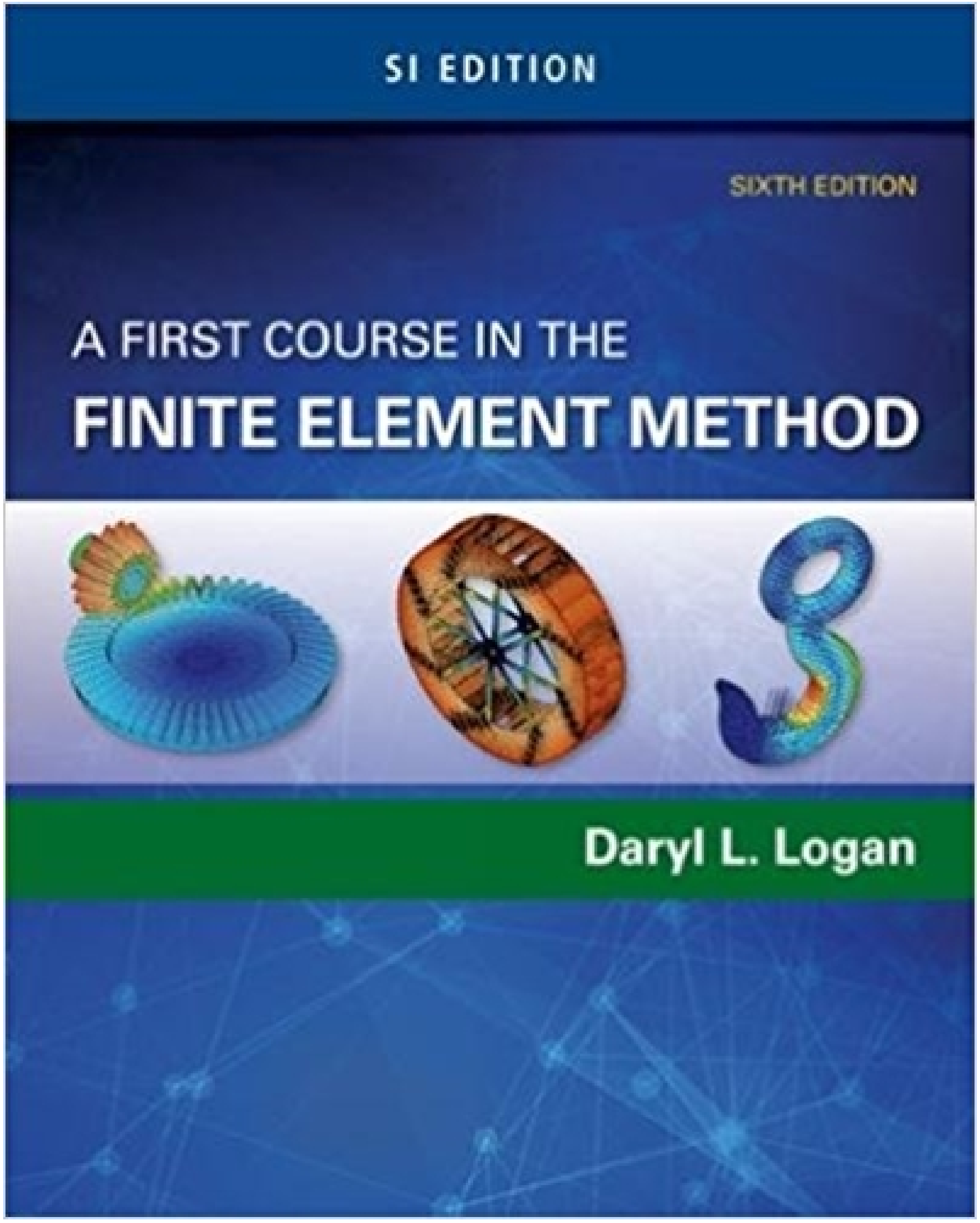
Solutions for A First Course in the Finite Element Method SI Edition 6th Edition by Logan

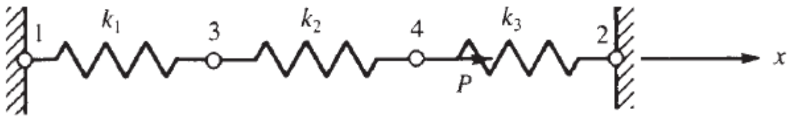


Solutions

Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k_3{}^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\} \Rightarrow [K]^{-1}\,\{F\} = [K]^{-1}\,[K]\,\{d\}$$

$$\Rightarrow \ [K]^{-1}\,\{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$$C_{11} = k_2 + k_3 \qquad C_{21} = (-1)^3\,(-k_2)$$

$$C_{12} = (-1)^{1+2}\,(-k_2) = k_2 \qquad C_{22} = k_1 + k_2$$

$$[C] = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}$$

$$\det [K] = |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - (-k_2)\,(-k_2)$$

$$\Rightarrow \quad |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - k_2^2$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{(k_1 + k_2)(k_2 + k_3) - k_2^{\,2}} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow \quad u_3 = \frac{k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow \quad u_4 = \frac{(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K]\,\{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

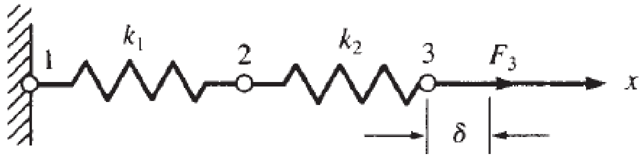
$$F_{1x} = -k_1\,u_3 = -k_1\,\frac{k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1\,k_2\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$F_{2x} = -k_3\,u_4 = -k_3\,\frac{(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3\,(k_1 + k_2)\,P}{k_1\,k_2 + k_1\,k_3 + k_2\,k_3}$$

2.2



$$k_1 = k_2 = k_3 = 100\ \frac{\text{N}}{\text{mm}} = 1000\ \frac{\text{N}}{\text{cm}}$$

$$[k^{(1)}] = \begin{array}{cc} \begin{array}{cc} (1) & (2) \end{array} & \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{array}{c} (1) \\ (2) \end{array} \end{array};\quad [k^{(2)}] = \begin{array}{cc} \begin{array}{cc} (2) & (3) \end{array} & \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{array}{c} (2) \\ (3) \end{array} \end{array}$$

By the method of superposition the global stiffness matrix is constructed.

$$[K] = \begin{array}{cc} \begin{array}{ccc} (1) & (2) & (3) \end{array} & \\ \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} & \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \end{array} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} k & k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = \delta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 & = & 2k\,u_2 - k\delta \\ F_{3x} & = & -k\,u_2 + k\delta \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{k\delta}{2k} = \frac{\delta}{2} = \frac{20\text{ mm}}{2} \Rightarrow u_2 = 10\text{ mm} = 1\text{ cm}$$

$$F_{3x} = -k\,(1\text{ cm}) + k\,(2\text{ cm})$$

$$F_{3x} = (-1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) + (1000\ \frac{\text{N}}{\text{cm}})\,(2\text{ cm})$$

$$F_{3x} = 1000\text{ N}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 1\text{ cm} \end{Bmatrix}$$

