This shows that resonance would occur if the driving force had the frequency $\omega(2\pi) = 3/(2\pi)$.

30. $C^*(\omega)$ is given by (16), Sec. 2.8. The maximum is obtained by equating the derivative to zero; this gives (15*) in Sec. 2.8, which for our numerical values becomes

$$16 = 2(24 - \omega^2),$$

so that $\omega = 4$. Equation (16) in Sec. 2.8 then gives the maximum amplitude

$$C^*(\omega_{\max}) = \frac{2 \cdot 1 \cdot 10}{4\sqrt{4 \cdot 1^2 \cdot 24 - 16}} = 0.5590.$$

To check this result, we determine the general solution, using the method of undetermined coefficients, finding

$$y(t) = e^{-2t}(A \cos 2\sqrt{5}t + B \sin 2\sqrt{5}t) + 0.25 \cos 4t + 0.5 \sin 4t,$$

and confirm the result by calculating the amplitude

$$\sqrt{0.25^2 + 0.5^2} = 0.5590.$$