Solutions for Elementary Differential Equations 1st Edition by Kuttler

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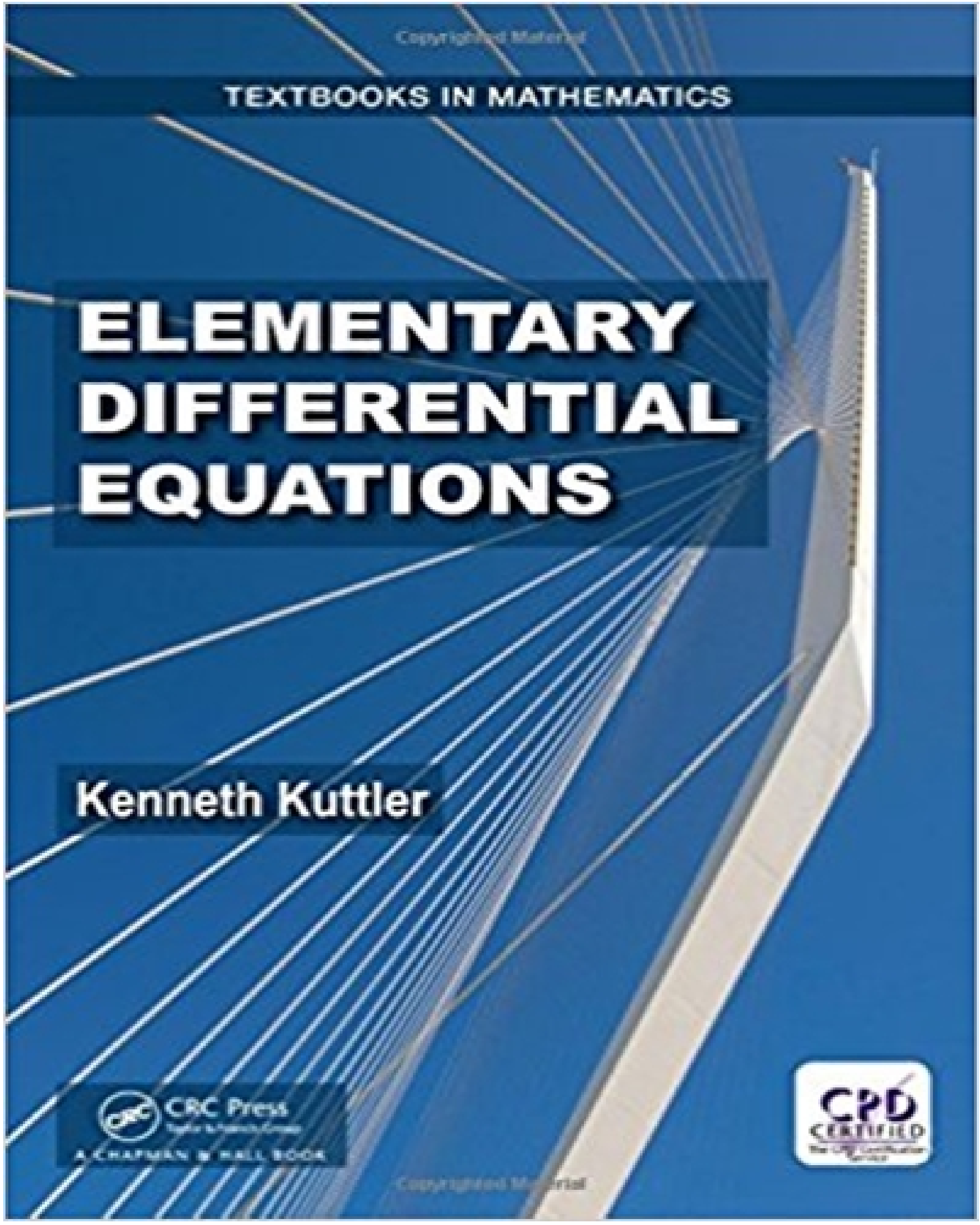

Solutions

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27. Give the solutions to the following quadratic equations having complex coefficients. Note how the solutions do not come in conjugate pairs as they do when the equation has real coefficients.