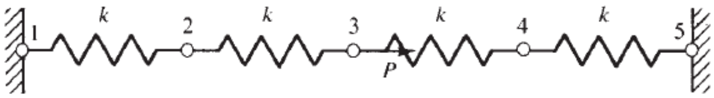
$$\Rightarrow \quad f_{1x}^{(1)} = (-1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) \Rightarrow f_{1x}^{(1)} = -1000\text{ N}$$

$$f_{2x}^{(1)} = (1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) \Rightarrow f_{2x}^{(1)} = 1000\text{ N}$$

Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 = 1\text{ cm} \\ u_3 = 2\text{ cm} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -1000\text{ N} \\ f_{3x}^{(2)} = 1000\text{ N} \end{matrix}$$

2.3



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K]\{d\}$ we have

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix}$$

(b) $$\begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{matrix} 0 = 2ku_2 - ku_3 & (1) \\ P = -ku_2 + 2ku_3 - ku_4 & (2) \\ 0 = -ku_3 + 2ku_4 & (3) \end{matrix}$$

$$\Rightarrow \quad u_2 = \frac{u_3}{2} \; ; u_4 = \frac{u_3}{2}$$

Substituting in the second equation above

$$P = -k\, u_2 + 2k\, u_3 - k\, u_4$$

$$\Rightarrow \quad P = -k\left(\frac{u_3}{2}\right) + 2k\, u_3 - k\left(\frac{u_3}{2}\right)$$

$$\Rightarrow \quad P = k\, u_3$$

$$\Rightarrow \quad u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k} \; ; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K]\{d\}$

$$F_{1x} = -k\, u_2 = -k\,\frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k\, u_4 = -k\,\frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

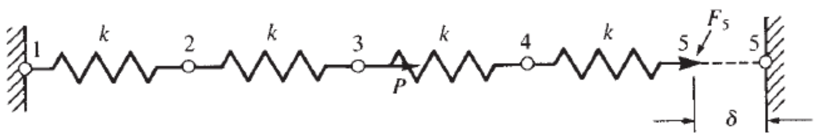
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2}\right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K]\,\{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{Bmatrix}$$

(b) $0 = 2k\, u_2 - k\, u_3$ (1)

$0 = -ku_2 + 2k\, u_3 - k\, u_4$ (2)

$0 = -k\, u_3 + 2k\, u_4 - k\, \delta$ (3)

From (2)

$$u_3 = 2\, u_2$$

From (3)

$$u_4 = \frac{\delta + 2\, u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -k\,(u_2) + 2k\,(2\, u_2) - k\left(\frac{\delta + 2u_2}{2}\right)$$

$$\Rightarrow -u_2 + 4\, u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2\frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\,\frac{\delta}{4}}{2} \Rightarrow u_4 = \frac{3\,\delta}{4}$$

(c) Going back to the global equation

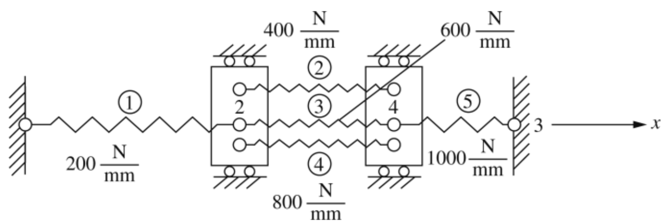
$$\{F\} = [K]\,\{d\}$$

$$F_{1x} = -k\ u_2 = -k\,\frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k\,\delta}{4}$$

$$F_{5x} = -k\,u_4 + k\,\delta = -k\left(\frac{3\,\delta}{4}\right) + k\,\delta$$

$$\Rightarrow\ F_{5x} = \frac{k\,\delta}{4}$$

2.5



$$[k^{(1)}] = \begin{array}{c} \begin{matrix} d_1 & \quad d_2 \end{matrix} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array}; \quad [k^{(2)}] = \begin{array}{c} \begin{matrix} d_2 & \quad d_4 \end{matrix} \\ \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \begin{array}{c} \begin{matrix} d_2 & \quad d_4 \end{matrix} \\ \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \end{array}; \quad [k^{(4)}] = \begin{array}{c} \begin{matrix} d_2 & \quad d_4 \end{matrix} \\ \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \begin{array}{c} \begin{matrix} d_4 & \quad d_3 \end{matrix} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{array}$$

Assembling global $[K]$ using direct stiffness method

$$[K] = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 200+400+600+800 & 0 & -400-600-800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -400-600-800 & -1000 & 400+600+800+1000 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 2000 & 0 & -1800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -1800 & -1000 & 2800 \end{bmatrix} \frac{\text{N}}{\text{mm}}$$

2.6 Now apply + 10,000 N at node 2 in spring assemblage of P 2.5.

$$\therefore\ F_{2x} = 10{,}000\ \text{N}$$

$$[K]\{d\} = \{F\}$$

$[K]$ from P 2.5

$$\begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 2000 & 0 & -1800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -1800 & -1000 & 2800 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 10,000 \\ F_3 \\ 0 \end{Bmatrix} \quad \text{(A)}$$

where $u_1 = 0$, $u_3 = 0$ as nodes 1 and 3 are fixed.

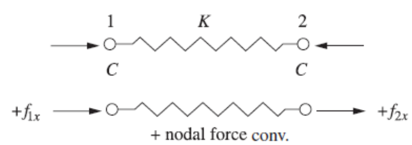
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 2000 & -1800 \\ -1800 & 2800 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 10,000 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 11.86 \text{ mm}, \quad u_4 = 7.63 \text{ mm}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

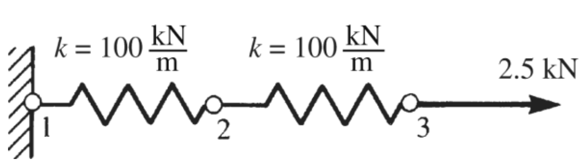
$$\therefore \quad f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{array}{l} \text{same as for} \\ \text{tensile element} \end{array}$$

2.8



$$k_1 = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \; k_2 = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

$$[K] = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\Rightarrow \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 2500 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 2 \times 10^5\ u_2 - 10^5\ u_3 \qquad (1)$$

$$2500 = -10^5\, u_2 + 10^5\, u_3 \qquad (2)$$

From (1)

$$u_2 = \frac{10^5}{2 \times 10^5}\ u_3 \Rightarrow u_2 = 0.5\, u_3 \qquad (3)$$

Substituting (3) into (2)

$$\Rightarrow \quad 2500 = -10^5\,(0.5\, u_3) + 10^5\, u_3$$

$$\Rightarrow \quad 2500 = 0.5 \times 10^5\ u_3$$

$$\Rightarrow \quad u_3 = 0.05 \text{ m} = 5 \text{ cm}$$

$$\Rightarrow \quad u_2 = (0.5)\,(5 \text{ cm}) \Rightarrow u_2 = 2.5 \text{ cm}$$

