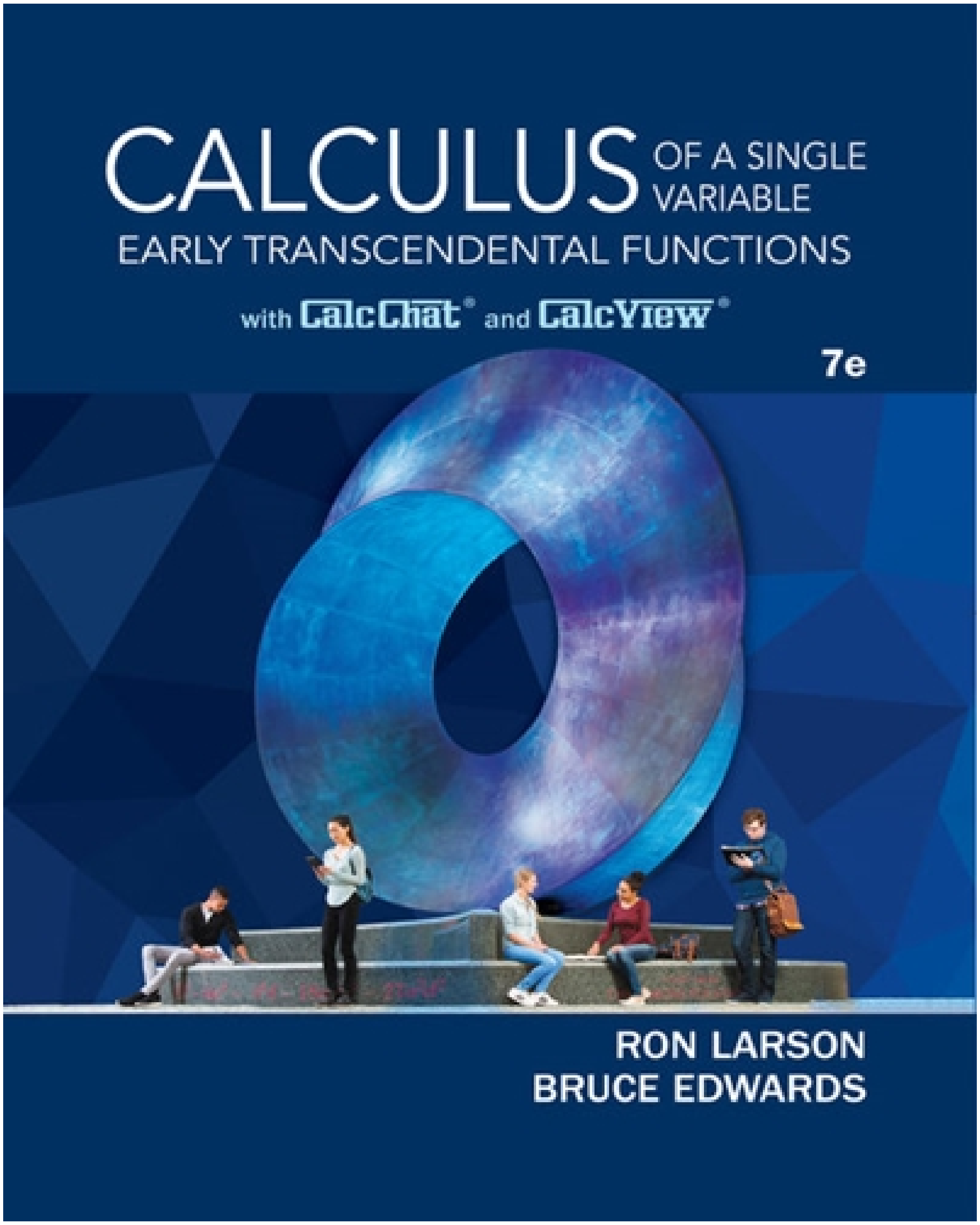
Solutions for Calculus of a Single Variable Early Transcendental Functions 7th Edition by Larson



Solutions

CHAPTER 2
Limits and Their Properties

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Section 2.1 A Preview of Calculus

1. Calculus is the mathematics of change. Precalculus is more static. Answers will vary. *Sample answer:*

Precalculus: Area of a rectangle
Calculus: Area under a curve

Precalculus: Work done by a constant force
Calculus: Work done by a variable force

Precalculus: Center of a rectangle
Calculus: Centroid of a region

2. A secant line through a point P is a line joining P and another point Q on the graph.

The slope of the tangent line P is the limit of the slopes of the secant lines joining P and Q, as Q approaches P.

3. Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

4. Calculus required: Velocity is not constant.

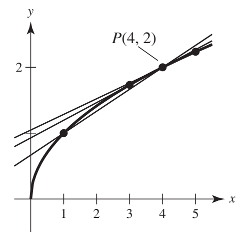
Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

5. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

6. Precalculus: rate of change $=$ slope $= 0.08$

7. $f(x) = \sqrt{x}$

(a)



(b) $$\text{slope} = m = \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2}, \ x \neq 4$$

$$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$$

$$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$$

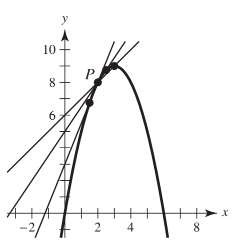
$$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$$

(c) At $P(4, 2)$ the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25.$

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

8. $f(x) = 6x - x^2$

(a)



(b) $$\text{slope} = m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x), \ x \neq 2$$

For $x = 3$, $m = 4 - 3 = 1$

For $x = 2.5$, $m = 4 - 2.5 = 1.5 = \dfrac{3}{2}$

For $x = 1.5$, $m = 4 - 1.5 = 2.5 = \dfrac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

9. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. Answers will vary. *Sample answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

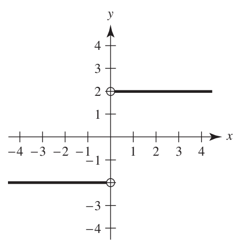
$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

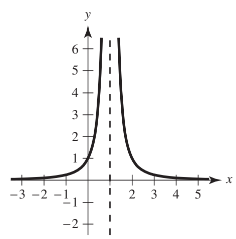
Section 2.2 Finding Limits Graphically and Numerically

1. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.

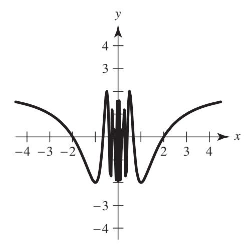
2. (i) The values of f approach different numbers as x approaches c from different sides of c:



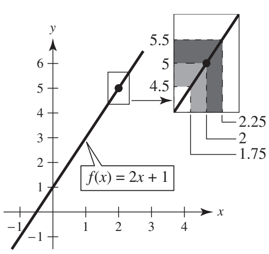
(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



3.



4. No. For example, consider Example 2 from this section.

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x \to 2} f(x) = 1$, but $f(2) = 0$

5.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	0.3448	0.3344	0.3334	?	0.3332	0.3322	0.3226

$\lim_{x \to 4} \frac{x-4}{x^2 - 5x - 4} \approx 0.3333$ $\left(\text{Actual limit is } \frac{1}{3}.\right)$

6.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

7.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x \to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

8.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	–0.0005	–0.0050	–0.0500

$\lim_{x \to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

9.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9516	0.9950	0.9995	1.0005	1.0050	1.0517

$\lim_{x \to 0} \frac{e^x - 1}{x} \approx 1.0000$ (Actual limit is 1.)

10.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.0536	1.0050	1.0005	0.9995	0.9950	0.9531

$\lim_{x \to 0} \frac{\ln(x + 1)}{x} \approx 1.0000$ (Actual limit is 1.)

11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x \to 1} \frac{x - 2}{x^2 + x - 6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

12.

x	–4.1	–4.01	–4.001	–4	–3.999	–3.99	–3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$\lim_{x \to -4} \frac{x + 4}{x^2 + 9x + 20} \approx 1.0000$ (Actual limit is 1.)

13.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x \to 1} \frac{x^4 - 1}{x^6 - 1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

14.

x	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$\lim_{x \to -3} \frac{x^3 + 27}{x + 3} \approx 27.0000$ (Actual limit is 27.)

15.

x	–6.1	–6.01	–6.001	–6	–5.999	–5.99	–5.9
$f(x)$	–0.1248	–0.1250	–0.1250	?	–0.1250	–0.1250	–0.1252

$$\lim_{x\to -6} \frac{\sqrt{10 - x} - 4}{x + 6} \approx -0.1250 \quad \left(\text{Actual limit is } -\frac{1}{8}.\right)$$

16.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$$\lim_{x\to 2} \frac{x/(x + 1) - 2/3}{x - 2} \approx 0.1111 \quad \left(\text{Actual limit is } \frac{1}{9}.\right)$$

17.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000 \quad (\text{Actual limit is 2.}) \; (\text{Make sure you use radian mode.})$$

18.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000 \quad \left(\text{Actual limit is } \frac{1}{2}.\right)$$

19.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.5129	0.5013	0.5001	0.4999	0.4988	0.4879

$$\lim_{x\to 2} \frac{\ln x - \ln 2}{x - 2} \approx 0.5000 \quad \left(\text{Actual limit is } \frac{1}{2}.\right)$$

20.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.3866	0.3697	0.3681	0.3677	0.3660	0.3498

$$\lim_{x\to 1} \frac{1 - x}{e - e^x} \approx 0.3679 \quad \left(\text{Actual limit is } \frac{1}{e}.\right)$$

21.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	3.99982	4	4	0	0	0.00018

$$\lim_{x\to 0} \frac{4}{1 + e^{1/x}} \text{ does not exist.}$$

22. $f(x) = \dfrac{3|x|}{x^2}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	30	300	3000	?	3000	300	30

As x approaches 0 from either side, the function increases without bound.

23. $\lim_{x\to 3}(4 - x) = 1$

24. $\lim_{x\to 0} \sec x = 1$

25. $\lim_{x\to 2} f(x) = \lim_{x\to 2}(4 - x) = 2$

26. $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2 + 3) = 4$

27. $\lim_{x\to 2} \dfrac{|x - 2|}{x - 2}$ does not exist.

For values of x to the left of 2, $\dfrac{|x - 2|}{(x - 2)} = -1$, whereas for values of x to the right of 2, $\dfrac{|x - 2|}{(x - 2)} = 1$.

28. $\lim_{x\to 0} \dfrac{4}{2 + e^{1/x}}$ does not exist. The function approaches 2 from the left side of 0 and it approaches 0 from the right side of 0.

29. $\lim_{x\to 0} \cos(1/x)$ does not exist because the function oscillates between –1 and 1 as x approaches 0.

30. $\lim_{x\to \pi/2} \tan x$ does not exist because the function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

31. (a) $f(1)$ exists. The black dot at (1, 2) indicates that $f(1) = 2$.

(b) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x\to 4} f(x) = 2$.

32. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at –2.

(b) $\lim_{x\to -2} f(x)$ does not exist. As x approaches –2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

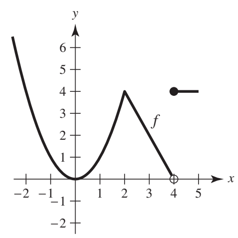
(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x\to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x\to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

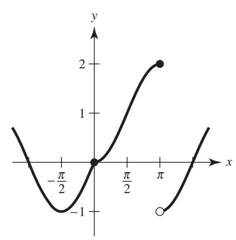
(h) $\lim_{x\to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

33.



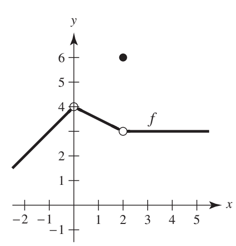
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

34.

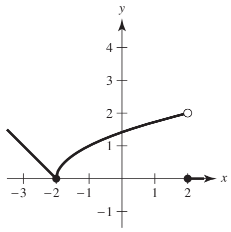


$\lim_{x\to c} f(x)$ exists for all values of $c \neq \pi$.

35. One possible answer is



36. One possible answer is



37. You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

38. You need $|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < |x - 2| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow |x - 1| > \frac{100}{101}$$

and you have

$$|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

39. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

40. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|2 - \frac{1}{x} - 1\right| = \left|1 - \frac{1}{x}\right| < \varepsilon$.

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$
$$1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$$
$$\frac{1}{1 - \varepsilon} > x > \frac{1}{1 + \varepsilon}$$
$$\frac{1}{1 - \varepsilon} - 1 > x - 1 > \frac{1}{1 + \varepsilon} - 1$$
$$\frac{\varepsilon}{1 - \varepsilon} > x - 1 > \frac{-\varepsilon}{1 + \varepsilon}$$

For $\varepsilon = 0.05$, take $\delta = \frac{0.05}{1 - 0.05} \approx 0.05$.

For $\varepsilon = 0.01$, take $\delta = \frac{0.01}{1 - 0.01} \approx 0.01$.

For $\varepsilon = 0.005$, take $\delta = \frac{0.005}{1 - 0.005} \approx 0.005$.

As ε decreases, so does δ.