    (a) $x^2 + 2x + 1 + i = 0$
    $\sqrt{4 - 4(1+i)} = (1-i)\sqrt{2}$
    $x = \frac{-2 \pm (1-i)\sqrt{2}}{2}$
    (b) $4x^2 + 4ix - 5 = 0,\ x = 1 - \frac{1}{2}i, -1 - \frac{1}{2}i$
    (c) $4x^2 + (4+4i)x + 1 + 2i = 0,\ x = -\frac{1}{2}, -\frac{1}{2} - i$
    (d) $x^2 - 4ix - 5 = 0,\ x = 1 + 2i, -1 + 2i$
    (e) $3x^2 + (1-i)x + 3i = 0$
    $x = \frac{-(1-i) \pm (1-i)\sqrt{19}}{6}$

28. Prove the fundamental theorem of algebra for quadratic polynomials having coefficients in $\mathbb{C}$. That is, show that an equation of the form $ax^2 + bx + c = 0$ where a, b, c are complex numbers, $a \neq 0$ has a complex solution. **Hint:** Consider the fact, noted earlier that the expressions given from the quadratic formula do in fact serve as solutions.

    You can just complete the square. Thus

    $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

    and so

    $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = 0$$

    Thus

    $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

    Pick one of the square roots of $b^2 - 4ac$. If this is 0, you have nothing left. You just have a repeated root of $x = -\frac{b}{2a}$. Calling this $\sqrt{b^2 - 4ac}$ a square root of the right side is $\frac{\sqrt{b^2-4ac}}{2a}$ and the other will be $-\frac{\sqrt{b^2-4ac}}{2a}$ and so the solution will be

    $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

    which was to be shown.

H.2 The Idea Of A Differential Equation

1. Find $\lim_{t\to\infty} y(t)$ for $y(t)$ a solution to the differential equation, whenever it exists. That is, you need to determine initial conditions which lead to the limit.

    (a) $y' = -y$
    On this one, $\lim_{t\to\infty} y(t) = 0$.

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(b) $y' = y - \frac{1}{4}y^3$

Here $y(t) \to 2$ if initial condition is in $(0,5)$ and converges to -2 if the initial condition is in $(-5,0)$

(c) $y' = \frac{1}{4}y^3 - y$

It appears that $y(t) \to 0$ if the initial condition is sufficiently close to 0. Not entirely clear how close it needs to be based on the picture. However, you can graph the right side and see that 0 is going to be stable and that in fact, if the initial condition is in $(-2,2)$, the solution will converge to 0. However, this is the only possible limit unless you start with -2 or 2 which will result in $y = -2$ or $y = 2$.

(d) $y' = (1-y)(y-2)(y^2)$

By looking at the slope which is defined by the right side, you see that unless the initial condition starts at 0, the solution to the equation will not converge to 0. Nor will the solution converge to 1 unless it starts at 1. However, if the initial condition starts in $(1,\infty)$ it will be drawn towards 2.

(e) $y' = -y + 3$

In this case, there is only one possibility for a limit of the solution and it is $x = 3$. Indeed, if the initial condition is anything the solution will move toward 3.

(f) $y' = \frac{y-1}{y^2+1}$

There is only one possible limit and it is 1. However, for y larger than 1, the derivative is positive so it moves away from 1. If $y < 1$, the derivative is negative so it also moves away from 1.

(g) $y' = \frac{1-y}{y^2+1}$

This is like the above except here the solutions move toward 1 because if they start off larger than 1, the derivative is negative so they move toward 1 and if they start less than 1, the derivative is positive so the solution moves toward 1. Thus the limit is 1.

(h) $y' = -\tan(y)$. On this one, only consider initial conditions where y is in $[-3/2, 3/2]$ because $\tan(\pi/2)$ is not defined.