Element 1–2

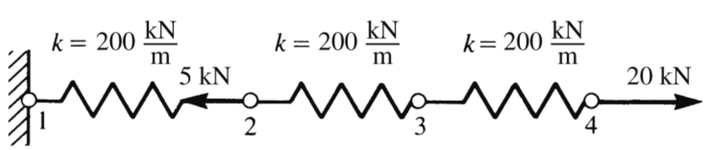
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.025 \text{ m} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -2500 \text{ N} \\ f_{2x}^{(1)} = \ \ 2500 \text{ N} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.025 \text{ m} \\ 0.05 \text{ m} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -2500 \text{ N} \\ f_{3x}^{(2)} = \ \ 2500 \text{ N} \end{matrix}$$

$$F_{1x} = 10^5\,[1\ \ -1\ \ 0] \begin{bmatrix} 0 \\ 0.025 \text{ m} \\ 0.05 \text{ m} \end{bmatrix} \Rightarrow F_{1x} = -2500 \text{ N}$$

2.9



$$[k^{(1)}] = 10^3 \begin{array}{c} \begin{matrix} \ \ (1) & \ \ (2) \end{matrix} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array}$$

$$[k^{(2)}] = 10^3 \begin{array}{cc} (2) & (3) \\ \end{array}\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[k^{(3)}] = 10^3 \begin{array}{cc} (3) & (4) \\ \end{array}\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[K] = 10^3 \begin{array}{cccc} (1) & (2) & (3) & (4) \\ \end{array}\begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & 0 \\ 0 & -200 & 400 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -5000 \\ F_{3x} = 0 \\ F_{4x} = 20{,}000 \end{Bmatrix} = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & 0 \\ 0 & -200 & 400 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_1 &= 0 \\ u_2 &= 0.075 \text{ m} = 7.5 \text{ cm} \\ u_3 &= 0.175 \text{ m} = 17.5 \text{ cm} \\ u_4 &= 0.275 \text{ m} = 27.5 \text{ cm} \end{aligned}$$

Reactions

$$F_{1x} = 10^3 \; [200 \quad -200 \quad 0 \quad 0] \begin{Bmatrix} u_1 = & 0 \\ u_2 = & 0.075 \\ u_3 = & 0.175 \\ u_4 = & 0.275 \end{Bmatrix} \Rightarrow F_{1x} = -15000 \text{ N}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.075 \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x}^{(1)} &= -15{,}000 \text{ N} \\ f_{2x}^{(1)} &= 15{,}000 \text{ N} \end{aligned}$$

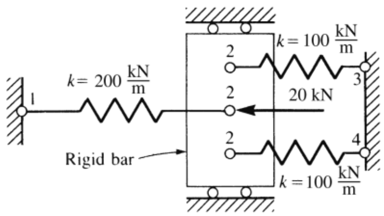
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.075 \\ 0.175 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2x}^{(2)} &= -20{,}000 \text{ N} \\ f_{3x}^{(2)} &= 20{,}000 \text{N} \end{aligned}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.175 \\ 0.275 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= -20{,}000 \\ f_{4x}^{(3)} &= 20{,}000 \end{aligned}$$

2.10



$$[k^{(1)}] = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[k^{(2)}] = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}$$

$$[k^{(3)}] = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -20,000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -100 & -100 \\ 0 & -100 & 100 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \qquad u_2 = \frac{-20,000}{400 \times 10^3} = -0.05 \text{ m} = -5 \text{ cm}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -100 & -100 \\ 0 & -100 & 100 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.05 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 10,000 \\ -20,000 \\ 5000 \\ 5000 \end{Bmatrix} \text{N}$$

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.05 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 10,000 \\ -10,000 \end{Bmatrix} \text{N}$$

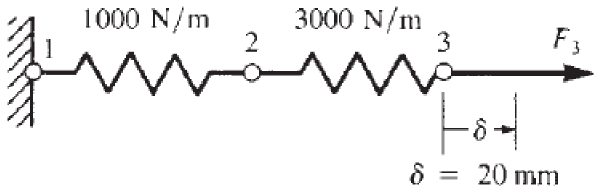
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} -0.05 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -5000 \\ 5000 \end{Bmatrix} \text{N}$$

Element (3)

$$\begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} -0.05 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} \begin{Bmatrix} -5000 \\ 5000 \end{Bmatrix} \text{N}$$

2.11



$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 4000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.02 \text{ m} \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0.015 \text{ m}$$

Reactions

$$F_{1x} = (-1000)\,(0.015) \Rightarrow F_{1x} = -15 \text{ N}$$

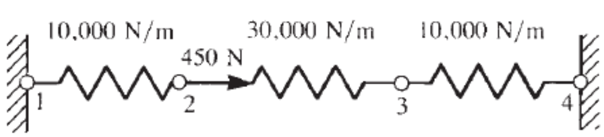
Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.015 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 3 & -3 \\ -3 & 3 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 450 \text{ N} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$0 = -3\,u_2 + 4\,u_3 \Rightarrow u_2 = \frac{4}{3}\,u_3 \Rightarrow u_2 = 1.33\,u_3$$

$$450 \text{ N} = 40000\,(1.33\,u_3) - 30000\,u_3$$

$$\Rightarrow\ 450 \text{ N} = (23200\ \frac{\text{N}}{\text{m}})\,u_3 \Rightarrow u_3 = 1.93 \times 10^{-2}\text{ m}$$

$$\Rightarrow\ u_2 = 1.5\,(1.94 \times 10^{-2}) \Rightarrow u_2 = 2.57 \times 10^{-2}\text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}{}^{(1)} = -257\text{ N} \\ f_{2x}{}^{(1)} = 257\text{ N} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 30000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.57 \times 10^{-2} \\ 1.93 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}{}^{(2)} = 193\text{ N} \\ f_{3x}{}^{(2)} = -193\text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}{}^{(3)} = 193\text{ N} \\ f_{4x}{}^{(3)} = -193\text{ N} \end{matrix}$$

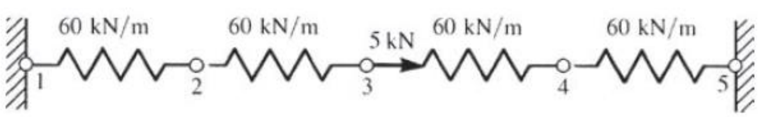
Reactions

$$\{F_{1x}\} = (10000\ \frac{\text{N}}{\text{m}})\,[1 - 1] \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -257\text{ N}$$

$$\{F_{4x}\} = (10000\ \frac{\text{N}}{\text{m}})\,[-1\ \ 1] \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ F_{4x} = -193\text{ N}$$