41. $\lim_{x\to 2}(3x+2) = 3(2) + 2 = 8 = L$

(a) $|(3x+2) - 8| < 0.01$

$|3x - 6| < 0.01$

$3|x - 2| < 0.01$

$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$

So, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$3|x - 2| < 0.01$

$|3x - 6| < 0.01$

$|(3x+2) - 8| < 0.01$

$|f(x) - L| < 0.01.$

(b) $|(3x+2) - 8| < 0.005$

$|3x - 6| < 0.005$

$3|x - 2| < 0.005$

$0 < |x - 2| < \dfrac{0.005}{3} \approx 0.00167 = \delta$

Finally, as in part (a), if $0 < |x - 2| < \dfrac{0.005}{3}$, you have $|(3x+2) - 8| < 0.005.$

42. $\lim_{x\to 6}\left(6 - \dfrac{x}{3}\right) = 6 - \dfrac{6}{3} = 4 = L$

(a) $\left|\left(6 - \dfrac{x}{3}\right) - 4\right| < 0.01$

$\left|2 - \dfrac{x}{3}\right| < 0.01$

$\left|-\frac{1}{3}(x - 6)\right| < 0.01$

$|x - 6| < 0.03$

$0 < |x - 6| < 0.03 = \delta$

So, if $0 < |x - 6| < \delta = 0.03$, you have

$\left|-\frac{1}{3}(x - 6)\right| < 0.01$

$\left|2 - \dfrac{x}{3}\right| < 0.01$

$\left|\left(6 - \dfrac{x}{3}\right) - 4\right| < 0.01$

$|f(x) - L| < 0.01.$

(b) $\left|\left(6 - \dfrac{x}{3}\right) - 4\right| < 0.005$

$\left|2 - \dfrac{x}{3}\right| < 0.005$

$\left|-\dfrac{1}{3}(x - 6)\right| < 0.005$

$|x - 6| < 0.015$

$0 < |x - 6| < 0.015 = \delta$

As in part (a), if $0 < |x - 6| < 0.015$, you have

$\left|\left(6 - \dfrac{x}{3}\right) - 4\right| < 0.005.$

43. $\lim_{x\to 2}\left(x^2 - 3\right) = 2^2 - 3 = 1 = L$

(a) $\left|\left(x^2 - 3\right) - 1\right| < 0.01$

$|x^2 - 4| < 0.01$

$|(x+2)(x-2)| < 0.01$

$|x+2||x-2| < 0.01$

$|x - 2| < \dfrac{0.01}{|x+2|}$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002.$

So, if $0 < |x - 2| < \delta \approx 0.002$, you have

$|x - 2| < 0.002 = \dfrac{1}{5}(0.01) < \dfrac{1}{|x+2|}(0.01)$

$|x+2||x-2| < 0.01$

$|x^2 - 4| < 0.01$

$\left|\left(x^2 - 3\right) - 1\right| < 0.01$

$|f(x) - L| < 0.01.$

(b) $\left|\left(x^2 - 3\right) - 1\right| < 0.005$

$|x^2 - 4| < 0.005$

$|(x+2)(x-2)| < 0.005$

$|x+2||x-2| < 0.005$

$|x - 2| < \dfrac{0.005}{|x+2|}$

If you assume $1 < x < 3$, then

$\delta = \dfrac{0.005}{5} = 0.001.$

Finally, as in part (a), if $0 < |x - 2| < 0.001$, you have $\left|\left(x^2 - 3\right) - 1\right| < 0.005.$

44. $\lim_{x\to 4}(x^2 + 6) = 4^2 + 6 = 22 = L$

(a) $\left|(x^2 + 6) - 22\right| < 0.01$

$\left|x^2 - 16\right| < 0.01$

$\left|(x + 4)(x - 4)\right| < 0.01$

$|x + 4||x - 4| < 0.01$

$|x - 4| < \dfrac{0.01}{|x + 4|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.01}{9} \approx 0.00111.$

So, if $0 < |x - 4| < \delta \approx \dfrac{0.01}{9}$, you have

$|x - 4| < \dfrac{0.01}{9} < \dfrac{0.01}{|x + 4|}$

$\left|(x + 4)(x - 4)\right| < 0.01$

$\left|x^2 - 16\right| < 0.01$

$\left|(x^2 + 6) - 22\right| < 0.01$

$\left|f(x) - L\right| < 0.01.$

(b) $\left|(x^2 + 6) - 22\right| < 0.005$

$\left|x^2 - 16\right| < 0.005$

$\left|(x - 4)(x + 4)\right| < 0.005$

$|x - 4||x + 4| < 0.005$

$|x - 4| < \dfrac{0.05}{|x + 4|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.005}{9} \approx 0.00056.$

Finally, as in part (a), if $0 < |x - 4| < \dfrac{0.005}{9}$,

you have $\left|(x^2 + 6) - 22\right| < 0.005.$

45. $\lim_{x\to 4}(x^2 - x) = 16 - 4 = 12 = L$

(a) $\left|(x^2 - x) - 12\right| < 0.01$

$\left|(x - 4)(x + 3)\right| < 0.01$

$|x - 4||x + 3| < 0.01$

$|x - 4| < \dfrac{0.01}{|x + 3|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.01}{8} = 0.00125.$

So, if $0 < |x - 4| < \dfrac{0.01}{8}$, you have

$|x - 4| < \dfrac{0.01}{|x + 3|}$

$|x - 4||x + 3| < 0.01$

$\left|x^2 - x - 12\right| < 0.01$

$\left|(x^2 - x) - 12\right| < 0.01$

$\left|f(x) - L\right| < 0.01$

(b) $\left|(x^2 - x) - 12\right| < 0.005$

$\left|(x - 4)(x + 3)\right| < 0.005$

$|x - 4||x + 3| < 0.005$

$|x - 4| < \dfrac{0.005}{|x + 3|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.005}{8} = 0.000625.$

Finally, as in part (a), if $0 < |x - 4| < \dfrac{0.005}{8}$,

you have $\left|(x^2 - x) - 12\right| < 0.005.$

46. $\lim_{x \to 3} x^2 = 3^2 = 9 = L$

(a) $|x^2 - 9| < 0.01$

$|(x - 3)(x + 3)| < 0.01$

$|x - 3||x + 3| < 0.01$

$|x - 3| < \dfrac{0.01}{|x + 3|}$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.01}{7} \approx 0.0014.$

So, if $0 < |x - 3| < \dfrac{0.01}{7}$, you have

$|x - 3| < \dfrac{0.01}{|x + 3|}$

$|x - 3||x + 3| < 0.01$

$|x^2 - 9| < 0.01$

$|f(x) - L| < 0.01$

(b) $|x^2 - 9| < 0.005$

$|(x - 3)(x + 3)| < 0.005$

$|x - 3||x + 3| < 0.005$

$|x - 3| < \dfrac{0.005}{|x + 3|}$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.005}{7} \approx 0.00071.$

Finally, as in part (a), if $0 < |x - 3| < \dfrac{0.005}{7}$, you have $|x^2 - 9| < 0.005.$

47. $\lim_{x \to 4} (x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$|(x + 2) - 6| < \varepsilon$

$|x - 4| < \varepsilon = \delta$

So, let $\delta = \varepsilon$. So, if $0 < |x - 4| < \delta = \varepsilon$, you have

$|x - 4| < \varepsilon$

$|(x + 2) - 6| < \varepsilon$

$|f(x) - L| < \varepsilon.$

48. $\lim_{x \to -2} (4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$|(4x + 5) - (-3)| < \varepsilon$

$|4x + 8| < \varepsilon$

$4|x + 2| < \varepsilon$

$|x + 2| < \dfrac{\varepsilon}{4} = \delta$

So, let $\delta = \dfrac{\varepsilon}{4}.$

So, if $0 < |x + 2| < \delta = \dfrac{\varepsilon}{4}$, you have

$|x + 2| < \dfrac{\varepsilon}{4}$

$|4x + 8| < \varepsilon$

$|(4x + 5) - (-3)| < \varepsilon$

$|f(x) - L| < \varepsilon.$

49. $\lim_{x \to -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$

$\left|\frac{1}{2}x + 2\right| < \varepsilon$

$\frac{1}{2}|x - (-4)| < \varepsilon$

$|x - (-4)| < 2\varepsilon$

So, let $\delta = 2\varepsilon$.

So, if $0 < |x - (-4)| < \delta = 2\varepsilon$, you have

$|x - (-4)| < 2\varepsilon$

$\left|\frac{1}{2}x + 2\right| < \varepsilon$

$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \varepsilon$

$|f(x) - L| < \varepsilon.$

50. $\lim_{x\to 3}\left(\frac{3}{4}x+1\right) = \frac{3}{4}(3)+1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{3}{4}x+1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\tfrac{3}{4}|x-3| < \varepsilon$$
$$|x-3| < \tfrac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < |x-3| < \delta = \frac{4}{3}\varepsilon$, you have

$$|x-3| < \tfrac{4}{3}\varepsilon$$
$$\tfrac{3}{4}|x-3| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\left|\left(\tfrac{3}{4}x+1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

51. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3-3| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|3-3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

52. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $|-1-(-1)| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|(-1)-(-1)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

53. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x-0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

54. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x-4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x-4| < \delta = 3\varepsilon \Rightarrow |x-4| < \varepsilon\left|\sqrt{x}+2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

55. $\lim_{x\to -5}|x-5| = |(-5)-5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x-5| - 10\big| < \varepsilon$

$$|-(x-5) - 10| < \varepsilon \qquad (x-5 < 0)$$
$$|-x-5| < \varepsilon$$
$$|x-(-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x-(-5)| < \delta = \varepsilon$, you have

$$|-(x+5)| < \varepsilon$$
$$|-(x-5) - 10| < \varepsilon$$
$$\big||x-5| - 10\big| < \varepsilon \qquad (\text{because } x - 5 < 0)$$
$$|f(x) - L| < \varepsilon.$$

56. $\lim\limits_{x\to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\big||x - 3| - 0\big| < \varepsilon$

$|x - 3| < \varepsilon$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$|x - 3| < \varepsilon$

$\big||x - 3| - 0\big| < \varepsilon$

$|f(x) - L| < \varepsilon.$

57. $\lim\limits_{x\to 1} (x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$: $|(x^2 + 1) - 2| < \varepsilon$

$|x^2 - 1| < \varepsilon$

$|(x + 1)(x - 1)| < \varepsilon$

$|x - 1| < \dfrac{\varepsilon}{|x + 1|}$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$|x - 1| < \dfrac{1}{3}\varepsilon < \dfrac{1}{|x + 1|}\varepsilon$

$|x^2 - 1| < \varepsilon$

$|(x^2 + 1) - 2| < \varepsilon$

$|f(x) - 2| < \varepsilon.$

58. $\lim\limits_{x\to -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$: $|(x^2 + 4x) - 0| < \varepsilon$

$|x(x + 4)| < \varepsilon$

$|x + 4| < \dfrac{\varepsilon}{|x|}$

If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$|x + 4| < \dfrac{\varepsilon}{5} < \dfrac{1}{|x|}\varepsilon$

$|x(x + 4)| < \varepsilon$

$|(x^2 + 4x) - 0| < \varepsilon$

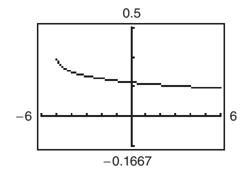
$|f(x) - L| < \varepsilon.$

59. $\lim\limits_{x\to \pi} f(x) = \lim\limits_{x\to \pi} 4 = 4$

60. $\lim\limits_{x\to \pi} f(x) = \lim\limits_{x\to \pi} x = \pi$

61. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

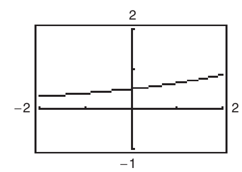
$\lim\limits_{x\to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \dfrac{1}{6}\right)$.

62. $f(x) = \dfrac{e^{x/2} - 1}{x}$

$\lim\limits_{x\to 0} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 0$. The graphing utility does not show the hole at $\left(0, \dfrac{1}{2}\right)$.

63. $C(t) = 9.99 - 0.79[\![1 - t]\!], t > 0$

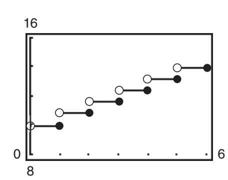
(a) $C(10.75) = 9.99 - 0.79[\![1 - 10.75]\!]$

$= 9.99 - 0.79(-10)$

$= \$17.89$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b)



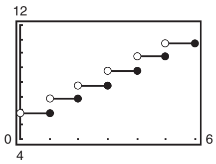
(c) The limit does not exist because the limits from the left and right are not equal.

64. $C(t) = 5.79 - 0.99[\![1 - t]\!], t > 0$

(a) $C(10.75) = 5.79 - 0.99[\![1 - 10.75]\!]$
$= 5.79 - 0.99(-10)$
$= \$15.69$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b)



(c) The limit does not exist because the limits from the left and right are not equal.

65. Choosing a smaller positive value of δ will still satisfy the inequality $|f(x) - L| < \varepsilon$.

66. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

67. No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

68. No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

69. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x\to 3/\pi} (2\pi r) = 6;\ \varepsilon = 0.5;\ \delta \approx 0.0796$

70. $V = \dfrac{4}{3}\pi r^3, V = 2.48$

(a) $2.48 = \dfrac{4}{3}\pi r^3$

$r^3 = \dfrac{1.86}{\pi}$

$r \approx 0.8397$ in.

(b) $2.45 \le V \le 2.51$

$2.45 \le \dfrac{4}{3}\pi r^3 \le 2.51$

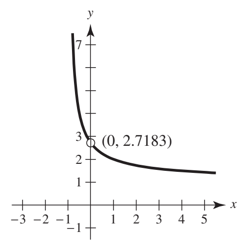
$0.5849 \le r^3 \le 0.5992$

$0.8363 \le r \le 0.8431$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03, \delta \approx 0.003$

71. $f(x) = (1 + x)^{1/x}$

$\lim_{x\to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

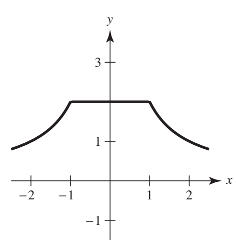
72. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

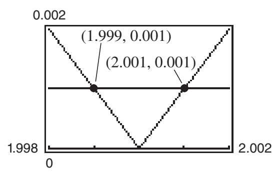
$\lim_{x\to 0} f(x) = 2$

Note that for

$$-1 < x < 1, x \ne 0, f(x) = \frac{(x + 1) + (x - 1)}{x} = 2.$$



73.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

74. (a) $\lim_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

75. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

76. True

77. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$f(2) = 0$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \neq 0$$

78. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \text{ and } f(2) = 0 \neq 2$$

79. $f(x) = \sqrt{x}$

$\lim_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

80. $f(x) = \sqrt{x}$

$\lim_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

81. Using a graphing utility, you see that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2, \text{ etc.}$$

$$\text{So, } \lim_{x \to 0} \frac{\sin nx}{x} = n.$$

82. Using a graphing utility, you see that

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{\tan 2x}{x} = 2, \quad \text{etc.}$$

$$\text{So, } \lim_{x \to 0} \frac{\tan(nx)}{x} = n.$$

83. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

84. $f(x) = mx + b, m \neq 0$. Let $\varepsilon > 0$ be given.

Take $\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$

$$|mx - mc| < \varepsilon$$

$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

85. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

86. (a) $(3x+1)(3x-1)x^2 + 0.01 = (9x^2-1)x^2 + \frac{1}{100}$

$= 9x^4 - x^2 + \frac{1}{100}$

$= \frac{1}{100}(10x^2-1)(90x^2-1)$

So, $(3x+1)(3x-1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive.

You can verify this with a graphing utility.

(b) You are given $\lim_{x\to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$.