Solutions move toward 0 by noticing that the slope is negative for y larger than 0 and positive if y is less than 0.

2. Consider $y' = ay + b$. Explain why if $a < 0$, then every solution y to the equation appears to have the property that $\lim_{t\to\infty} y(t) = -b/a$.

   It is because $ay+b$ is positive for y less than the equilibrium point and negative for y larger. Thus, if y is larger than the equilibrium point it gets smaller and if y is smaller, then it gets larger.

3. Consider $y' = ay + b$. Explain why if $a > 0$, then the only solution to this equation which has a limit as $t \to \infty$ is the constant solution $y = -b/a$.

   It is just like the above only this time, when y is too large, it gets even larger and when it is too small, it gets even smaller. In fact, the absolute value of the slope gets increasingly large as y moves away from the point $-b/a$.

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4. On the next page are graphs of slope fields which are labeled **A**, **B**, **C**, **D**, **E**, **F**. What follows are some differential equations. Label each with the appropriate slope field.

   (a) $y' = 1 - 2y$
   C

   (b) $y' = (1 - y)(y - 2)$
   B

   (c) $y' = 1 + y$
   A

   (d) $y' = y(y - 1)(y - 2)$
   E

   (e) $y' = y(1 - y)(y - 2)$
   D

   (f) $y' = \sin(y)$
   F

5. Assume that a rain drop is a perfect ball. It evaporates at a rate which is proportional to its surface area. Find a differential equation in terms of its volume V and dV/dt which expresses this fact.

   The area of a sphere is $4\pi r^2$ and so when you solve for r in the volume, you get $r = \left(\frac{3V}{4\pi}\right)^{1/3}$ and so the differential equation is $\frac{dV}{dt} = k4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.

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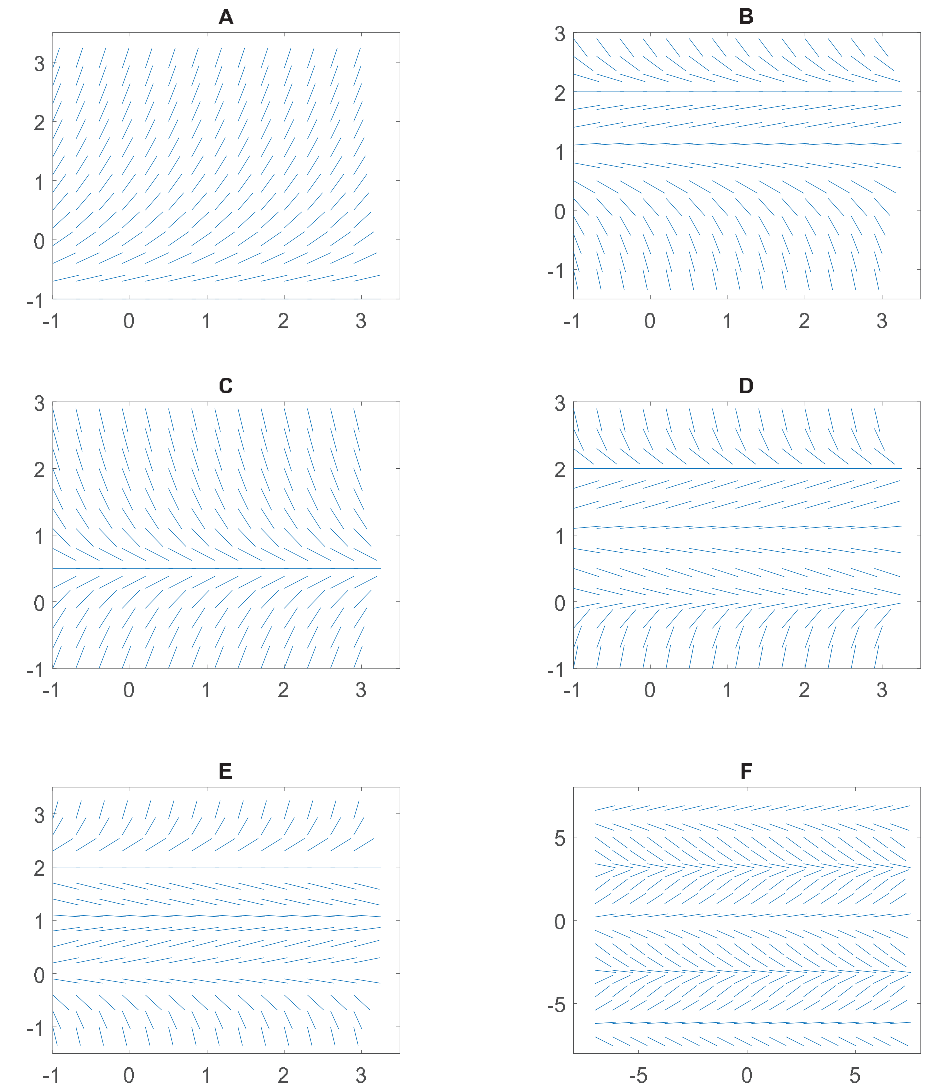