2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 5\text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 60 \begin{Bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

$$\left.\begin{array}{l} 0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5\,u_3 \\ 0 = -u_3 + 2u_4 \Rightarrow u_4 = 0.5\,u_3 \end{array}\right\} \Rightarrow u_2 = u_4$$

$$\Rightarrow \quad 5 \text{ kN} = -60\,u_2 + 120\,(2\,u_2) - 60\,u_2$$

$$\Rightarrow \quad 5 = 120\,u_2 \Rightarrow u_2 = 0.042 \text{ m}$$

$$\Rightarrow \quad u_4 = 0.042 \text{ m}$$

$$\Rightarrow \quad u_3 = 2(0.042) \Rightarrow u_3 = 0.084 \text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{1x}^{(1)} = -2.5 \text{ kN} \\ f_{2x}^{(1)} = 2.5 \text{ kN} \end{array}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0.084 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2x}^{(2)} = -2.5 \text{ kN} \\ f_{3x}^{(2)} = 2.5 \text{ kN} \end{array}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.084 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{3x}^{(3)} = 2.5 \text{ kN} \\ f_{4x}^{(3)} = -2.5 \text{ kN} \end{array}$$

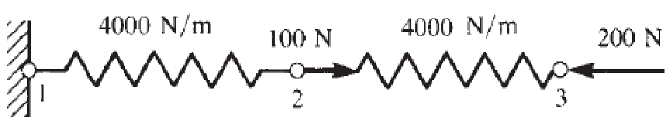
Element (4)

$$\begin{Bmatrix} f_{4x} \\ f_{5x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{4x}^{(4)} = 2.5 \text{ kN} \\ f_{5x}^{(4)} = -2.5 \text{ kN} \end{array}$$

$$F_{1x} = 60\,[1 \quad -1] \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow F_{1x} = -2.5 \text{ kN}$$

$$F_{5x} = 60\,[-1 \quad 1] \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5 \text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 100 \\ F_{3x} = -200 \end{Bmatrix} = 4000 \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{0} \\ \cancel{1} & 2 & -1 \\ \cancel{0} & -1 & 1 \end{bmatrix} \begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$100 = 8000\,u_2 - 4000\,u_3$$

$$-200 = -4000\,u_2 + 4000\,u_3$$

$$-100 = 4000\, u_2 \Rightarrow u_2 = -0.025 \text{ m}$$

$$100 = 8000\,(-0.025) - 4000\, u_3 \Rightarrow u_3 = -0.075 \text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.025 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{1x}^{(1)} = 100 \text{ N} \\ f_{2x}^{(1)} = -100 \text{ N} \end{array}$$

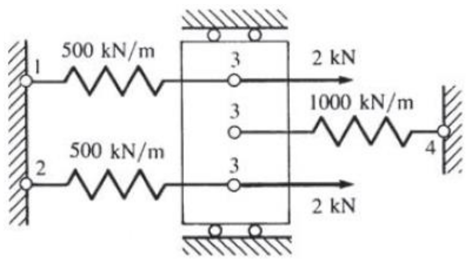
Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.025 \\ -0.075 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2x}^{(2)} = 200 \text{ N} \\ f_{3x}^{(2)} = -200 \text{ N} \end{array}$$

Reaction

$$\{F_{1x}\} = 4000\,[1 \quad -1] \begin{Bmatrix} 0 \\ -0.025 \end{Bmatrix} \Rightarrow F_{1x} = 100 \text{ N}$$

2.15



$$[k^{(1)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix};\ [k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix};\ [k^{(3)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = ? \\ F_{3x} = 4 \text{ kN} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & 0 & -500 & 0 \\ 0 & 500 & -500 & 0 \\ -500 & -500 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_3 = 0.002 \text{ m}$$

Reactions

$$F_{1x} = (-500)\,(0.002) \Rightarrow F_{1x} = -1.0 \text{ kN}$$

$$F_{2x} = (-500)\,(0.002) \Rightarrow F_{2x} = -1.0 \text{ kN}$$

$$F_{4x} = (-1000)\,(0.002) \Rightarrow F_{4x} = -2.0 \text{ kN}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -1.0 \text{ kN} \\ 1.0 \text{ kN} \end{Bmatrix}$$

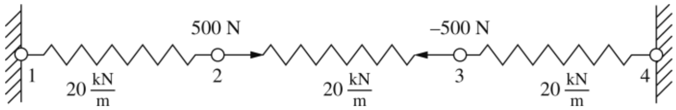
Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -1.0 \text{ kN} \\ 1.0 \text{ kN} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0.002 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{Bmatrix} 2.0 \text{ kN} \\ -2.0 \text{ kN} \end{Bmatrix}$$

2.16



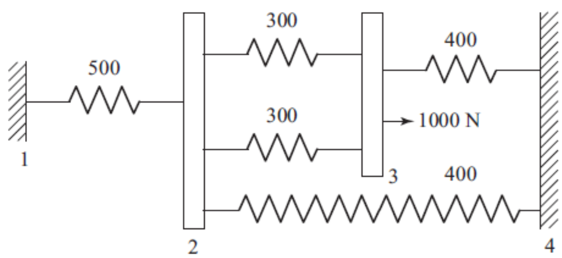
$$\begin{Bmatrix} F_{1x} \\ 500 \\ -500 \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 20 & -200 & 0 & 0 \\ -20 & 20+20 & -20 & 0 \\ 0 & -20 & 20+20 & -20 \\ 0 & 0 & -20 & 20 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 500 \\ -500 \end{Bmatrix} = 10^3 \begin{Bmatrix} 40 & -20 \\ -20 & 40 \end{Bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_2 = 8.33 \times 10^{-3}\,\text{m} = 8.33\,\text{mm}$$

$$u_3 = -8.33 \times 10^{-3}\,\text{m} = -8.33\,\text{mm}$$

2.17



$$\begin{Bmatrix} F_{1x} = ? \\ 0 \\ 1000\text{ N} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} \cancel{500} & \cancel{-500} & \cancel{0} & \cancel{0} \\ \cancel{-500} & \begin{pmatrix} 400+300 \\ 500+300 \end{pmatrix} & -300-300 & \cancel{-400} \\ \cancel{0} & -300-300 & (300+300+400) & \cancel{-400} \\ \cancel{0} & -400 & -400 & \cancel{400+400} \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$0 = 1500\, u_2 - 600\, u_3$$

$$1000 = -600\, u_2 + 1000\, u_3$$

$$u_3 = \frac{15\cancel{0}\cancel{0}}{6\cancel{0}\cancel{0}}\, u_2 = 2.5\, u_2$$

$$1000 = -600\, u_2 + 1000\,(2.5\, u_2)$$

$$1000 = 1900\, u_2$$

$$u_2 = \frac{1000}{1900} = \frac{1}{1.9}\,\text{mm} = 0.526\,\text{mm}$$

$$u_3 = 2.5\left(\frac{1}{1.9}\right)\,\text{mm} = 1.316\,\text{mm}$$

$$F_{1x} = -500\left(\frac{1}{1.9}\right) = -263.16 \text{ N}$$

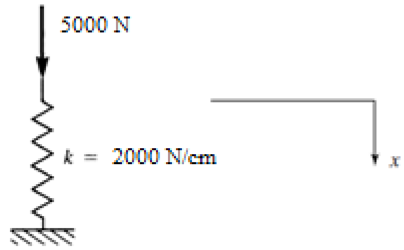
$$F_{4x} = -400\left(\frac{1}{1.9}\right) - 400\left(2.5\left(\frac{1}{1.9}\right)\right)$$

$$= -400\left(\frac{1}{1.9} + \frac{2.5}{1.9}\right) = -736.84 \text{ N}$$

$$\Sigma F_x = -263.16 + 1000 - 736.84 = 0$$

2.18

(a)