There exists $\delta > 0$ such that $0 < |x - c| < \delta$

implies that $|g(x) - L| < \varepsilon = \frac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$

$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \neq c$, you

have $g(x) > \frac{L}{2} > 0$, as desired.

87. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. So, $z = 1 - \frac{h}{2}$.

Area triangle $= \frac{1}{2}b\left(1 - \frac{h}{2}\right)$

Area rectangle $= bh$

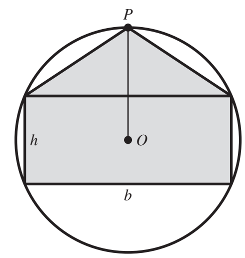
Because these are equal,

$$\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}.$$



88. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

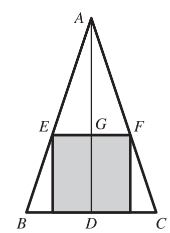
Let x be the length of a side of the cube.

Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3-x}{3}$$



Solving for x,

$$3\sqrt{2}x = 6 - 2x$$

$$(3\sqrt{2} + 2)x = 6$$

$$x = \frac{6}{3\sqrt{2}+2} = \frac{9\sqrt{2}-6}{7} \approx 0.96.$$

Section 2.3 Evaluating Limits Analytically

1. For polynomial functions $p(x)$, substitute c for x, and simplify.

2. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

3. If a function f is squeezed between two functions h and g, $h(x) \leq f(x) \leq g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals L

4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$

$\lim_{x\to 0} (1 + x)^{1/x} = e$

5. $\lim_{x\to -3} (2x + 5) = 2(-3) + 5 = -1$

6. $\lim_{x\to 9} (4x - 1) = 4(9) - 1 = 36 - 1 = 35$

7. $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

8. $\lim_{x\to 1} (2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5$

$= 2 - 6 + 5 = 1$

9. $\lim_{x\to 3} \sqrt{x+8} = \sqrt{3+8} = \sqrt{11}$

10. $\lim_{x\to 2} \sqrt[3]{12x+3} = \sqrt[3]{12(2)+3}$

$= \sqrt[3]{24+3} = \sqrt[3]{27} = 3$

11. $\lim_{x\to -4} (1-x)^3 = \left[1-(-4)\right]^3 = 5^3 = 125$

12. $\lim_{x\to 0} (3x-2)^4 = (3(0)-2)^4 = (-2)^4 = 16$

13. $\lim_{x\to 2} \dfrac{3}{2x+1} = \dfrac{3}{2(2)+1} = \dfrac{3}{5}$

14. $\lim_{x\to -5} \dfrac{5}{x+3} = \dfrac{5}{-5+3} = -\dfrac{5}{2}$

15. $\lim_{x\to 1} \dfrac{x}{x^2+4} = \dfrac{1}{1^2+4} = \dfrac{1}{5}$

16. $\lim_{x\to 1} \dfrac{3x+5}{x+1} = \dfrac{3(1)+5}{1+1} = \dfrac{3+5}{2} = \dfrac{8}{2} = 4$

17. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x+2}} = \dfrac{3(7)}{\sqrt{7+2}} = \dfrac{21}{3} = 7$

18. $\lim_{x\to 3} \dfrac{\sqrt{x+6}}{x+2} = \dfrac{\sqrt{3+6}}{3+2} = \dfrac{\sqrt{9}}{5} = \dfrac{3}{5}$

19. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

20. (a) $\lim_{x\to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x\to 4} g(x) = 4^2 = 16$

(c) $\lim_{x\to -3} g(f(x)) = g(4) = 16$

21. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

22. (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

23. $\lim_{x\to \pi/2} \sin x = \sin\dfrac{\pi}{2} = 1$

24. $\lim_{x\to \pi} \tan x = \tan \pi = 0$

25. $\lim_{x\to 1} \cos\dfrac{\pi x}{3} = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

26. $\lim_{x\to 2} \sin\dfrac{\pi x}{12} = \sin\dfrac{\pi(2)}{12} = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

27. $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

28. $\lim_{x\to \pi} \cos 3x = \cos 3\pi = -1$

29. $\lim_{x\to 5\pi/6} \sin x = \sin\dfrac{5\pi}{6} = \dfrac{1}{2}$

30. $\lim_{x\to 5\pi/3} \cos x = \cos\dfrac{5\pi}{3} = \dfrac{1}{2}$

31. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right) = \tan\dfrac{3\pi}{4} = -1$

32. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec\dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

33. $\lim_{x\to 0} e^x \cos 2x = e^0 \cos 0 = 1$

34. $\lim_{x\to 0} e^{-x} \sin \pi x = e^0 \sin 0 = 0$

35. $\lim_{x\to 1} \left(\ln 3x + e^x\right) = \ln 3 + e$

36. $\lim_{x\to 1} \ln\left(\dfrac{x}{e^x}\right) = \ln\left(\dfrac{1}{e}\right) = \ln e^{-1} = -1$

37. $\lim_{x\to c} f(x) = \dfrac{2}{5},\ \lim_{x\to c} g(x) = 2$

(a) $\lim_{x\to c} \left[5g(x)\right] = 5\lim_{x\to c} g(x) = 5(2) = 10$

(b) $\lim_{x\to c} \left[f(x) + g(x)\right] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

$= \dfrac{2}{5} + 2$

$= \dfrac{12}{5}$

(c) $\lim_{x\to c} \left[f(x)g(x)\right] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right]$

$= \dfrac{2}{5}(2)$

$= \dfrac{4}{5}$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{2/5}{2} = \dfrac{1}{5}$

38. $\lim_{x \to c} f(x) = 2, \lim_{x \to c} g(x) = \dfrac{3}{4}$

(a) $\lim_{x \to c} \left[4f(x)\right] = 4 \lim_{x \to c} f(x) = 4(2) = 8$

(b) $\lim_{x \to c} \left[f(x) + g(x)\right] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 2 + \dfrac{3}{4} = \dfrac{11}{4}$

(c) $\lim_{x \to c} \left[f(x)g(x)\right] = \left[\lim_{x \to c} f(x)\right]\left[\lim_{x \to c} g(x)\right] = 2\left(\dfrac{3}{4}\right) = \dfrac{3}{2}$

(d) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{2}{(3/4)} = \dfrac{8}{3}$

39. $\lim_{x \to c} f(x) = 16$

(a) $\lim_{x \to c} \left[f(x)\right]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (16)^2 = 256$

(b) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)} = \sqrt{16} = 4$

(c) $\lim_{x \to c} \left[3f(x)\right] = 3\left[\lim_{x \to c} f(x)\right] = 3(16) = 48$

(d) $\lim_{x \to c} \left[f(x)\right]^{3/2} = \left[\lim_{x \to c} f(x)\right]^{3/2} = (16)^{3/2} = 64$

40. $\lim_{x \to c} f(x) = 27$

(a) $\lim_{x \to c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \to c} f(x)} = \sqrt[3]{27} = 3$

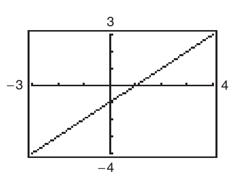
(b) $\lim_{x \to c} \dfrac{f(x)}{18} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} 18} = \dfrac{27}{18} = \dfrac{3}{2}$

(c) $\lim_{x \to c} \left[f(x)\right]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x \to c} \left[f(x)\right]^{2/3} = \left[\lim_{x \to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

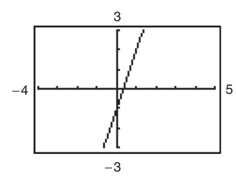
41. $f(x) = \dfrac{x^2 - 1}{x + 1} = \dfrac{(x + 1)(x - 1)}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2$



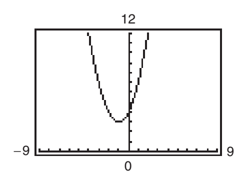
42. $f(x) = \dfrac{3x^2 + 5x - 2}{x + 2} = \dfrac{(x + 2)(3x - 1)}{x + 2}$ and $g(x) = 3x - 1$ agree except at $x = -2$.

$\lim_{x \to -2} f(x) = \lim_{x \to -2} g(x) = \lim_{x \to -2} (3x - 1)$
$= 3(-2) - 1 = -7$



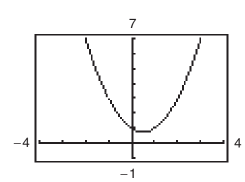
43. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} \left(x^2 + 2x + 4\right)$
$= 2^2 + 2(2) + 4 = 12$



44. $f(x) = \dfrac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

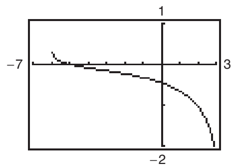
$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} \left(x^2 - x + 1\right)$
$= (-1)^2 - (-1) + 1 = 3$



45. $f(x) = \dfrac{(x+4)\ln(x+6)}{x^2 - 16}$ and $g(x) = \dfrac{\ln(x+6)}{x-4}$

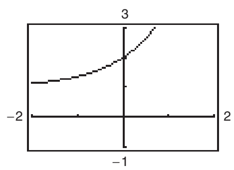
agree except at $x = -4$.

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} g(x) = \frac{\ln 2}{-8} \approx -0.0866$$



46. $f(x) = \dfrac{e^{2x} - 1}{e^x - 1}$ and $g(x) = e^x + 1$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = e^0 + 1 = 2$$



47. $\displaystyle\lim_{x\to 0} \frac{x}{x^2 - x} = \lim_{x\to 0} \frac{x}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = \frac{1}{0-1} = -1$

48. $\displaystyle\lim_{x\to 0} \frac{7x^3 - x^2}{x} = \lim_{x\to 0} \left(7x^2 - x\right) = 0 - 0 = 0$

49. $\displaystyle\lim_{x\to 4} \frac{x-4}{x^2-16} = \lim_{x\to 4} \frac{x-4}{(x+4)(x-4)} = \lim_{x\to 4} \frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$

50. $\displaystyle\lim_{x\to 5} \frac{5-x}{x^2-25} = \lim_{x\to 5} \frac{-(x-5)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{-1}{x+5} = \frac{-1}{5+5} = -\frac{1}{10}$

51. $\displaystyle\lim_{x\to -3} \frac{x^2+x-6}{x^2-9} = \lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)(x-3)} = \lim_{x\to -3} \frac{x-2}{x-3} = \frac{-3-2}{-3-3} = \frac{-5}{-6} = \frac{5}{6}$

52. $\displaystyle\lim_{x\to 2} \frac{x^2+2x-8}{x^2-x-2} = \lim_{x\to 2} \frac{(x-2)(x+4)}{(x-2)(x+1)} = \lim_{x\to 2} \frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$

53. $\displaystyle\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$\displaystyle = \lim_{x\to 4} \frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}$

$\displaystyle = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

54. $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$

$\displaystyle = \lim_{x\to 3} \frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]}$

$\displaystyle = \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

55. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$\displaystyle = \lim_{x\to 0} \frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

56. $\displaystyle\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$\displaystyle = \lim_{x\to 0} \frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

57. $\displaystyle\lim_{x\to 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x} = \lim_{x\to 0} \frac{3-(3+x)}{(3+x)3(x)} = \lim_{x\to 0} \frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0} \frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$

58. $\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x\to 0} \frac{\frac{4-(x+4)}{4(x+4)}}{x} = \lim_{x\to 0} \frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$

59. $\lim_{\Delta x\to 0} \frac{2(x+\Delta x) - 2x}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x\to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

60. $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$

61. $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - 2(x+\Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x}$

$= \lim_{\Delta x\to 0} (2x + \Delta x - 2) = 2x - 2$

62. $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$

63. $\lim_{x\to 0} \frac{\sin x}{5x} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

64. $\lim_{x\to 0} \frac{3(1-\cos x)}{x} = \lim_{x\to 0} \left[3\left(\frac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

65. $\lim_{x\to 0} \frac{(\sin x)(1-\cos x)}{x^2} = \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{1-\cos x}{x}\right]$

$= (1)(0) = 0$

66. $\lim_{\theta\to 0} \frac{\cos\theta \tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

67. $\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \left[\frac{\sin x}{x} \sin x\right] = (1)\sin 0 = 0$

68. $\lim_{x\to 0} \frac{\tan^2 x}{x} = \lim_{x\to 0} \frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

69. $\lim_{h\to 0} \frac{(1-\cos h)^2}{h} = \lim_{h\to 0} \left[\frac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

70. $\lim_{\phi\to\pi} \phi \sec\phi = \pi(-1) = -\pi$

71. $\lim_{x\to 0} \frac{6 - 6\cos x}{3} = \frac{6 - 6\cos 0}{3} = \frac{6-6}{3} = 0$

72. $\lim_{x\to 0} \frac{\cos x - \sin x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x} + \lim_{x\to 0} \frac{\cos x - 1}{2x}$

$= -\frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} - \frac{1}{2}\lim_{x\to 0} \frac{1-\cos x}{x}$

$= -\frac{1}{2}(1) - \frac{1}{2}(0) = -\frac{1}{2}$

73. $\lim_{x\to 0} \frac{1-e^{-x}}{e^x - 1} = \lim_{x\to 0} \frac{1-e^{-x}}{e^x - 1} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to 0} \frac{(1-e^{-x})e^{-x}}{1-e^{-x}}$

$= \lim_{x\to 0} e^{-x} = 1$

74. $\lim_{x\to 0} \frac{4(e^{2x} - 1)}{e^x - 1} = \lim_{x\to 0} \frac{4(e^x - 1)(e^x + 1)}{e^x - 1}$

$= \lim_{x\to 0} 4(e^x + 1) = 4(2) = 8$

75. $\lim_{t\to 0} \frac{\sin 3t}{2t} = \lim_{t\to 0} \left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

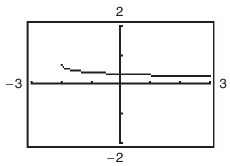
76. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$

$= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

77. $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

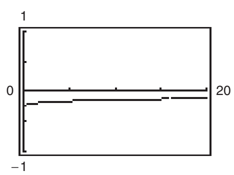
Analytically, $\lim\limits_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \dfrac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$

$$= \lim_{x\to 0} \frac{x + 2 - 2}{x\left(\sqrt{x+2} + \sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

78. $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.



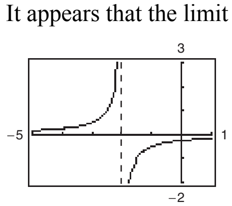
The graph has a hole at $x = 16$.