6. Newton's law of cooling states that the temperature of an object changes in proportion to the difference between the temperature of the object and its surroundings. Denoting the constant of proportionality as k, a positive constant and the surrounding temperature as T_0, write a differential equation which describes the temperature T of the object.

   $\frac{dT}{dt} = k\,(T_0 - T)$

7. A small spherical object falling through the atmosphere has air resistance of the form kv^2 where v is the speed of the falling object. If the object has mass m, find a differential equation which describes its speed v. The force on the object from gravity is mg where g is acceleration of gravity. The object will reach a terminal speed. Find it in terms of m, g, k.

   $m\frac{dv}{dt} = mg - kv^2$

   Terminal speed is $\sqrt{\frac{mg}{k}}$.

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8. Chemicals such as DDT degrade at a rate proportional to the amount present. Suppose the constant of proportionality is .5 and you begin with an amount A_0. Also suppose that the chemical is added to a field at the rate of 100 pounds per year. Write a differential equation satisfied by A. After a long time, how much of the chemical will be on the field? Does this answer depend on A_0?

   $\frac{dA}{dt} = -.5A + 100,\ A(0) = A_0$

   After a long time, you will have $\frac{100}{.5} = 200.0$ pounds.

9. The amount of money in a bank grows at the rate of $.05A$ where time is measured in years. At the same time, a person is spending this money at the rate of r dollars per year in a continuous manner. Let the person start with A_0. What must A_0 be if the person is to never run out of money? Show that if he has enough, he will not only not run out, he will become increasingly wealthy but if he has too little, then he will lose it all. You might want to consider graphing a slope field.

   $\frac{dA}{dt} = .05A - r$

   If $A_0 > r/.05$ then the slope is positive and so the amount of money continues to grow. If $A_0 < r/.05$ the slope is negative and so A gets smaller and so the slope keeps on getting more and more negative so he will eventually run out. Thus, unless he is the US government, he will have to quit spending and so the equation will no longer hold.

10. Consider the equation $y' = y$ with the initial condition $y(0) = 1$. From the equation,

$$y(0) = 1, y'(0) = y(0), y'' = y'$$

   so $y''(0) = y'(0) = 1$. Continue this way. Explain why $y^{(n)}(0) = 1$. Now obtain a power series for $y(t)$. Explain why this process shows that there is only one solution to this equation and it is the power series you just obtained. Do you recognize this series from calculus? If not, do the following. Explain each step.

$$\begin{aligned} y' - y &= 0, e^{-t}(y' - y) = 0 \\ \frac{d}{dt}\left(e^{-t}y\right) &= 0,\ e^{-t}y = C \\ y(t) &= Ce^{t},\ y(t) = e^{t}. \end{aligned}$$

   The above steps are valid because of the chain rule and the product rule. As to the derivatives, you have $y^{(n+1)} = y^{(n)}$ and so by induction, all of these evaluated at 0 give 1. Thus the power series obtained is nothing more than $\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t$.