As in Example 2.4

$$\pi_p = U + \Omega$$

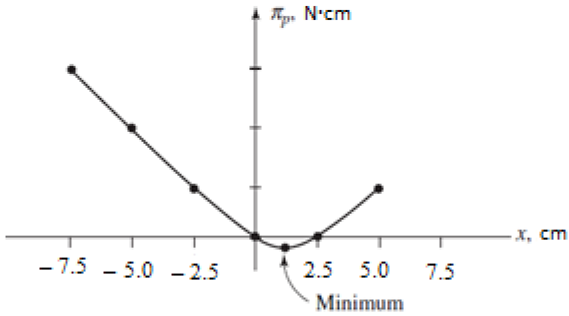
$$U = \frac{1}{2}\, k\, x^2,\ \Omega = -Fx$$

Set up table

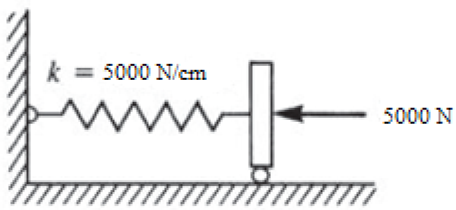
$$\pi_p = \frac{1}{2}(2000)\, x^2 - 5000\, x = 1000\, x^2 - 5000\, x$$

Deformation x, cm	π_p, N·cm
– 7.5	93,750
– 5.0	50,000
– 2.5	18,750
0.0	0
0.5	6250
2.5	0
7.5	18,750

$$\frac{\partial \pi_p}{\partial x} = 2000\, x - 5000 = 0 \Rightarrow x = 2.5 \text{ cm yields minimum } \pi_p \text{ as table verifies.}$$



(b)



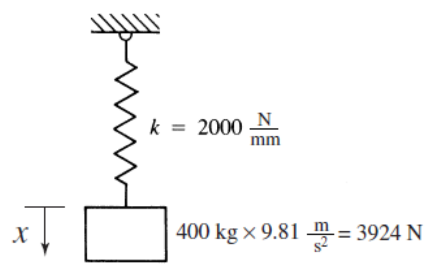
$$\pi_p = \frac{1}{2}\ kx^2 - F_x = 2500\ x^2 - 5000\ x$$

x, cm	π_p, N·cm
– 3.0	37,5000
– 2.0	20,000
– 1.0	7500
0	0
1.0	– 2500
2.0	0
3.0	7500

$$\frac{\partial \pi_p}{\partial x} = 5000\ x - 5000 = 0$$

$\Rightarrow \quad x = 1.0$ in. yields π_p minimum

(c)



$$\pi_p = \frac{1}{2}\ (2000)\ x^2 - 3924\ x = 1000\ x^2 - 3924\ x$$

$$\frac{\partial \pi_p}{\partial x} = 2000\,x - 3924 = 0$$

$$\Rightarrow \quad x = 1.962 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}\,(2000)\,(1.962)^2 - 3924\,(1.962)$$

$$\Rightarrow \pi_{p\,\min} = -\,3849.45 \text{ N·mm}$$

(d) $$\pi_p = \frac{1}{2}\,(400)\,x^2 - 981\,x$$

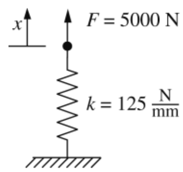
$$\frac{\partial \pi_p}{\partial x} = 400\,x - 981 = 0$$

$$\Rightarrow \quad x = 2.4525 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}\,(400)\,(2.4525)^2 - 981\,(2.4525)$$

$$\Rightarrow \pi_{p\,\min} = -\,1202.95 \text{ N·mm}$$

2.19



Now let positive x be upward

$$\pi_p = \frac{1}{2}\,kx^2 - Fx$$

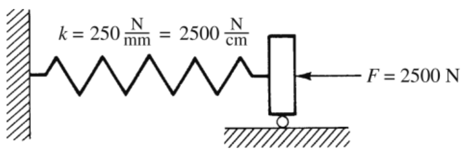
$$\pi_p = \frac{1}{2}\,(125)\,x^2 - 5000\,x$$

$$\pi_p = (62.5)x^2 - 5000\,x$$

$$\frac{\partial \pi_p}{\partial x} = (125)x - 5000 = 0$$

$$\Rightarrow x = 40.0 \text{ mm} \uparrow$$

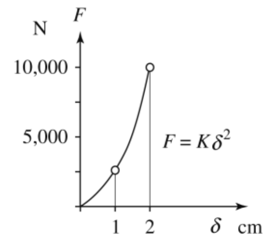
2.20



$$F = k\delta^2 \qquad (x = \delta)$$

$$dU = F\,dx$$

$$U = \int_0^x (kx^2)\, dx$$



$$U = \frac{k x^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3}\ kx^3 - 2500\ x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 2500$$

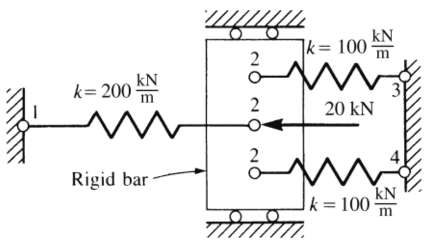
$$0 = 2500\ x^2 - 2500$$

$$\Rightarrow \quad x = 1.0 \text{ cm} \quad \text{(equilibrium value of displacement)}$$

$$\pi_{p\ \min} = \frac{1}{3}\ (2500)\ (1.0)^3\ (2500)\ (7.0)$$

$$\pi_{p\ \min} = -1666.7 \text{ N} \cdot \text{cm}$$

2.21 Solve Problem 2.10 using P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2}\ k_1 (u_2 - u_1)^2 + \frac{1}{2}\ k_2 (u_3 - u_2)^2 + \frac{1}{2}\ k_3 (u_4 - u_2)^2$$

$$- f_{1x}^{(1)} u_1 - f_{2x}^{(1)} u_2 - f_{2x}^{(2)} u_2$$

$$- f_{3x}^{(2)} u_3 - f_{2x}^{(3)} u_2 - f_{4x}^{(3)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = -k_1 u_2 + k_1 u_1 - f_{1x}^{(1)} = 0 \qquad (1)$$

$$\frac{\partial \pi_p}{\partial u_2} = k_1 u_2 - k_1 u_1 - k_2 u_3 + k_2 u_2 - k_3 u_4$$

$$+ k_3 u_2 - f_{2x}^{(1)} - f_{2x}^{(2)} - f_{2x}^{(3)} = 0 \qquad (2)$$

$$\frac{\partial \pi_p}{\partial u_3} = k_2 u_3 - k_2 u_2 - f_{3x}^{(2)} = 0 \qquad (3)$$

$$\frac{\partial \pi_p}{\partial u_4} = k_3\, u_4 - k_3\, u_2 - f_{4x}^{(3)} = 0 \tag{4}$$