Analytically, $\lim\limits_{x\to 16} \dfrac{4 - \sqrt{x}}{x - 16} = \lim\limits_{x\to 16} \dfrac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim\limits_{x\to 16} \dfrac{-1}{\sqrt{x} + 4} = -\dfrac{1}{8}.$

79. $f(x) = \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



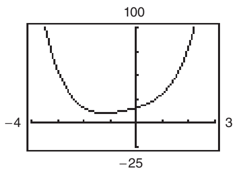
The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} = \lim\limits_{x\to 0} \dfrac{2 - (2 + x)}{2(2 + x)} \cdot \dfrac{1}{x} = \lim\limits_{x\to 0} \dfrac{-x}{2(2 + x)} \cdot \dfrac{1}{x} = \lim\limits_{x\to 0} \dfrac{-1}{2(2 + x)} = -\dfrac{1}{4}.$

80. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

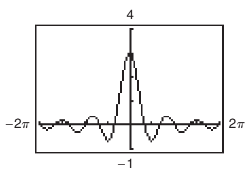
Analytically, $\lim\limits_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim\limits_{x\to 2} \dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim\limits_{x\to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

81. $f(t) = \dfrac{\sin 3t}{t}$

t	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



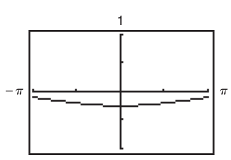
The graph has a hole at $t = 0$.

Analytically, $\lim\limits_{t\to 0} \dfrac{\sin 3t}{t} = \lim\limits_{t\to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

82. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	–1	–0.1	–0.01	0.01	0.1	1
$f(x)$	–0.2298	–0.2498	–0.25	–0.25	–0.2498	–0.2298

It appears that the limit is –0.25.



The graph has a hole at $x = 0$.

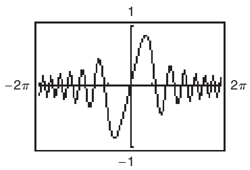
Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$$\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

83. $f(x) = \dfrac{\sin x^2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.099998	–0.01	–0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



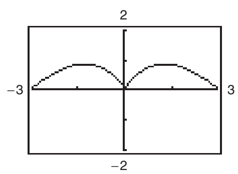
The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0} \dfrac{\sin x^2}{x} = \lim_{x\to 0} x\left(\dfrac{\sin x^2}{x^2}\right) = 0(1) = 0.$

84. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

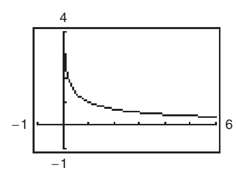
It appears that the limit is 0.



The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

85. $f(x) = \dfrac{\ln x}{x - 1}$



x	0.5	0.9	0.99	1.01	1.1	1.5
$f(x)$	1.3863	1.0536	1.0050	0.9950	0.9531	0.8109

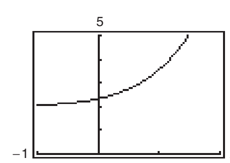
It appears that the limit is 1.

Analytically, $\lim_{x\to 1} \dfrac{\ln x}{x - 1} = \lim_{x\to 1}\left(\dfrac{1}{x - 1}\right)\ln x = \lim_{x\to 1} \ln x^{1/(x-1)}$

Let $y = x - 1$, then $x = y + 1$ and $y \to 0$ as $x \to 1$.

So, $\lim_{x\to 1} \ln x^{1/(x-1)} = \lim_{y\to 0} \ln(y + 1)^{1/y} = \ln e = 1.$

86. $f(x) = \dfrac{e^{3x} - 8}{e^{2x} - 4}$



x	0.5	0.6	0.69	0.70	0.8	0.9
$f(x)$	2.7450	2.8687	2.9953	3.0103	3.1722	3.3565

It appears that the limit is 3.

Analytically, $\lim_{x\to \ln 2} \dfrac{e^{3x} - 8}{e^{2x} - 4} = \lim_{x\to \ln 2} \dfrac{(e^x - 2)(e^{2x} + 2e^x + 4)}{(e^x - 2)(e^x + 2)} = \lim_{x\to \ln 2} \dfrac{e^{2x} + 2e^x + 4}{e^x + 2} = \dfrac{4 + 4 + 4}{2 + 2} = 3.$

87. $f(x) = 3x - 2$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

88. $f(x) = -6x + 3$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[-6(x + \Delta x) + 3] - [-6x + 3]}{\Delta x} = \lim_{\Delta x \to 0} \frac{-6x - 6\Delta x + 3 + 6x - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (-6) = -6$$

89. $f(x) = x^2 - 4x$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

90. $f(x) = 3x^2 + 1$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[3(x + \Delta x)^2 + 1\right] - \left[3x^2 + 1\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(3x^2 + 6x\Delta x + 3(\Delta x)^2 + 1\right) - \left(3x^2 + 1\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 6x + 3\Delta x = 6x$$

91. $f(x) = 2\sqrt{x}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sqrt{x + \Delta x} - 2\sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\left(\sqrt{x + \Delta x} - \sqrt{x}\right)}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{2(x + \Delta x - x)}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-1/2}$$

92. $f(x) = \sqrt{x} - 5$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x} - 5\right) - \left(\sqrt{x} - 5\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

93. $f(x) = \dfrac{1}{x + 3}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x + 3} - \frac{1}{x + 3}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$$

94. $f(x) = \dfrac{1}{x^2}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - \left[x^2 + 2x\Delta x + (\Delta x)^2\right]}{x^2(x + \Delta x)^2 \Delta x} = \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2(x + \Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

95. $\lim_{x \to 0} \left(4 - x^2\right) \le \lim_{x \to 0} f(x) \le \lim_{x \to 0} \left(3 + x^2\right)$

$$4 \le \lim_{x \to 0} f(x) \le 4$$

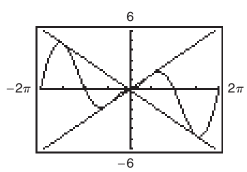
Therefore, $\lim_{x \to 0} f(x) = 4$.

96. $\lim_{x \to a} \left[b - |x - a|\right] \le \lim_{x \to a} f(x) \le \lim_{x \to a} \left[b + |x - a|\right]$

$$b \le \lim_{x \to a} f(x) \le b$$

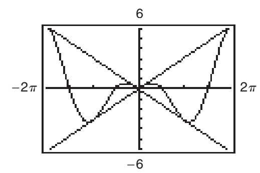
Therefore, $\lim_{x \to a} f(x) = b$.

97. $f(x) = |x| \sin x$



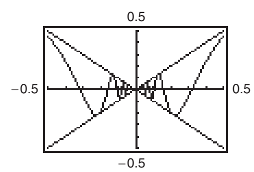
$\lim_{x \to 0} |x| \sin x = 0$

98. $f(x) = |x| \cos x$



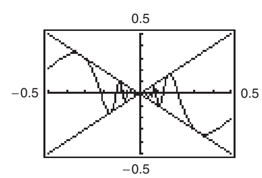
$\lim_{x \to 0} |x| \cos x = 0$

99. $f(x) = x \sin \dfrac{1}{x}$



$\lim_{x \to 0} \left(x \sin \dfrac{1}{x}\right) = 0$

100. $f(x) = x \cos \dfrac{1}{x}$



$\lim_{x \to 0} \left(x \cos \dfrac{1}{x}\right) = 0$

101. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x + 1)(x - 1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

(Other answers possible.)

102. Answers will vary. *Sample answers:*

(a) linear: $f(x) = \dfrac{1}{2}x$; $\lim_{x \to 8} \dfrac{1}{2}x = \dfrac{1}{2}(8) = 4$

(b) polynomial of degree 2: $f(x) = x^2 - 60$; $\lim_{x \to 8} \left(x^2 - 60\right) = 8^2 - 60 = 4$

(c) rational: $f(x) = \dfrac{x}{2x - 14}$; $\lim_{x \to 8} \dfrac{x}{2x - 14} = \dfrac{8}{2(8) - 14} = \dfrac{8}{2} = 4$

(d) radical: $f(x) = \sqrt{x + 8}$; $\lim_{x \to 8} \sqrt{x + 8} = \sqrt{8 + 8} = \sqrt{16} = 4$

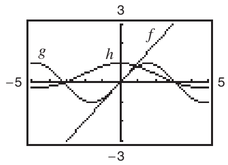
(e) cosine: $f(x) = 4\cos(\pi x)$; $\lim_{x \to 8} 4\cos(\pi x) = 4\cos 8\pi = 4(1) = 4$

(f) sine: $f(x) = 4\sin\left(\dfrac{\pi}{16}x\right)$; $\lim_{x \to 8} 4\sin\left(\dfrac{\pi}{16}x\right) = 4\sin\dfrac{\pi}{2} = 4(1) = 4$

(g) $f(x) = 4e^{x-8}$; $\lim_{x\to 8} 4e^{x-8} = 4e^0 = 4$

(h) $f(x) = 4\ln[e(x-7)]$; $\lim_{x\to 8} 4\ln[e(x-7)] = 4\ln e = 4$

103. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

104. (a) Use the dividing out technique because the numerator and denominator have a common factor.

(b) Use the rationalizing technique because the numerator involves a radical expression.

105. $s(t) = -16t^2 + 500$

$$\lim_{t\to 2}\frac{s(2)-s(t)}{2-t} = \lim_{t\to 2}\frac{-16(2)^2 + 500 - (-16t^2 + 500)}{2-t}$$
$$= \lim_{t\to 2}\frac{436 + 16t^2 - 500}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t^2-4)}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t-2)(t+2)}{2-t}$$
$$= \lim_{t\to 2} -16(t+2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

106. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{0 - (-16t^2 + 500)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\frac{5\sqrt{5}}{2}}\left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec} \approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

107. $s(t) = -4.9t^2 + 200$

$$\lim_{t\to 3}\frac{s(3)-s(t)}{3-t} = \lim_{t\to 3}\frac{-4.9(3)^2 + 200 - (-4.9t^2 + 200)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9(t^2-9)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9(t-3)(t+3)}{3-t}$$
$$= \lim_{t\to 3}[-4.9(t+3)]$$
$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

108. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\begin{aligned}\lim_{t\to a}\frac{s(a)-s(t)}{a-t} &= \lim_{t\to a}\frac{0-\left[-4.9t^2+200\right]}{a-t}\\ &= \lim_{t\to a}\frac{4.9(t+a)(t-a)}{a-t}\\ &= \lim_{t\to\frac{20\sqrt{5}}{7}}\left[-4.9\left(t+\frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5}\text{ m/sec}\\ &\approx -62.6\text{ m/sec.}\end{aligned}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

109. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However,

$$\lim_{x\to 0}\left[f(x)+g(x)\right] = \lim_{x\to 0}\left[\frac{1}{x}+\left(-\frac{1}{x}\right)\right] = \lim_{x\to 0}[0] = 0 \text{ and therefore does not exist.}$$

110. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x)+g(x)\right]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

111. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f(x)-b\right| < \varepsilon$ whenever $\left|x-c\right| < \delta$. Because $\left|f(x)-b\right| = \left|b-b\right| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

112. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

113. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $\left|f(x)-L\right| < \varepsilon/\left|b\right|$ whenever $0 < \left|x-c\right| < \delta$. So, whenever $0 < \left|x-c\right| < \delta$, we have

$$\left|b\right|\left|f(x)-L\right| < \varepsilon \quad\text{or}\quad \left|bf(x)-bL\right| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

114. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x)-0\right| < \varepsilon$ whenever $0 < \left|x-c\right| < \delta$.

Now $\left|f(x)-0\right| = \left|f(x)\right| = \left|\left|f(x)\right|-0\right| < \varepsilon$ for $\left|x-c\right| < \delta$. Therefore, $\lim_{x\to c}\left|f(x)\right| = 0$.

115.

$$\begin{aligned}-M\left|f(x)\right| \le f(x)g(x) &\le M\left|f(x)\right|\\ \lim_{x\to c}\left(-M\left|f(x)\right|\right) \le \lim_{x\to c}\left[f(x)g(x)\right] &\le \lim_{x\to c}\left(M\left|f(x)\right|\right)\\ -M(0) \le \lim_{x\to c}\left[f(x)g(x)\right] &\le M(0)\\ 0 \le \lim_{x\to c}\left[f(x)g(x)\right] &\le 0\end{aligned}$$

Therefore, $\lim_{x\to c}\left[f(x)g(x)\right] = 0$.

116. (a) If $\lim_{x\to c}\left|f(x)\right| = 0$, then $\lim_{x\to c}\left[-\left|f(x)\right|\right] = 0$.

$$\begin{aligned}-\left|f(x)\right| \le f(x) &\le \left|f(x)\right|\\ \lim_{x\to c}\left[-\left|f(x)\right|\right] \le \lim_{x\to c} f(x) &\le \lim_{x\to c}\left|f(x)\right|\\ 0 \le \lim_{x\to c} f(x) &\le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x)-L\right| < \varepsilon$ whenever $0 < \left|x-c\right| < \delta$. Since $\left|\left|f(x)\right|-\left|L\right|\right| \le \left|f(x)-L\right| < \varepsilon$ for $\left|x-c\right| < \delta$, then $\lim_{x\to c}\left|f(x)\right| = \left|L\right|$.

117. Let

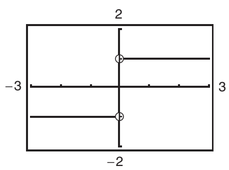
$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0,\ f(x) = -4$ and for $x \geq 0,\ f(x) = 4$.

118. The graphing utility was set in degree mode, instead of *radian* mode.

119. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0.



120. False. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

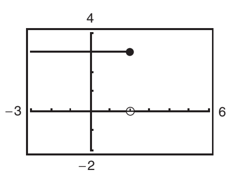
121. True.

122. False. Let

$$f(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}, \quad c = 1.$$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

123. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.



124. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

125.
$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$
$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$
$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right]$$
$$= (1)(0) = 0$$

126.
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

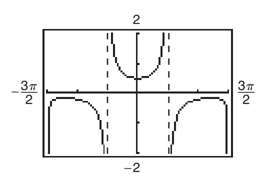
$\lim_{x\to 0} g(x) = 0$

when x is "close to" 0, both parts of the function are "close to" 0.

127. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1}$$
$$= \frac{\sec^2 x - 1}{x^2(\sec x + 1)} = \frac{\tan^2 x}{x^2(\sec x + 1)}$$
$$= \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

So,
$$\lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$
$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

128. (a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x}$

$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$

(b) From part (a),

$\frac{1-\cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1-\cos x$

$\approx \frac{1}{2}x^2 \Rightarrow \cos x$

$\approx 1 - \frac{1}{2}x^2$ for x

$\approx 0.$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 2.4 Continuity and One-Sided Limits

1. A function f is continuous at a point c if there is no interruption of the graph at c.

2. $c = -1$ because $\lim_{x\to -1^+} 2\sqrt{x+1} = 2\sqrt{-1+1} = 0$

3. The limit exists because the limit from the left and the limit from the right and equivalent.

4. If f is continuous on a close interval $[a, b]$ and $f(a) \neq f(b)$, then f takes on all values between $f(a)$ and $f(b)$.

5. (a) $\lim_{x\to 4^+} f(x) = 3$

(b) $\lim_{x\to 4^-} f(x) = 3$

(c) $\lim_{x\to 4} f(x) = 3$

The function is continuous at $x = 4$ and is continuous on $(-\infty, \infty)$.

6. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

7. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

8. (a) $\lim_{x\to -3^+} f(x) = 3$

(b) $\lim_{x\to -3^-} f(x) = 3$

(c) $\lim_{x\to -3} f(x) = 3$

The function is NOT continuous at $x = -3$ because $f(-3) = 4 \neq \lim_{x\to -3} f(x)$.

9. (a) $\lim_{x\to 2^+} f(x) = -3$

(b) $\lim_{x\to 2^-} f(x) = 3$

(c) $\lim_{x\to 2} f(x)$ does not exist

The function is NOT continuous at $x = 2$.

10. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

11. $\lim_{x\to 8^+} \frac{1}{x+8} = \frac{1}{8+8} = \frac{1}{16}$

12. $\lim_{x\to 3^+} \frac{2}{x+3} = \frac{2}{3+3} = \frac{1}{3}$

13. $\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x+5)(x-5)}$

$= \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$

14. $\lim_{x\to 4^+} \frac{4-x}{x^2-16} = \lim_{x\to 4^+} \frac{-(x-4)}{(x+4)(x-4)} = \lim_{x\to 4^+} \frac{-1}{x+4}$

$= \frac{-1}{4+4} = -\frac{1}{8}$

15. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

16. $\lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}\cdot\dfrac{\sqrt{x}+2}{\sqrt{x}+2}$

$= \lim_{x\to 4^-} \dfrac{x-4}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4^-} \dfrac{1}{\sqrt{x}+2} = \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{4}$

17. $\lim_{x\to 0^-} \dfrac{|x|}{x} = \lim_{x\to 0^-} \dfrac{-x}{x} = -1$

18. $\lim_{x\to 10^+} \dfrac{|x-10|}{x-10} = \lim_{x\to 10^+} \dfrac{x-10}{x-10} = 1$

19. $\lim_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \dfrac{x-(x+\Delta x)}{x(x+\Delta x)}\cdot\dfrac{1}{\Delta x}$

$= \lim_{\Delta x\to 0^-} \dfrac{-\Delta x}{x(x+\Delta x)}\cdot\dfrac{1}{\Delta x}$

$= \lim_{\Delta x\to 0^-} \dfrac{-1}{x(x+\Delta x)}$

$= \dfrac{-1}{x(x+0)} = -\dfrac{1}{x^2}$

20. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+(x+\Delta x)-(x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \dfrac{x^2+2x(\Delta x)+(\Delta x)^2+x+\Delta x-x^2-x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} \dfrac{2x(\Delta x)+(\Delta x)^2+\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} (2x+\Delta x+1)$

$= 2x+0+1 = 2x+1$

21. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x+2}{2} = \dfrac{5}{2}$

22. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-4x+6) = 9-12+6 = 3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-x^2+4x-2) = -9+12-2 = 1$

Since these one-sided limits disagree, $\lim_{x\to 3} f(x)$ does not exist.

23. $\lim_{x\to\pi} \cot x$ does not exist because $\lim_{x\to\pi^+} \cot x$ and $\lim_{x\to\pi^-} \cot x$ do not exist.

24. $\lim_{x\to\pi/2} \sec x$ does not exist because $\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

25. $\lim_{x\to 4^-} (5[\![x]\!]-7) = 5(3)-7 = 8$

$([\![x]\!] = 3 \text{ for } 3 \le x < 4)$

26. $\lim_{x\to 2^+} (2x-[\![x]\!]) = 2(2)-2 = 2$

27. $\lim_{x\to -1} \left(\left[\!\!\left[\dfrac{x}{3}\right]\!\!\right]+3\right) = \left[\!\!\left[-\dfrac{1}{3}\right]\!\!\right]+3 = -1+3 = 2$

28. $\lim_{x\to 1} \left(1-\left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right) = 1-(-1) = 2$

29. $\lim_{x\to 3^+} \ln(x-3) = \ln 0$

does not exist.

30. $\lim_{x\to 6^-} \ln(6-x) = \ln 0$

does not exist.

31. $\lim_{x\to 2^-} \ln\left[x^2(3-x)\right] = \ln\left[4(1)\right] = \ln 4$

32. $\lim_{x\to 5^+} \ln\dfrac{x}{\sqrt{x-4}} = \ln\dfrac{5}{1} = \ln 5$

33. $f(x) = \dfrac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

34. $f(x) = \dfrac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

35. $f(x) = \dfrac{[\![x]\!]}{2}+x$

has discontinuities at each integer k because $\lim_{x\to k^-} f(x) \ne \lim_{x\to k^+} f(x)$.

36. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$

because $f(1) = 2 \neq \lim_{x \to 1} f(x) = 1$.

37. $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

38. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

39. $\lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$. f is continuous on $[-1, 4]$.

40. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

41. $f(x) = \dfrac{4}{x - 6}$ has a nonremovable discontinuity at $x = 6$ because $\lim_{x \to 6} f(x)$ does not exist.

42. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

43. $f(x) = 3x - \cos x$ is continuous for all real x.

44. $f(x) = \sin x - 8x$ is continuous for all real x.

45. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

46. $f(x) = \dfrac{x}{x^2 - 4}$ has nonremovable discontinuities at $x = 2$ and $x = -2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10} = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$

has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 3} = -\frac{1}{5}.$$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

has a nonremovable discontinuity at $x = -7$ because $\lim_{x \to -7} f(x)$ does not exist.

50. $f(x) = \dfrac{2|x - 3|}{x - 3}$ has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist.

51. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} \left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (3 - x) = 1 \end{aligned}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

52. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{aligned}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

53. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim_{x\to -1} f(x) = -1$ $\qquad \lim_{x\to 1} f(x) = 1$

3. $f(-1) = \lim_{x\to -1} f(x)$ $\qquad f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

54. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim_{x\to 1} f(x) = 2$ $\qquad \lim_{x\to 5} f(x) = 2$

3. $f(1) = \lim_{x\to 1} f(x)$ $\qquad f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

55. $f(x) = \begin{cases} \ln(x + 1), & x \ge 0 \\ 1 - x^2, & x < 0 \end{cases}$

has a **possible** discontinuity at $x = 0$.

1. $f(0) = \ln(0 + 1) = \ln 1 = 0$

2. $\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= 1 - 0 = 1 \\ \lim_{x\to 0^+} f(x) &= 0 \end{aligned}\right\} \lim_{x\to 0} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 0$.

56. $f(x) = \begin{cases} 10 - 3e^{5-x}, & x > 5 \\ 10 - \frac{3}{5}x, & x \le 5 \end{cases}$

has a **possible** discontinuity at $x = 5$.

1. $f(5) = 7$

2. $\left.\begin{aligned} \lim_{x\to 5^+} f(x) &= 10 - 3e^{5-5} = 7 \\ \lim_{x\to 5^-} f(x) &= 10 - \tfrac{3}{5}(5) = 7 \end{aligned}\right\} \lim_{x\to 5} f(x) = 7$

3. $f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 5$, so, f is continuous for all real x.

57. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

58. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

59. $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer k.

60. $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

61. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2$.

62. $f(1) = 3$

Find a so that $\lim_{x\to 1^-} (ax - 4) = 3$

$$\begin{aligned} a(1) - 4 &= 3 \\ a &= 7. \end{aligned}$$

63. $\lim_{x\to a} g(x) = \lim_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim_{x\to a} (x + a) = 2a$

Find a such $2a = 8 \Rightarrow a = 4.$

64. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (a - 2x) = a$

Let $a = 4.$

65. $f(1) = \arctan(1 - 1) + 2 = 2$

Find a such that $\lim_{x\to 1^-} \left(ae^{x-1} + 3\right) = 2$

$ae^{1-1} + 3 = 2$

$a + 3 = 2$

$a = -1.$

66. $f(4) = 2e^{4a} - 2$

Find a such that $\lim_{x\to 4^+} \ln(x - 3) + x^2 = 2e^{4a} - 2$

$\ln(4 - 3) + 4^2 = 2e^{4a} - 2$

$16 = 2e^{4a} - 2$

$9 = e^{4a}$

$\ln 9 = 4a$

$a = \dfrac{\ln 9}{4} = \dfrac{\ln 3^2}{4} = \dfrac{\ln 3}{2}.$

67. $f(g(x)) = \dfrac{1}{\left(x^2 + 5\right) - 6} = \dfrac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

68. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

69. $f(g(x)) = \tan\dfrac{x}{2}$

Not continuous at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ Continuous on the open intervals

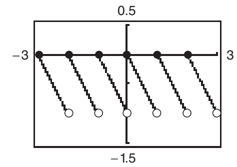
$\ldots, (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), \ldots$

70. $f(g(x)) = \sin x^2$

Continuous for all real x

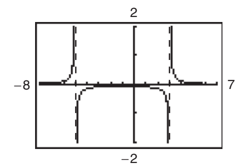
71. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



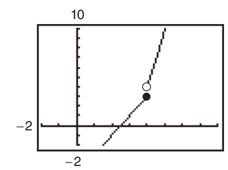
72. $h(x) = \dfrac{1}{x^2 + 2x - 15} = \dfrac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$



73. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$



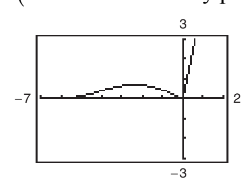
74. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5x) = 0$

Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line.
($x = 0$ was the only possible discontinuity.)



75. $f(x) = \dfrac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

76. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

77. $f(x) = 3 - \sqrt{x}$

Continuous on $[0, \infty)$

78. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty)$

79. $f(x) = \sec\dfrac{\pi x}{4}$

Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

80. $f(x) = \cos\dfrac{1}{x}$

Continuous on $(-\infty, 0)$ and $(0, \infty)$

81. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Since $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x\to 1} \dfrac{(x - 1)(x + 1)}{x - 1}$

$= \lim\limits_{x\to 1} (x + 1) = 2,$

f is continuous on $(-\infty, \infty)$.

82. $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

Since $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} (2x - 4) = 2 \neq 1,$

f is continuous on $(-\infty, 3)$ and $(3, \infty)$.

83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

84. $f(x) = x^3 + 5x - 3$ is continuous on the interval $[0, 1]$. $f(0) = -3$ and $f(1) = 3$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $f(c) = 0$.

85. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$. $f(0) = -3$ and $f(\pi) = \pi^2 - 1 \approx 8.87 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and π.

86. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\left(\dfrac{\pi}{10}\right) \approx -4.7$ and

$f(4) = -\dfrac{5}{4} + \tan\left(\dfrac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

87. $h(x) = -2e^{-x/2}\cos 2x$ is continuous on the interval $\left[0, \dfrac{\pi}{2}\right]$. $h(0) = -2 < 0$ and $h\left(\dfrac{\pi}{2}\right) \approx 0.91 > 0$.

By the Intermediate Value Theorem, there exists a number c in $\left[0, \dfrac{\pi}{2}\right]$ such that $h(c) = 0$.

88. $g(t) = \left(t^3 + 2t - 2\right)\ln\left(t^2 + 4\right)$ is continuous on the interval $[0, 1]$. $g(0) \approx -2.77 < 0$ and $g(1) \approx 1.61 > 0$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $g(c) = 0$.

89. Consider the intervals $[1, 3]$ and $[3, 5]$ for $f(x) = (x - 3)^2 = 2$.

$f(1) = 2 > 0$ and $f(3) = -2 < 0$, so f has at least one zero in $[1, 3]$.

$f(3) = -2 < 0$ and $f(5) = 2 > 0$, so f has at least one zero in $[3, 5]$.

So, f has at least two zeros in $[1, 5]$.

90. Consider the intervals $[1, 3]$ and $[3, 5]$ for $f(x) = 2\cos x$.

$f(1) = 2\cos 1 \approx 1.08 > 0$ and $f(3) = 2\cos 3 \approx -1.98 < 0$, so f has at least one zero in $[1, 3]$.

$f(3) = 2\cos 3 \approx -1.98 < 0$ and $f(5) = 2\cos 5 \approx 0.57 > 0$, so f has at least one zero in $[3, 5]$.

So, f has at least two zeros in $[1, 5]$.

91. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

92. $f(x) = x^4 - x^2 + 3x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

93. $f(x) = \sqrt{x^2 + 17x + 19} - 6$

f is continuous on $[0, 1]$.

$f(0) = \sqrt{19} - 6 \approx -1.64 < 0$

$f(1) = \sqrt{37} - 6 \approx 0.08 > 0$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.95$. Using the *root* feature, you find that $x \approx 0.9472$.

94. $f(x) = \sqrt{x^4 + 39x + 13} - 4$

f is continuous on $[0, 1]$.

$f(0) = \sqrt{13} - 4 \approx -0.39 < 0$

$f(1) = \sqrt{53} - 4 \approx 3.28 > 0$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.08$. Using the *root* feature, you find that $x \approx 0.0769$.

95. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the *root* feature, you find that $t \approx 0.5636$.

96. $h(\theta) = \tan\theta + 3\theta - 4$ is continuous on $[0, 1]$.

$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

97. $f(x) = x + e^x - 3$

f is continuous on $[0, 1]$.

$f(0) = e^0 - 3 = -2 < 0$ and

$f(1) = 1 + e - 3 = e - 2 > 0$.

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.79$. Using the *root* feature, you find that $x \approx 0.7921$.

98. $g(x) = 5\ln(x + 1) - 2$

g is continuous on $[0, 1]$.

$g(0) = 5\ln(0 + 1) - 2 = -2$ and

$g(1) = 5\ln(2) - 2 > 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(x)$, you find that $x \approx 0.49$. Using the *root* feature, you find that $x \approx 0.4918$.

99. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

100. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

101. $f(x) = \sqrt{x + 7} - 2$

f is continuous on $[0, 5]$.

$f(0) = \sqrt{7} - 2 \approx 0.6458 < 1$

$f(5) = \sqrt{12} - 2 \approx 1.4641 > 1$

The Intermediate Value Theorem applies.

$$\sqrt{x + 7} - 2 = 1$$
$$\sqrt{x + 7} = 3$$
$$x + 7 = 9$$
$$x = 2$$
$$c = 2$$

So, $f(2) = 1$.

102. $f(x) = \sqrt[3]{x} + 8$

f is continuous on $[-9, -6]$.

$f(-9) = (-9)^{1/3} + 8 \approx 5.9199 < 6$

$f(-6) = (-6)^{1/3} + 8 \approx 6.1829 > 6$

The Intermediate Value Theorem applies.

$$\sqrt[3]{x} + 8 = 6$$
$$\sqrt[3]{x} = -2$$
$$x = (-2)^3 = -8$$
$$c = -8$$

So, $f(-8) = 6$.

103. $f(x) = \dfrac{x - x^3}{x - 4}$

f is continuous on $[1, 3]$. The nonremovable discontinuity, $x = 4$, lies outside the interval.

$f(1) = \dfrac{1 - 1}{1 - 4} = 0 < 3$

$f(3) = 24 > 3$

The Intermediate Value Theorem applies.

$$\frac{x - x^3}{x - 4} = 3$$
$$x - x^3 = 3x - 12$$
$$x^3 + 2x - 12 = 0$$
$$(x - 2)(x^2 + 2x + 6) = 0$$
$$x = 2$$

($x^2 + 2x + 6$ has no real solution.)

$$c = 2$$

So, $f(2) = 3$.

104. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\dfrac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$\dfrac{35}{6} < 6 < \dfrac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

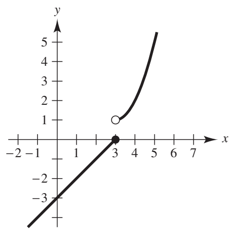
$c = 3$ ($x = 2$ is not in the interval.)

So, $f(3) = 6$.

105. Answers will vary. *Sample answer:*

$f(x) = \dfrac{1}{(x - a)(x - b)}$

106. Answers will vary. *Sample answer*:



The function is not continuous at $x = 3$ because $\lim_{x\to 3^+} f(x) = 1 \neq 0 = \lim_{x\to 3^-} f(x)$.

107. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

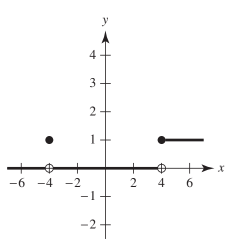
108. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x-4|}{x-4}$

(b) $f(x) = \dfrac{\sin(x+4)}{x+4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



109. True

1. $f(c) = L$ is defined.

2. $\lim_{x\to c} f(x) = L$ exists.

3. $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

110. True. If $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

111. False. $f(x) = \cos x$ has two zeros in $[0, 2\pi]$. However, $f(0)$ and $f(2\pi)$ have the same sign.

112. True. For $x \in (-1, 0)$, $[\![x]\!] = -1$, which implies that $\lim_{x\to 0^-} [\![x]\!] = -1$.

113. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

114. False. $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

115. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example,

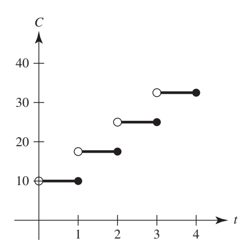
$f\left(\frac{1}{2}\right) = 3 + 0 = 3,\ g\left(\frac{1}{2}\right) = 3 - (-1) = 4.$

116. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 ounces. At the beginning of day 4, more chlorine was added, and the amount is now about 56 ounces.

117. $C(t) = 10 - 7.5[\![1 - t]\!],\ t > 0$

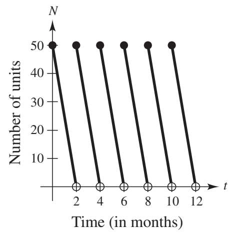


There is a nonremovable discontinuity at every integer value of t, or gigabyte.

118. $N(t) = 25\left(2\left[\!\left[\dfrac{t+2}{2}\right]\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

There is a nonremovable discontinuity at every positive even integer. The company replenishes its inventory every two months.



119. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

120. Let $V = \dfrac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and

$V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because

$523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

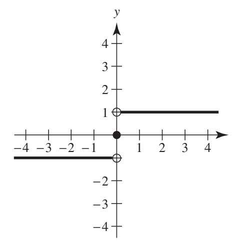
121. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

122. Let c be any real number. Then $\lim_{x\to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

123. If $x = 0$, then $f(0) = 0$ and $\lim_{x\to 0} f(x) = 0$. So, f is continuous at $x = 0$.

If $x \ne 0$, then $\lim_{t\to x} f(t) = 0$ for x rational, whereas $\lim_{t\to x} f(t) = \lim_{t\to x} kt = kx \ne 0$ for x irrational. So, f is not continuous for all $x \ne 0$.

124. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

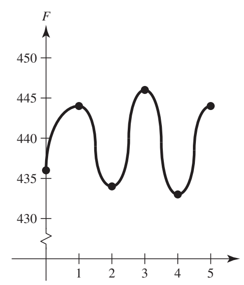


(a) $\lim_{x\to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$

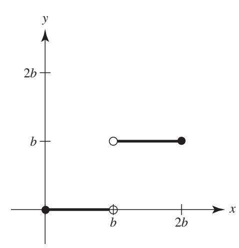
(c) $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist.

125. (a)



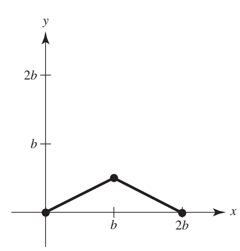
(b) No. The frequency is oscillating.

126. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

127. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

128. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

129. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \ge 0 \Rightarrow x \ge -c^2$ and $x \ne 0, [-c^2, 0) \cup (0, \infty)$

$$\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x \to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

130. **1.** $f(c)$ is defined.

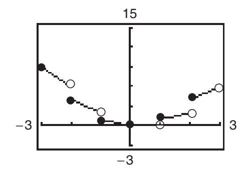
2. $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

3. $\lim_{x \to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

131. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

132. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

133. The statement is true.

If $y \ge 0$ and $y \le 1$, then $y(y - 1) \le 0 \le x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1:

If $x \le y - \frac{1}{2}$, then $2x + 1 \le 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\le (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\le x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2:

If $x \ge y - \frac{1}{2}$

$$\begin{aligned} x^2 &\ge \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \le x^2$.

134. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of x.

So, the finite degree polynomial must be constant: $P(x) = x$ for all x.

Section 2.5 Infinite Limits

1. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x\to c} f(x) = \infty$$

Says how the limit fails to exist.

2. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

3. $\lim_{x\to -2^+} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

$\lim_{x\to -2^-} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

4. $\lim_{x\to -2^+} \dfrac{1}{x + 2} = \infty$

$\lim_{x\to -2^-} \dfrac{1}{x + 2} = -\infty$

5. $\lim_{x\to -2^+} \tan\dfrac{\pi x}{4} = -\infty$

$\lim_{x\to -2^-} \tan\dfrac{\pi x}{4} = \infty$

6. $\lim_{x\to -2^+} \sec\dfrac{\pi x}{4} = \infty$

$\lim_{x\to -2^-} \sec\dfrac{\pi x}{4} = -\infty$

7. $f(x) = \dfrac{1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$$\lim_{x\to 4^-} f(x) = -\infty$$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = \infty$$

8. $f(x) = \dfrac{-1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$$\lim_{x\to 4^-} f(x) = \infty.$$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = -\infty.$$

9. $f(x) = \dfrac{1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = \infty.$$

10. $f(x) = \dfrac{-1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

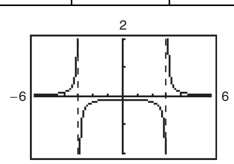
$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = -\infty.$$

11. $f(x) = \dfrac{1}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	0.308	1.639	16.64	166.6	–166.7	–16.69	–1.695	–0.364

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

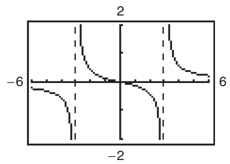


12. $f(x) = \dfrac{x}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	–1.077	–5.082	–50.08	–500.1	499.9	49.92	4.915	0.9091

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

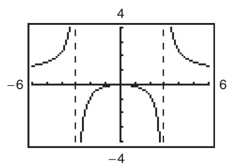


13. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	3.769	15.75	150.8	1501	–1499	–149.3	–14.25	–2.273

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

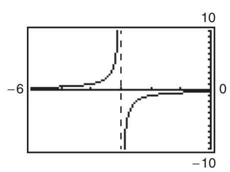


14. $f(x) = -\dfrac{1}{3 + x}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	2	10	100	1000	–1000	–100	–10	–2

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

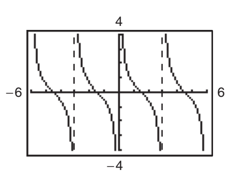


15. $f(x) = \cot \dfrac{\pi x}{3}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	–1.7321	–9.514	–95.49	–954.9	954.9	95.49	9.514	1.7321

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$



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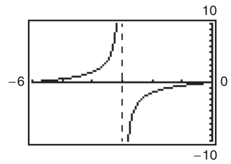
16. $f(x) = \tan \dfrac{\pi x}{6}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	3.73	19.08	190.98	1909.9	–11909.9	–190.98	–19.08	–3.73

$\lim_{x \to -3^-} f(x) = \infty$

$\lim_{x \to -3^+} f(x) = -\infty$



17. $f(x) = \dfrac{x^2}{x^2 - 4} = \dfrac{x^2}{(x+2)(x-2)}$

$\lim_{x \to -2^-} \dfrac{x^2}{x^2 - 4} = \infty$ and $\lim_{x \to -2^+} \dfrac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x \to 2^-} \dfrac{x^2}{x^2 - 4} = -\infty$ and $\lim_{x \to 2^+} \dfrac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

18. $g(t) = \dfrac{t - 1}{t^2 + 1}$

No vertical asymptotes because the denominator is never zero.

19. $f(x) = \dfrac{3}{x^2 + x - 2} = \dfrac{3}{(x+2)(x-1)}$

$\lim_{x \to -2^-} \dfrac{3}{x^2 + x - 2} = \infty$ and $\lim_{x \to -2^+} \dfrac{3}{x^2 + x - 2} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x \to 1^-} \dfrac{3}{x^2 + x - 2} = -\infty$ and $\lim_{x \to 1^+} \dfrac{3}{x^2 + x - 2} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

20. $g(x) = \dfrac{x^2 - 5x + 25}{x^3 + 125}$

$= \dfrac{x^2 - 5x + 25}{(x+5)(x^2 - 5x + 25)}$

$= \dfrac{1}{x + 5}$

$\lim_{x \to -5^-} \dfrac{1}{x + 5} = -\infty$ and $\lim_{x \to -5^+} \dfrac{1}{x + 5} = \infty$

Therefore, $x = -5$ is a vertical asymptote.

21. $f(x) = \dfrac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)}$

$= \dfrac{4(x+3)(x-2)}{x(x-2)(x^2 - 9)}$

$= \dfrac{4}{x(x-3)}, x \neq -3, 2$

$\lim_{x \to 0^-} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = -\infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

$\lim_{x \to 2} f(x) = \dfrac{4}{2(2-3)} = -2$

and

$\lim_{x \to -3} f(x) = \dfrac{4}{-3(-3-3)} = \dfrac{2}{9}$

Therefore, the graph has holes at $x = 2$ and $x = -3$.

22. $h(x) = \dfrac{x^2 - 9}{x^3 + 3x^2 - x - 3}$

$= \dfrac{(x-3)(x+3)}{(x-1)(x+1)(x+3)}$

$= \dfrac{x - 3}{(x+1)(x-1)}, x \neq -3$

$\lim_{x \to -1^-} h(x) = -\infty$ and $\lim_{x \to -1^+} h(x) = \infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim_{x \to 1^-} h(x) = \infty$ and $\lim_{x \to 1^+} h(x) = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

$\lim_{x \to -3} h(x) = \dfrac{-3 - 3}{(-3+1)(-3-1)} = -\dfrac{3}{4}$

Therefore, the graph has a hole at $x = -3$.

23. $f(x) = \dfrac{e^{-2x}}{x - 1}$

$\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

24. $g(x) = xe^{-2x}$

The function is continuous for all x. Therefore, there are no vertical asymptotes.

25. $h(t) = \dfrac{\ln(t^2 + 1)}{t + 2}$

$\lim\limits_{t \to -2^-} h(t) = -\infty$ and $\lim\limits_{t \to -2^+} = \infty$

Therefore, $t = -2$ is a vertical asymptote.

26. $f(z) = \ln(z^2 - 4) = \ln[(z + 2)(z - 2)]$

$= \ln(z + 2) + \ln(z - 2)$

The function is undefined for $-2 < z < 2$.

$\lim\limits_{z \to -2^-} f(z) = -\infty$ and $\lim\limits_{z \to -2^+} f(z) = -\infty$

Therefore, the graph vertical asymptotes at $z = \pm 2$.

27. $f(x) = \dfrac{1}{e^x - 1}$

$\lim\limits_{x \to 0^-} f(x) = -\infty$ and $\lim\limits_{x \to 0^+} f(x) = \infty$

Therefore, $x = 0$ is a vertical asymptote.

28. $f(x) = \ln(x + 3)$

$\lim\limits_{x \to -3} f(x) = -\infty$

Therefore, $x = -3$ is a vertical asymptote.

29. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

Let n be any integer.

$\lim\limits_{x \to n} f(x) = -\infty$ or ∞

Therefore, the graph has vertical asymptotes at $x = n$.

30. $f(x) = \tan \pi x = \dfrac{\sin \pi x}{\cos \pi x}$

$\cos \pi x = 0$ for $x = \dfrac{2n + 1}{2}$, where n is an integer.

$\lim\limits_{x \to \frac{2n+1}{2}} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

$x = \dfrac{2n + 1}{2}$.

31. $s(t) = \dfrac{t}{\sin t}$

$\sin t = 0$ for $t = n\pi$, where n is an integer.

$\lim\limits_{t \to n\pi} s(t) = \infty$ or $-\infty$ (for $n \neq 0$)

Therefore, the graph has vertical asymptotes at $t = n\pi$, for $n \neq 0$.