11. Consider $y'' + y = 0, y(0) = 0, y'(0) = 1$. Find a formula for $y^{(k)}(0)$ directly from the differential equation and obtain a power series for the solution to this equation. Do you recognize this series from calculus?

   From the initial condition, $y(0) = 0, y'(0) = 1$, and now from the equation, $y''(0) = -y(0) = 0, y''' = -y'$ so $y'''(0) = -y'(0) = -1$. Then as before,

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$y^{(4)}(0) = 0, y^{(5)}(0) = 1$ etc. Thus $y^{(2n)}(0) = 0, y^{(2n-1)}(0) = (-1)^{n-1}$. It follows that the power series obtained is

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sin x$$

Then you can verify right away that in fact this solves the equation and initial condition.

12. Do the same thing for $y'' + y = 0, y(0) = 1, y'(0) = 0$. Do you recognize this series from calculus?

    It is exactly like the above only this time, you have $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$. Then you can verify that this does solve the initial value problem.

13. You have a cement lined hole so water can only evaporate. If y is the depth of water in the hole, the area is $A(y)$ where $A'(y) > 0, \lim_{y\to\infty} A(y) = \infty, \lim_{y\to 0} A(y) = 0$. Water evaporates at a rate proportional to the surface area exposed. Suppose water is added slowly to this hole at a small but constant rate. Show that eventually, the level of the water must stabilize.

    The differential equation is $\frac{dV}{dt} = -kA(y) + r$ where r is the small constant rate. Of course this will not do. We need to write in terms of a single unknown function. However, $V(y) = \int_0^y A(x)\,dx$ and so $\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}$. Thus the differential equation is

$$\begin{aligned} A(y)\frac{dy}{dt} &= -kA(y) + r \\ \frac{dy}{dt} &= -k + \frac{r}{A(y)} \end{aligned}$$

    You start adding water the level cannot continue increasing because eventually if y large enough, the right side of the equation is less than 0 and so y' will then be negative. On the other hand, if y is too small, then y' will be positive and so the depth should stabilize.

14. The equation

$$\frac{dy}{dt} = k(p-y)(q-y), \quad k > 0, y(0) < p.$$

    occurs in chemical reactions. To be specific, let $p = 1, q = 2$. Have matlab produce a graph of the slope field and determine the behavior of solutions to this differential equation. Explain why in the general case, if $p < q$, the reaction will continue until $y = p$.

    This will happen because if y is smaller than p, then the derivative $y' > 0$ so it will tend to converge to p.

15. A lake holding V cubic meters, is being polluted at the rate of r kg. per month, the pollutant being disolved in run off from a factory which flows into the lake at the rate of U cubic meters per month. The pollutant is thoroughly mixed by fish swimming in the lake and a stream flows out of the lake at the rate of U cubic meters per month. Assuming the rate of evaporation is about the same as the rate water enters because of rain, write a differential equation for A the amount of pollutant in the lake.

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16. Let T_1 be a closed tank of water containing V_1 cubic meters. Let T_2 also be a closed tank of water containing V_2 cubic meters. Polluted water containing k kg. of pollutant per cubic meter enters T_1 at the rate of U cubic meters per minute where it is stirred thoroughly. Then the water flows at the rate of U cubic meters per minute from T_1 to T_2 where it is also stirred and then out of tank T_2 at the same rate. If x is the amount of pollutant in T_1 and y the amount in T_2, write a pair of differential equations for x and y.