In matrix form (1) through (4) become

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \\ f_{3x}^{(2)} \\ f_{4x}^{(3)} \end{Bmatrix} \tag{5}$$

or using numerical values

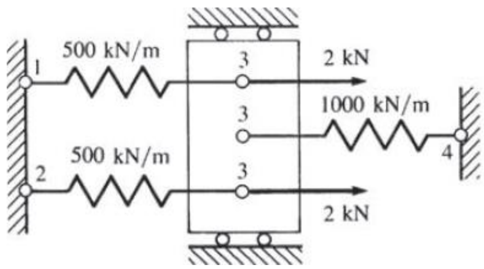
$$10^3 \times \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & -200 \\ 0 & -200 & 200 & 0 \\ 0 & -200 & 0 & 200 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ -20{,}000 \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{6}$$

Solution now follows as in Problem 2.10

Solve 2nd of Equations (6) for $u_2 = -\,0.05$ m = 5 cm

For reactions and element forces, see solution to Problem 2.10

2.22 Solve Problem 2.15 by P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2}\, k_1\,(u_3 - u_1)^2 + \frac{1}{2}\, k_2\,(u_3 - u_2)^2$$

$$+ \frac{1}{2}\, k_3\,(u_4 - u_3)^2 - f_{1x}^{(1)}\, u_1$$

$$- f_{3x}^{(1)}\, u_3 - f_{2x}^{(2)}\, u_2 - f_{3x}^{(2)}\, u_3$$

$$- f_{3x}^{(3)}\, u_3 - f_{3x}^{(4)}\, u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -\,k_1\, u_3 + k_1\, u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = -\,k_2\, u_3 + k_2\, u_2 - f_{2x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1\, u_3 + k_2\, u_3 - k_2\, u_2 - k_3\, u_4 + k_3\, u_3 - f_{3x}^{(2)} - f_{3x}^{(3)} - f_{3x}^{(1)} - k_1\, u_1$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_3 u_4 - k_3 u_3 - f_{3x}^{(4)}$$

In matrix form

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1 + k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} = 4\text{ kN} \\ F_{4x} \end{Bmatrix}$$

For rest of solution, see solution of Problem 2.15.

2.23

$$I = a_1 + a_2 x$$

$$I(0) = a_1 = I_1$$

$$I(L) = a_1 + a_2 L = I_2$$

$$a_2 = \frac{I_2 - I_1}{L}$$

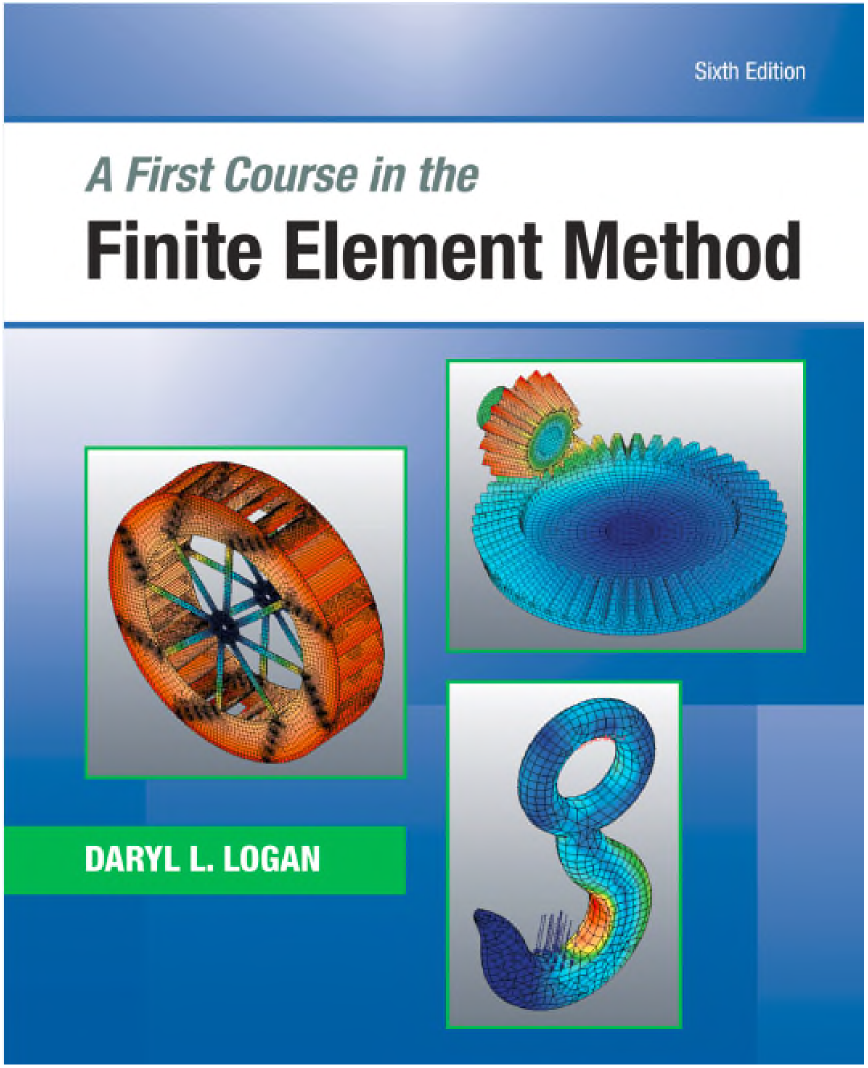
$$\therefore \qquad I = I_1 + \frac{I_2 - I_1}{L} x$$

Now $V = IR$

$$V = -V_1 = R(I_2 - I_1)$$

$$V = V_2 = R(I_2 - I_1)$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = R \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$



Chapter 2

Introduction to the Stiffness (Displacement) Method

Learning Objectives

- To define the stiffness matrix
- To derive the stiffness matrix for a spring element
- To demonstrate how to assemble stiffness matrices into a global stiffness matrix
- To illustrate the concept of direct stiffness method to obtain the global stiffness matrix and solve a spring assemblage problem
- To describe and apply the different kinds of boundary conditions relevant for spring assemblages
- To show how the potential energy approach can be used to both derive the stiffness matrix for a spring and solve a spring assemblage problem

Definition of the Stiffness Matrix

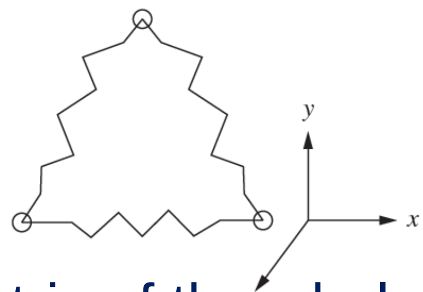
- For an element, a stiffness matrix [k] is a matrix such that:

$$\{f\} = [k]\{d\}$$

Where [k] relates nodal displacements {d} to nodal forces {f} of a single element, such as to the single spring element below