$\lim\limits_{t \to 0} s(t) = 1$

Therefore, the graph has a hole at $t = 0$.

32. $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$

$\cos \theta = 0$ for $\theta = \dfrac{\pi}{2} + n\pi$, where n is an integer.

$\lim\limits_{\theta \to \frac{\pi}{2} + n\pi} g(\theta) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

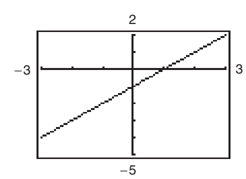
$\theta = \dfrac{\pi}{2} + n\pi$.

$\lim\limits_{\theta \to 0} g(\theta) = 1$

Therefore, the graph has a hole at $\theta = 0$.

33. $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x \to -1} (x - 1) = -2$

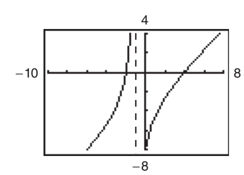
Removable discontinuity at $x = -1$



34. $\lim\limits_{x \to -1^-} \dfrac{x^2 - 2x - 8}{x + 1} = \infty$

$\lim\limits_{x \to -1^+} \dfrac{x^2 - 2x - 8}{x + 1} = -\infty$

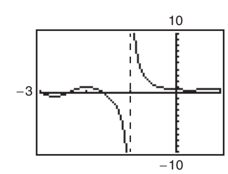
Vertical asymptote at $x = -1$



35. $\lim\limits_{x \to -1^-} \dfrac{\cos(x^2 - 1)}{x + 1} = -\infty$

$\lim\limits_{x \to -1^+} \dfrac{\cos(x^2 - 1)}{x + 1} = \infty$

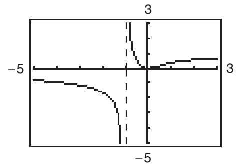
Vertical asymptote at $x = 1$



36. $\lim_{x\to -1^+} \dfrac{\ln(x^2+1)}{x+1} = \infty$

$\lim_{x\to -1^-} \dfrac{\ln(x^2+1)}{x+1} = -\infty$

Vertical asymptote at $x = -1$



37. $\lim_{x\to 2^+} \dfrac{x}{x-2} = \infty$

38. $\lim_{x\to 2^-} \dfrac{x^2}{x^2+4} = \dfrac{4}{4+4} = \dfrac{1}{2}$

39. $\lim_{x\to -3^-} \dfrac{x+3}{(x^2+x-6)} = \lim_{x\to -3^-} \dfrac{x+3}{(x+3)(x-2)}$

$= \lim_{x\to -3^-} \dfrac{1}{x-2} = -\dfrac{1}{5}$

40. $\lim_{x\to -(1/2)^+} \dfrac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+} \dfrac{(3x-1)(2x+1)}{(2x-3)(2x+1)}$

$= \lim_{x\to -(1/2)^+} \dfrac{3x-1}{2x-3} = \dfrac{5}{8}$

41. $\lim_{x\to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

42. $\lim_{x\to 0^+} \left(6 - \dfrac{1}{x^3}\right) = -\infty$

43. $\lim_{x\to -4^-} \left(x^2 + \dfrac{2}{x+4}\right) = -\infty$

44. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x} + 3\right) = -\infty$

45. $\lim_{x\to 0^+} \left(\sin x + \dfrac{1}{x}\right) = \infty$

46. $\lim_{x\to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

47. $\lim_{x\to 8^-} \dfrac{e^x}{(x-8)^3} = -\infty$

48. $\lim_{x\to 4^+} \ln(x^2-16) = -\infty$

49. $\lim_{x\to (\pi/2)^-} \ln|\cos x| = \ln\left|\cos \dfrac{\pi}{2}\right| = \ln 0 = -\infty$

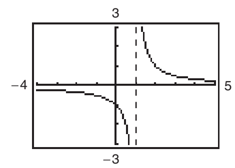
50. $\lim_{x\to 0^+} e^{-0.5x} \sin x = 1(0) = 0$

51. $\lim_{x\to (1/2)^-} x \sec \pi x = \lim_{x\to (1/2)^-} \dfrac{x}{\cos \pi x} = \infty$

52. $\lim_{x\to (1/2)^+} x^2 \tan \pi x = -\infty$

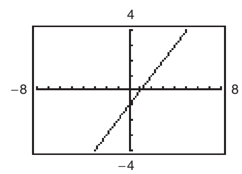
53. $f(x) = \dfrac{x^2+x+1}{x^3-1} = \dfrac{x^2+x+1}{(x-1)(x^2+x+1)}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$



54. $f(x) = \dfrac{x^3-1}{x^2+x+1} = \dfrac{(x-1)(x^2+x+1)}{x^2+x+1}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 0$



55. $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = -2$

(a) $\lim_{x\to c} \left[f(x) + g(x)\right] = \infty - 2 = \infty$

(b) $\lim_{x\to c} \left[f(x)g(x)\right] = \infty(-2) = -\infty$

(c) $\lim_{x\to c} \dfrac{g(x)}{f(x)} = \dfrac{-2}{\infty} = 0$

56. $\lim_{x\to c} f(x) = -\infty$ and $\lim_{x\to c} g(x) = 3$

(a) $\lim_{x\to c} \left[f(x) + g(x)\right] = -\infty + 3 = -\infty$

(b) $\lim_{x\to c} \left[f(x)g(x)\right] = (-\infty)(3) = -\infty$

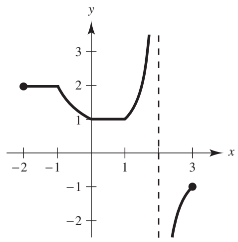
(c) $\lim_{x\to c} \dfrac{g(x)}{f(x)} = \dfrac{3}{-\infty} = 0$

57. One answer is

$f(x) = \dfrac{x-3}{(x-6)(x+2)} = \dfrac{x-3}{x^2-4x-12}.$

58. No. For example, $f(x) = \dfrac{1}{x^2+1}$ has no vertical asymptote.

59.

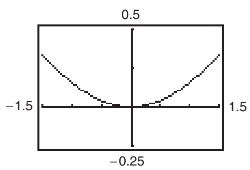


60. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

61. (a)

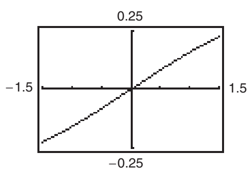
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x} = 0$$

(b)

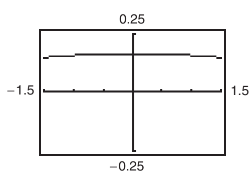
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

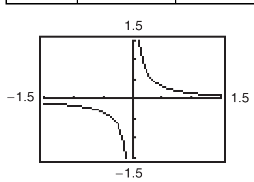
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1667 = 1/6$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$\lim_{x \to 0^+} \dfrac{x - \sin x}{x^4} = \infty$ or $n > 3$, $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^n} = \infty$.

62. $\lim_{V \to 0^+} P = \infty$

As the volume of the gas decreases, the pressure increases.

63. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\lim_{x \to 25^-} \dfrac{2x}{\sqrt{625 - x^2}} = \infty$

64. (a) $\text{Average speed} = \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\displaystyle\lim_{x\to 25^+} \frac{25x}{\sqrt{x - 25}} = \infty$

As x gets close to 25 miles per hour, y becomes larger and larger.

65. (a) $A = \dfrac{1}{2}bh - \dfrac{1}{2}r^2\theta$

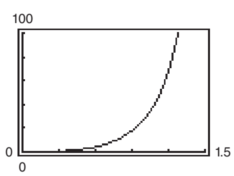
$$= \frac{1}{2}(10)(10\tan\theta) - \frac{1}{2}(10)^2\theta$$

$$= 50\tan\theta - 50\theta$$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\displaystyle\lim_{\theta\to\pi/2^-} A = \infty$

66. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20\cot\phi) + 2(10\cot\phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

So, the length of the belt around the pulleys is $20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi)$.

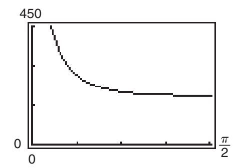
$\text{Total length} = 60\cot\phi + 30(\pi + 2\phi)$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\displaystyle\lim_{\phi\to(\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) When the angle θ approaches $\left(\dfrac{\pi}{2}\right)^-$, the total length of the belt approaches the sum of the circumferences of the two pulleys, which is $20\pi + 40\pi = 60\pi$ cm.

(h) $\displaystyle\lim_{\phi\to 0^+} L = \infty$

67. True. The function is undefined at a vertical asymptote.

68. True

69. False. The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$ and $y = \csc x$ have vertical asymptotes.

70. False. Let

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

71. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$$\lim_{x\to 0}\frac{1}{x^2} = \infty \text{ and } \lim_{x\to 0}\frac{1}{x^4} = \infty, \text{ but } \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x\to 0}\left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

72. Given $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$:

(1) Difference:

Let $h(x) = -g(x)$. Then $\lim_{x\to c} h(x) = -L$, and $\lim_{x\to c}\left[f(x) - g(x)\right] = \lim_{x\to c}\left[f(x) + h(x)\right] = \infty$, by the Sum Property.

(2) Product:

If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_1$.
So, $L/2 < g(x) < 3L/2$. Because $\lim_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $\left|x - c\right| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$,
you have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < \left|x - c\right| < \delta_1$ and there exists $\delta_2 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$.

73. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

74. Given $\lim_{x\to c} \dfrac{1}{f(x)} = 0$. Suppose $\lim_{x\to c} f(x)$ exists and equals L.

Then, $\lim_{x\to c} \dfrac{1}{f(x)} = \dfrac{\lim_{x\to c} 1}{\lim_{x\to c} f(x)} = \dfrac{1}{L} = 0.$

This is not possible. So, $\lim_{x\to c} f(x)$ does not exist.

75. $f(x) = \dfrac{1}{x - 3}$ is defined for all $x > 3$. Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 3} > M$ whenever $3 < x < 3 + \delta$. Equivalently, $x - 3 < \dfrac{1}{M}$ whenever $\left|x - 3\right| < \delta$, $x > 3$. So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and $\left|x - 3\right| < \delta$, $\dfrac{1}{x - 3} > \dfrac{1}{8} = M$ and so $f(x) > M$. Thus, $\lim_{x\to 3^+} \dfrac{1}{x - 3} = \infty$.

76. $f(x) = \dfrac{1}{x - 5}$ is defined for all $x < 5$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 5} < N$ whenever $5 - \delta < x < 5$. Equivalently, $x - 5 > \dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta$, $x < 5$. Equivalently, $\dfrac{1}{\left|x - 5\right|} < -\dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta$, $x < 5$. So take $\delta = -\dfrac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $\left|x - 5\right| < \delta$ and $x < 5$, $\dfrac{1}{\left|x - 5\right|} > \dfrac{1}{\delta} = -N$, and $\dfrac{1}{x - 5} = -\dfrac{1}{\left|x - 5\right|} < N$. Thus, $\lim_{x\to 5^-} \dfrac{1}{x - 5} = -\infty$.

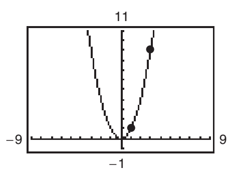
77. $f(x) = \dfrac{3}{8-x}$ is defined for all $x > 8$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{3}{8-x} < N$ whenever $8 < x < 8 + \delta$. Equivalently, $\dfrac{8-x}{3} > \dfrac{1}{N}$ whenever $|x - 8| < \delta, x > 8$. Equivalently, $|8 - x| < \dfrac{-3}{N}$ whenever $|x - 8| < \delta, x > 8$. So, let $\delta = \dfrac{-3}{N}$. Note that $\delta > 0$ because $N < 0$. Finally, for $|x - 8| < \delta$ and $x > 8$,

$\dfrac{1}{|x-8|} > \dfrac{1}{\delta} = \dfrac{N}{-3}, \dfrac{-3}{|x-8|} < N$, and $\dfrac{3}{8-x} < N$. Thus, $\displaystyle\lim_{x\to 8^+} f(x) = -\infty$.

78. $f(x) = \dfrac{6}{9-x}$ is defined for all $x < 9$. Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{6}{9-x} > M$ whenever $9 - \delta < x < 9$. Equivalently, $9 - x < \dfrac{6}{M}$ whenever $|x - 9| < \delta, x < 9$. So, let $\delta = \dfrac{6}{M}$. Finally, for $|x - 9| < \delta$ and $x < 9$, $|x - 9| < \dfrac{6}{M}, \dfrac{1}{|x-9|} > \dfrac{M}{6}$, and $\dfrac{6}{9-x} > M$. Thus, $\displaystyle\lim_{x\to 9^-} f(x) = \delta$.

Review Exercises for Chapter 2

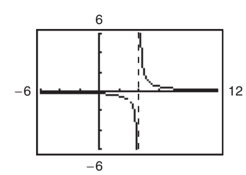
1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.



2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

3. $f(x) = \dfrac{x-3}{x^2 - 7x + 12}$

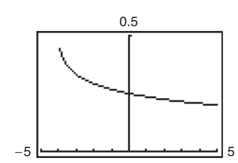
x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	–0.9091	–0.9901	–0.9990	?	–1.0010	–1.0101	–1.1111



$\displaystyle\lim_{x\to 3} f(x) \approx -1.0000$ (Actual limit is -1.)

4. $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.2516	0.2502	0.2500	?	0.2500	0.2498	0.2485



$\displaystyle\lim_{x\to 0} f(x) \approx 0.2500$ $\left(\text{Actual limit is } \tfrac{1}{4}.\right)$

5. $h(x) = \left[\!\left[-\dfrac{x}{2}\right]\!\right] + x^2$

(a) The limit does not exist at $x = 2$. The function approaches 3 from the left side of 2, but it approaches 2 from the right side of 2.

(b) $\displaystyle\lim_{x\to 1} h(x) = \left[\!\left[-\frac{1}{2}\right]\!\right] + x^2 = -1 + 1 = 0$

6. $f(t) = \dfrac{\ln(t+2)}{t}$

(a) $\lim_{t\to 0} f(t)$ does not exist because $\lim_{t\to 0^-} f(t) = -\infty$ and $\lim_{t\to 0^+} f(t) = \infty$.