    $\frac{dx}{dt} = kU - \frac{x}{V_1}U, \ \frac{dy}{dt} = \frac{x}{V_1}U - \frac{y}{V_2}U$

17. The next few problems involve a discrete version of differential equations. If you have an amount A in the bank for a payment period and r is the interest rate per payment period, then you will receive Ar in interest. This is what the preceding statement says. Thus at the end of one payment period, you have $A(1+r)$ which is the interest added to the original amount. Thus if you begin with an amount $A = A_0$ and it stays in the bank for n payment periods, how much will you have? Note that you have the equation $A_{k+1} = (1+r)A_k$ where A_k is the amount after k periods. You could look for a solution in the form $C\rho^k = A_k$. Find C and ρ.

    You need $C\rho^{k+1} = (1+r)C\rho^k$ and so $\rho = (1+r)$. Then the solution is $A_k = (1+r)^k A_0$

18. For an ordinary annuity, you have a sequence of payments which occur at the end of each payment period. Thus you begin with nothing and at the end of the first payment period, a payment of P is made. Then at the end of the next, another payment is made and so forth till n payment periods. The interest rate per payment period will be r. Determine a formula which will give the amount after n payment periods. **Hint:** Letting A_k be the amount after k periods, explain why

    $$A_{k+1} = A_k(1+r) + P$$

    Thus

    $$A_{k+2} = A_{k+1}(1+r) + P$$

    Subtracting these, you get

    $$A_{k+2} - A_{k+1} = (1+r)(A_{k+1} - A_k)$$

    $$A_{k+2} = (2+r)A_{k+1} - (1+r)A_k \qquad (*)$$

    Look for ρ such that ρ^k solves this. There should be two values. Explain why if ρ^k solves it, then so does $C\rho^k$ for C a constant. Also explain why if $\rho^k, \hat{\rho}^k$ both are solutions, then so is $C\rho^k + D\hat{\rho}^k$. This is called superposition. You have $A_0 = 0, A_1 = P$. These are initial conditions. Now find $\rho, \hat{\rho}, C, D$ to satisfy the difference equation $*$ and these initial conditions.

    You need $\rho^2 = (2+r)\rho - (1+r)$. Thus

    $$\begin{aligned} \rho &= \frac{(2+r) \pm \sqrt{(2+r)^2 - 4(1+r)}}{2} \\ &= \frac{(2+r) \pm \sqrt{r^2}}{2} = \frac{(2+r) \pm r}{2} \\ \rho &= r+1, 1 \end{aligned}$$

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When you plug in $C\rho^k + D\hat{\rho}^k$ into the difference equation, things work out just fine. Thus the solution is of the form

$$C(1+r)^k + D$$

and the constants need to be determined. When $k = 0$, you are supposed to have 0 and then you are supposed to have P when $k = 1$ so you have the two equations

$$C + D = 0$$

$$C(1+r) + D = P$$

Then you find that $C = \frac{P}{r}, D = -\frac{P}{r}$ and so the solution is

$$A_k = \frac{P}{r}(1+r)^k - \frac{P}{r} = P\left(\frac{(1+r)^k - 1}{r}\right)$$

19. You want to pay off a loan of amount Q with a sequence of equal payments, the first occuring at the end of the first payment period, usually month. The rule is that you pay interest on the unpayed balance which is the amount still owed. If you owe Q_k at the end of the k^{th} payment period, then in that payment period, you pay rQ_k where r is the interest rate per period. This is money thrown away. That which remains after removing rQ_k is what goes toward paying off the loan. Thus

$$Q_{k+1} = Q_k - (P - rQ_k) = Q_k(1+r) - P$$

So what is P if $Q_n = 0$ meaning that you pay it off in n payment periods? Determine this by using the technique of the above problem. First find Q_k and then write $Q_n = 0 = Q_{n-1}(1+r) - P$.