Definition of the Stiffness Matrix

- For a structure comprising of a series of elements such as the three-spring assemblage shown below:

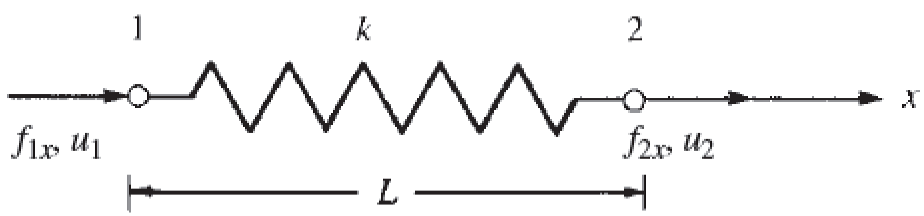


- The stiffness matrix of the whole spring assemblage [K] relates global-coordinate nodal displacements {d} to global forces {F} by the relation:

$$\{F\} = [K]\{d\}$$

Derivation of the Stiffness Matrix for a Spring Element

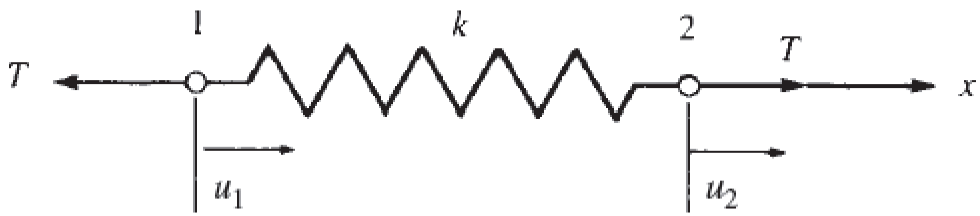
- Consider the following linear spring element:



- Points 1 and 2 are reference points called nodes
- f_{1x} and f_{2x} are the local nodal forces on the x-axis
- μ_1 and μ_2 are the local nodal displacements
- k is the spring constant or stiffness of the spring
- L is the distance between the nodes

Derivation of the Stiffness Matrix for a Spring Element

- We have selected our element type and now need to define the deformation relationships



- For the spring subject to tensile forces at each node:

  $\delta = \mu_2 - \mu_1$ & $T = k\delta$

  Where δ is the total deformation and T is the tensile force
- Combine to obtain: $T = k(\mu_2 - \mu_1)$

Derivation of the Stiffness Matrix for a Spring Element

- Performing a basic force balance yields:

$$f_{1x} = -T \qquad f_{2x} = T$$

- Combining these force eqs with the previous eqs:

$$f_{1x} = k(u_1 - u_2)$$

$$f_{2x} = k(u_2 - u_1)$$

- Express in matrix form:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Derivation of the Stiffness Matrix for a Spring Element

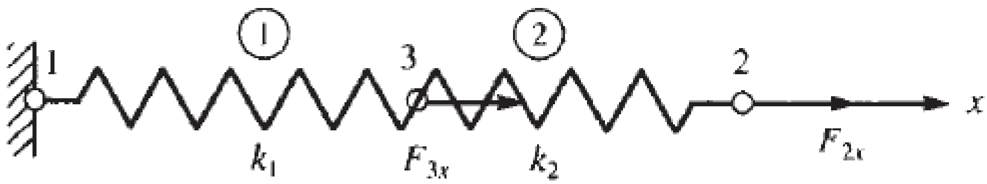
- The stiffness matrix for a linear element is derived as:

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

- Here [k] is called the local stiffness matrix for the element.
- Observe that this matrix is symmetric, is square, and is singular.
- This was the basic process of deriving the stiffness matrix for any element.

Establishing the Global Stiffness Matrix for a Spring Assemblage

- Consider the two-spring assemblage:



- Node 1 is fixed and axial forces are applied at nodes 3 and 2.
- The x-axis is the global axis of the assemblage.

Establishing the Global Stiffness Matrix for a Spring Assemblage

- For element 1:

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_3^{(1)} \end{Bmatrix}$$

- For element 2:

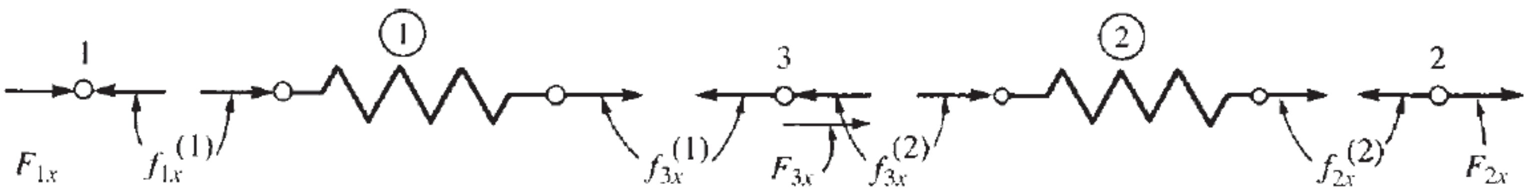
$$\begin{Bmatrix} f_{3x}^{(2)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3^{(2)} \\ u_2^{(2)} \end{Bmatrix}$$

- Elements 1 and 2 must remain connected at common node 3. The is called the <u>continuity or compatibility requirement</u> given by:

$$u_3^{(1)} = u_3^{(2)} = u_3$$

Establishing the Global Stiffness Matrix for a Spring Assemblage

- From the Free-body diagram of the assemblage:



- We can write the equilibrium nodal equations:

$$F_{3x} = f_{3x}^{(1)} + f_{3x}^{(2)} \qquad F_{2x} = f_{2x}^{(2)} \qquad F_{1x} = f_{1x}^{(1)}$$

Establishing the Global Stiffness Matrix for a Spring Assemblage

- Combining the nodal equilibrium equations with the elemental force/displacement/stiffness relations we obtain the global relationship:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

- Which takes the form: {F} = [K]{d}
- {F} is the global nodal force matrix
- {d} is the global nodal displacement matrix
- [K] is the total or global or system stiffness matrix

Direct Stiffness Method

- Reliable method of directly assembling individual element stiffness matrices to form the total structure stiffness matrix and the total set of stiffness equations
- Individual element stiffness matrices are superimposed to obtain the global stiffness matrix.
- To superimpose the element matrices, they must be expanded to the order (size) of the total structure stiffness matrix.

Boundary Conditions

- We must specify boundary (or support) conditions for structure models or [K] will be singular.
- This means that the structural system is unstable.
- Without specifying proper kinematic constraints or support conditions, the structure will be free to move as a rigid body and not resist any applied loads.
- In general, the number of boundary conditions necessary is equal to the number of possible rigid body modes.

Boundary Conditions

- Homogeneous boundary conditions
  - Most common type
  - Occur at locations completely prevented from moving
  - Zero degrees of freedom
- Nonhomogeneous boundary conditions
  - Occur where finite nonzero values of displacements are specified
  - Nonzero degree of freedom
  - i.e. the settlement of a support

Homogenous Boundary Conditions

- Where is the homogenous boundary condition for the spring assemblage?
- It is at the location which is fixed, Node 1
- Because Node 1 is fixed $\mu_1 = 0$
- The system relation can be written as:

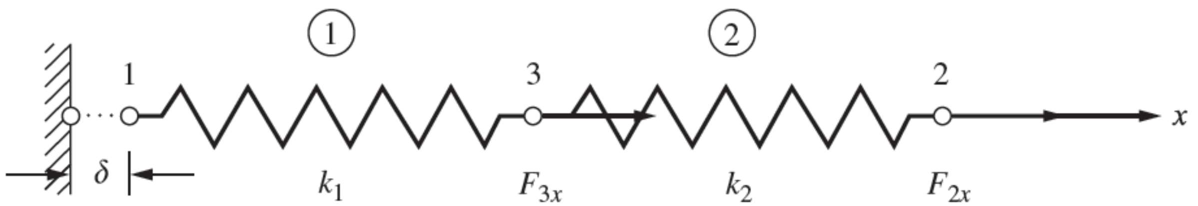
$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

Homogenous Boundary Conditions

- For all homogenous boundary conditions, we can delete the row and columns corresponding to the zero-displacement degrees of freedom.
- This makes solving for the unknown displacements possible.
- Appendix B.4 presents a practical, computer-assisted scheme for solving systems of simultaneous equations.

Nonhomogeneous Boundary Conditions

- Consider the case where there is a known displacement, δ, at Node 1



- Let $\mu_1 = \delta$

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

Nonhomogeneous Boundary Conditions

- By considering only the second and third force equations we can arrive at the equation:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ k_1\delta + F_{3x} \end{Bmatrix}$$

- It can be seen that for nonhomogeneous boundary conditions we cannot initially delete row 1 and column 1 like was done for homogenous boundary conditions.

Nonhomogeneous Boundary Conditions

- In general for nonhomogeneous boundary conditions, we must transform the terms associated with the known displacements to the force matrix before solving for the unknown nodal displacements.

Minimum Potential Energy Approach

- Alternative method often used to derive the element equations and stiffness matrix.
- More adaptable to the determination of element equations for complicated elements such as:
  - Plane stress/strain element
  - Axisymmetric stress element
  - Plate bending element
  - Three-dimensional solid stress element

Minimum Potential Energy Approach

- Principle of minimum potential energy is only applicable to elastic materials.
- Categorized as a “variational method” of FEM
- Use the potential energy approach to derive the spring element equations as we did earlier with the direct method.

Total Potential Energy

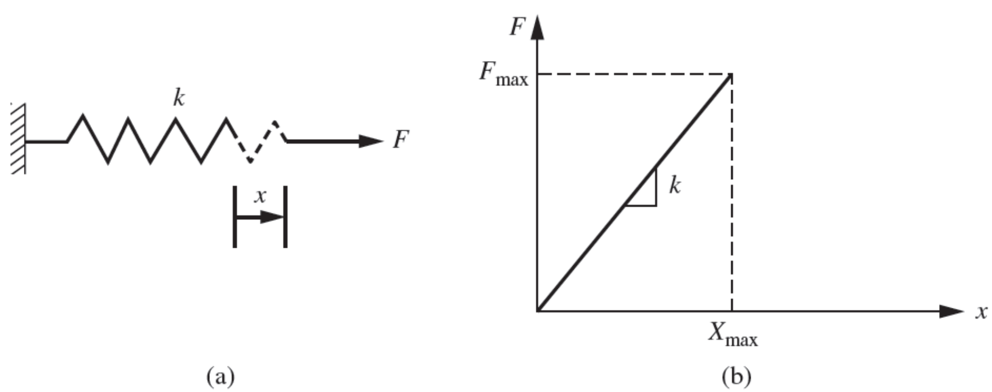
- Defined as the sum of the internal strain energy, U, and the potential energy of the external forces, Ω

$$\pi_p = U + \Omega$$

- Strain energy is the capacity of internal forces to do work through deformations in the structure.
- The potential energy of external forces is the capacity of forces such as body forces, surface traction forces, or applied nodal forces to do work through deformation of the structure.

Concept of External Work

- A force is applied to a spring and the force-deformation curve is given.
- The external work is given by the area under the force-deformation curve where the slope is equal to the spring constant k



External Work and Internal Strain Energy

- From basic mechanics principles the external work is expressed as:

$$W_e = \int F \cdot dx = \int_0^{x_{\max}} F_{\max}\left(\frac{x}{x_{\max}}\right) dx = F_{\max} x_{\max}/2$$

- From conservation of mechanical energy principle external work is expressed as:

$$W_e = U = F_{\max} x_{\max}/2$$

- For when the external work is transformed into the internal strain energy of the spring

Total Potential Energy of Spring

- The strain energy can be expressed as:

$$U = kx_{\max}^2/2$$

- The potential energy of the external force can be expressed as:

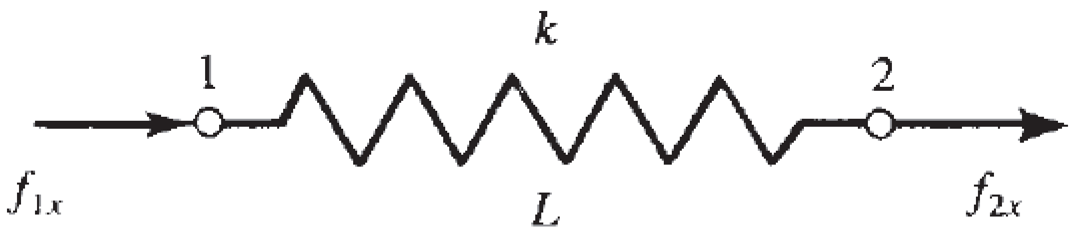
$$\Omega = -F_{\max}x_{\max}$$

- Therefore, the total potential energy of a spring is:

$$\pi_p = \frac{1}{2}kx_{\max}^2 - F_{\max}x_{\max}$$

Potential Energy Approach to Derive Spring Element Eqs.

- Consider the linear spring subject to nodal forces:



- The total potential energy is:

$$\pi_p = \frac{1}{2}k(u_2 - u_1)^2 - f_{1x}u_1 - f_{2x}u_2$$

Potential Energy Approach to Derive Spring Element Eqs.

- To minimize the total potential energy the partial derivatives of π_p with respect to each nodal displacement must be taken:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{1}{2}k(-2u_2 + 2u_1) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{1}{2}k(2u_2 - 2u_1) - f_{2x} = 0$$

Potential Energy Approach to Derive Spring Element Eqs.

- Simplify to:

$$k(-u_2 + u_1) = f_{1x}$$
$$k(u_2 - u_1) = f_{2x}$$

- In matrix form:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$$

- The results are identical to the direct method

Summary

- Defined the stiffness matrix
- Derived the stiffness matrix for a spring element
- Established the global stiffness matrix for a spring assemblage
- Discussed boundary conditions (homogenous & nonhomogeneous)
- Introduced the potential energy approach
- Reviewed minimum potential energy, external work, and strain energy
- Derived the spring element equations using the potential energy approach