It is similar. You have $Q_{k+2} = Q_{k+1}(1+r) - P$ and so subtracting these you get

$$\begin{aligned} Q_{k+2} - Q_{k+1} &= Q_{k+1}(1+r) - Q_k(1+r) \\ Q_{k+2} &= Q_{k+1}(2+r) - Q_k(1+r) \end{aligned}$$

which is exactly the same difference equation as the previous problem. Thus it has the same general solution $C(1+r)^k + D$ only now you want to pay it off when $k = n$ so you have

$$C(1+r)^n + D = 0$$

$$C + D = Q$$

Then $C = -\frac{Q}{(r+1)^n - 1}, D = \frac{Q}{(r+1)^n - 1}(r+1)^n$. Then the solution is $Q_k =$

$$\begin{aligned} &\left(-\frac{Q}{(r+1)^n - 1}\right)(1+r)^k + \left(\frac{Q}{(r+1)^n - 1}(r+1)^n\right) \\ = \; &Q\frac{(r+1)^n - (r+1)^k}{(r+1)^n - 1} \end{aligned}$$

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As to the payment P,

$$P = \left(Q\frac{(r+1)^n - (r+1)^{n-1}}{(r+1)^n - 1} \right)(1+r) = Q\frac{r}{(r+1)^n - 1}(r+1)^n$$
$$= \frac{rQ}{1-(1+r)^{-n}}$$

H.3 Solutions To First Order Scalar Equations

linear equations

1. Find all solutions to the following linear equations.

   (a) $y' + 2ty = e^{-t^2}$, Exact solution is: $\left\{Ce^{-t^2} + \frac{t}{e^{t^2}}\right\}$

   (b) $y' - ty = e^t$

   $\left(ye^{-t^2/2}\right)' = e^{-t^2/2}e^t$

   $y(t)\,e^{-t^2/2} = C + \int_0^t e^{-s^2/2}e^s ds$

   $y(t) = Ce^{t^2/2} + e^{t^2/2}\int_0^t e^{-s^2/2}e^s ds$

   (c) $y' + \cos(t)\,y = \cos(t)$, Exact solution is: $\left\{Ce^{-\sin t} + 1\right\}$

   (d) $y' + ty = \sin(t)$, Exact solution is: $\left\{\frac{1}{e^{\frac{1}{2}t^2}}\int_0^t (\sin s)\,e^{\frac{1}{2}s^2}ds + Ce^{-\frac{1}{2}t^2}\right\}$

   (e) $y' + \frac{1}{t-1}y = \frac{1}{(t-1)^2}$ Exact solution is: $\left\{\frac{\ln(1-t)}{t-1} + \frac{C}{t-1}\right\}$ for t near 0.

   (f) $y' + \tan(t)\,y = \cos(t)$, Exact solution is: $\{C\cos t + t\cos t\}$

   (g) $y' - \tan(t)\,y = \sec(t)$, Exact solution is: $\left\{\frac{C}{\cos t} + \frac{t}{\cos t}\right\}$

   (h) $y' - \tan(t)\,y = \sec^2(t)$, Exact solution is:

   $$\left\{\frac{C}{\cos t} - \frac{1}{2\cos t}\left(\ln(2-2\sin t) - \ln(2\sin t + 2)\right)\right\}$$

2. In the above linear equations find the solution to the initial value problems when $y(0)$ equals the following numbers.

   (a) 1 $e^{-t^2} + \frac{t}{e^{t^2}}$

   (b) 2 $2e^{t^2/2} + e^{t^2/2}\int_0^t e^{-s^2/2}e^s ds$

   (c) 3 $2e^{-\sin t} + 1$

   (d) 4 $\frac{1}{e^{\frac{1}{2}t^2}}\int_0^t (\sin s)\,e^{\frac{1}{2}s^2}ds + 4e^{-\frac{1}{2}t^2}$

   (e) -2 $\frac{\ln(1-t)}{t-1} + \frac{2}{t-1}$

   (f) 12 $12\cos t + t\cos t$

   (g) -3 $\frac{-3}{\cos t} + \frac{t}{\cos t}$

   (h) -2 $\frac{-2}{\cos t} - \frac{1}{2\cos t}\left(\ln(2-2\sin t) - \ln(2\sin t + 2)\right)$