(b) $\lim_{t\to -1} f(t) = \dfrac{\ln 1}{-1} = 0$

7. $\lim_{x\to 1}(x+4) = 1 + 4 = 5$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$|x - 1| < \varepsilon$$
$$|(x+4) - 5| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

8. $\lim_{x\to 9}\sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. You need

$$\left|\sqrt{x} - 3\right| < \varepsilon \Rightarrow \left|\sqrt{x} + 3\right|\left|\sqrt{x} - 3\right| < \varepsilon\left|\sqrt{x} + 3\right| \Rightarrow |x - 9| < \varepsilon\left|\sqrt{x} + 3\right|.$$

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

So, for $0 < |x - 9| < \delta = 5\varepsilon$, you have

$$|x - 9| < 5\varepsilon < \left|\sqrt{x} + 3\right|\varepsilon$$
$$\left|\sqrt{x} - 3\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

9. $\lim_{x\to 2}\left(1 - x^2\right) = 1 - 2^2 = -3$

Let $\varepsilon > 0$ be given. You need

$$\left|1 - x^2 - (-3)\right| < \varepsilon \Rightarrow \left|x^2 - 4\right| = |x - 2||x + 2| < \varepsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|}\varepsilon$$

Assuming $1 < x < 3$, you can choose $\delta = \dfrac{\varepsilon}{5}$.

So, for $0 < |x - 2| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x - 2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{|x + 2|}$$
$$|x - 2||x + 2| < \varepsilon$$
$$\left|x^2 - 4\right| < \varepsilon$$
$$\left|4 - x^2\right| < \varepsilon$$
$$\left|\left(1 - x^2\right) - (-3)\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

10. $\lim_{x\to 5} 9 = 9$. Let $\varepsilon > 0$ be given. δ can be any positive number. So, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

11. $\lim_{x\to -6} x^2 = (-6)^2 = 36$

12. $\lim_{x\to 0}(5x - 3) = 5(0) - 3 = -3$

13. $\lim_{x\to 27}\left(\sqrt[3]{x} - 1\right)^4 = \left(\sqrt[3]{27} - 1\right)^4 = (3 - 1)^4 = 2^4 = 16$

14. $\lim_{x\to 2}\sqrt{x^3 + 1} = \sqrt{2^3 + 1} = \sqrt{8 + 1} = \sqrt{9} = 3$

15. $\lim_{x\to 4}\frac{4}{x - 1} = \frac{4}{4 - 1} = \frac{4}{3}$

16. $\lim_{x\to 2}\frac{x}{x^2 + 1} = \frac{2}{2^2 + 1} = \frac{2}{4 + 1} = \frac{2}{5}$

17. $$\lim_{x\to -3}\frac{2x^2 + 11x + 15}{x + 3} = \lim_{x\to -3}\frac{(2x + 5)(x + 3)}{x + 3} = \lim_{x\to -3}(2x + 5) = 2(-3) + 5 = -1$$

18. $$\lim_{t\to 4}\frac{t^2 - 16}{t - 4} = \lim_{t\to 4}\frac{(t - 4)(t + 4)}{t - 4} = \lim_{t\to 4}(t + 4) = 4 + 4 = 8$$

19. $$\lim_{x\to 4}\frac{\sqrt{x - 3} - 1}{x - 4} = \lim_{x\to 4}\frac{\sqrt{x - 3} - 1}{x - 4}\cdot\frac{\sqrt{x - 3} + 1}{\sqrt{x - 3} + 1} = \lim_{x\to 4}\frac{(x - 3) - 1}{(x - 4)\left(\sqrt{x - 3} + 1\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x - 3} + 1} = \frac{1}{2}$$

20. $$\lim_{x\to 0}\frac{\sqrt{9 + x} - 3}{x} = \lim_{x\to 0}\frac{\sqrt{9 + x} - 3}{x}\cdot\frac{\sqrt{9 + x} + 3}{\sqrt{9 + x} + 3} = \lim_{x\to 0}\frac{(9 + x) - 9}{x\left(\sqrt{9 + x} + 3\right)} = \lim_{x\to 0}\frac{x}{x\left(\sqrt{9 + x} + 3\right)} = \lim_{x\to 0}\frac{1}{\sqrt{9 + x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

21. $$\lim_{x\to 0}\frac{[1/(x + 1)] - 1}{x} = \lim_{x\to 0}\frac{1 - (x + 1)}{x(x + 1)} = \lim_{x\to 0}\frac{-1}{x + 1} = -1$$

22. $$\lim_{s\to 0}\frac{\left(1/\sqrt{1 + s}\right) - 1}{s} = \lim_{s\to 0}\left[\frac{\left(1/\sqrt{1 + s}\right) - 1}{s}\cdot\frac{\left(1/\sqrt{1 + s}\right) + 1}{\left(1/\sqrt{1 + s}\right) + 1}\right] = \lim_{s\to 0}\frac{[1/(1 + s)] - 1}{s\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = \lim_{s\to 0}\frac{-1}{(1 + s)\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = -\frac{1}{2}$$

23. $\lim_{x\to 0}\frac{1 - \cos x}{\sin x} = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{1 - \cos x}{x}\right) = (1)(0) = 0$

24. $\lim_{x\to(\pi/4)}\frac{4x}{\tan x} = \frac{4(\pi/4)}{1} = \pi$

25. $\lim_{x\to 1} e^{x-1}\sin\frac{\pi x}{2} = e^0\sin\frac{\pi}{2} = 1$

26. $\lim_{x\to 2}\frac{\ln(x - 1)^2}{\ln(x - 1)} = \lim_{x\to 2}\frac{2\ln(x - 1)}{\ln(x - 1)} = \lim_{x\to 2} 2 = 2$

27. $$\lim_{\Delta x\to 0}\frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x + \cos(\pi/6)\sin\Delta x - (1/2)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x - 1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x} = 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}$$

28. $$\lim_{\Delta x \to 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \pi \cos \Delta x - \sin \pi \sin \Delta x + 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[-\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[\sin \pi \frac{\sin \Delta x}{\Delta x} \right]$$
$$= -0 - (0)(1) = 0$$

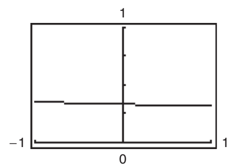
29. $$\lim_{x \to c} \left[f(x)g(x) \right] = \left[\lim_{x \to c} f(x) \right] \left[\lim_{x \to c} g(x) \right]$$
$$= (-6)\left(\tfrac{1}{2}\right) = -3$$

30. $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{-6}{\left(\frac{1}{2}\right)} = -12$$

31. $$\lim_{x \to c} \left[f(x) + 2g(x) \right] = \lim_{x \to c} f(x) + 2 \lim_{x \to c} g(x)$$
$$= -6 + 2\left(\tfrac{1}{2}\right) = -5$$

32. $$\lim_{x \to c} \left[f(x) \right]^2 = \left[\lim_{x \to c} f(x) \right]^2$$
$$= (-6)^2 = 36$$

33. $$f(x) = \frac{\sqrt{2x + 9} - 3}{x}$$



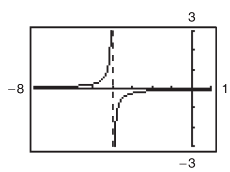
The limit appears to be $\frac{1}{3}$.

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	0.3335	0.3333	?	0.3333	0.331

$$\lim_{x \to 0} f(x) \approx 0.3333$$

$$\lim_{x \to 0} \frac{\sqrt{2x + 9} - 3}{x} \cdot \frac{\sqrt{2x + 9} + 3}{\sqrt{2x + 9} + 3} = \lim_{x \to 0} \frac{(2x + 9) - 9}{x\left[\sqrt{2x + 9} + 3\right]} = \lim_{x \to 0} \frac{2}{\sqrt{2x + 9} + 3} = \frac{2}{\sqrt{9} + 3} = \frac{1}{3}$$

34. $$f(x) = \frac{\left[1/(x + 4)\right] - (1/4)}{x}$$



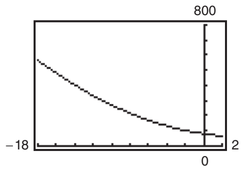
The limit appears to be $-\frac{1}{16}$

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	–0.0627	–0.0625	?	–0.0625	–0.0623

$$\lim_{x \to 0} f(x) \approx -0.0625 = -\frac{1}{16}$$

$$\lim_{x \to 0} \frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x \to 0} \frac{4 - (x + 4)}{(x + 4)4(x)} = \lim_{x \to 0} \frac{-1}{(x + 4)4} = -\frac{1}{16}$$

35. $f(x) = \dfrac{x^3 + 729}{x + 9}$



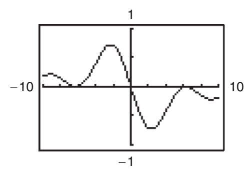
The limit appears to be 243.

x	−9.1	−9.01	−9.001	−9	−8.999	−8.99	−8.9
$f(x)$	245.7100	243.2701	243.0270	?	242.9730	242.7301	240.310

$$\lim_{x\to-9} \frac{x^3 + 729}{x + 9} \approx 243.00$$

$$\lim_{x\to-9} \frac{x^3 + 729}{x + 9} = \lim_{x\to-9} \frac{(x + 9)(x^2 - 9x + 81)}{x + 9} = \lim_{x\to-9} (x^2 - 9x + 81) = 81 + 81 + 81 = 243$$

36. $f(x) = \dfrac{\cos x - 1}{x}$



The limit appears to be 0.

x	−0.01	−0.001	0	0.001	0.01
$f(x)$	0.005	0.0005	0	−0.0005	−0.005

$$\lim_{x\to0} f(x) \approx 0.000$$

$$\begin{aligned}\lim_{x\to0} \frac{\cos x - 1}{x} &= \lim_{x\to0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}\\ &= \lim_{x\to0} \frac{\cos^2 x - 1}{x(\cos x + 1)}\\ &= \lim_{x\to0} \frac{-\sin^2 x}{x(\cos x + 1)}\\ &= \lim_{x\to0} \left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right)\\ &= (1)\left(\frac{0}{2}\right)\\ &= 0\end{aligned}$$

37. $$\begin{aligned}v &= \lim_{t\to4} \frac{s(4) - s(t)}{4 - t}\\ &= \lim_{t\to4} \frac{[-4.9(16) + 250] - [-4.9t^2 + 250]}{4 - t}\\ &= \lim_{t\to4} \frac{4.9(t^2 - 16)}{4 - t}\\ &= \lim_{t\to4} \frac{4.9(t - 4)(t + 4)}{4 - t}\\ &= \lim_{t\to4} [-4.9(t + 4)] = -39.2 \text{ m/sec}\end{aligned}$$

The object is falling at about 39.2 m/sec.

38. $-4.9t^2 + 250 = 0 \Rightarrow t = \dfrac{50}{7} \approx 7.143$

The object will hit the ground after about 7.1 seconds.

When $a = \dfrac{50}{7}$, the velocity is

$$\begin{aligned}\lim_{t\to a} \frac{s(a) - s(t)}{a - t} &= \lim_{t\to a} \frac{[-4.9a^2 + 250] - [-4.9t^2 + 250]}{a - t}\\ &= \lim_{t\to a} \frac{4.9(t^2 - a^2)}{a - t}\\ &= \lim_{t\to a} \frac{4.9(t - a)(t + a)}{a - t}\\ &= \lim_{t\to a} [-4.9(t + a)]\\ &= -4.9(2a) \qquad \left(a = \frac{50}{7}\right)\\ &= -70 \text{ m/sec.}\end{aligned}$$

39. $\displaystyle\lim_{x\to3^+} \frac{1}{x + 3} = \frac{1}{3 + 3} = \frac{1}{6}$

40. $$\begin{aligned}\lim_{x\to6^-} \frac{x - 6}{x^2 - 36} &= \lim_{x\to6^-} \frac{x - 6}{(x - 6)(x + 6)}\\ &= \lim_{x\to6^-} \frac{1}{x + 6} = \frac{1}{12}\end{aligned}$$

41. $$\begin{aligned}\lim_{x\to25^+} \frac{\sqrt{x} - 5}{x - 25} &= \lim_{x\to25^+} \frac{\sqrt{x} - 5}{(\sqrt{x} + 5)(\sqrt{x} - 5)}\\ &= \lim_{x\to25^+} \frac{1}{\sqrt{x} + 5}\\ &= \frac{1}{\sqrt{25} + 5} = \frac{1}{5 + 5} = \frac{1}{10}\end{aligned}$$

42. $\displaystyle\lim_{x\to3^-} \frac{|x - 3|}{x - 3} = \lim_{x\to3^-} \frac{-(x - 3)}{x - 3} = -1$

43. $\lim_{x\to 2} f(x) = 0$

44. $\lim_{x\to 1^+} g(x) = 1 + 1 = 2$

45. $\lim_{t\to 1} h(t)$ does not exist because $\lim_{t\to 1^-} h(t) = 1 + 1 = 2$ and $\lim_{t\to 1^+} h(t) = \frac{1}{2}(1 + 1) = 1$.

46. $\lim_{s\to -2} f(s) = 2$

47. $\lim_{x\to 2^-} \left(2[\![x]\!] + 1\right) = 2(1) + 1 = 3$

48. $\lim_{x\to 4} [\![x - 1]\!]$ does not exist. There is a break in the graph at $x = 4$.

49. $\lim_{x\to 2^-} \frac{x^2 - 4}{|x - 2|} = \lim_{x\to 2^-} \frac{(x - 2)(x + 2)}{2 - x}$
$= \lim_{x\to 2^-} -(x + 2) = -(2 + 2) = -4$

50. $\lim_{x\to 1^+} \sqrt{x(x - 1)} = \sqrt{1(1 - 1)} = 0$

51. The function $g(x) = \sqrt{8 - x^3}$ is continuous on $[-2, 2]$ because $8 - x^3 \ge 0$ on $[-2, 2]$.

52. The function $h(x) = \frac{3}{5 - x}$ is not continuous on $[0, 5]$ because $h(5)$ is not defined.

53. $f(x) = x^4 - 81x$ is continuous for all real x.

54. $f(x) = x^2 - x + 20$ is continuous for all real x.

55. $f(x) = \frac{4}{x - 5}$ has a nonremovable discontinuity at $x = 5$ because $\lim_{x\to 5} f(x)$ does not exist.

56. $f(x) = \frac{1}{x^2 - 9} = \frac{1}{(x - 3)(x + 3)}$

has nonremovable discontinuities at $x = \pm 3$ because $\lim_{x\to 3} f(x)$ and $\lim_{x\to -3} f(x)$ do not exist.

57. $f(x) = \frac{x}{x^3 - x} = \frac{x}{x(x^2 - 1)} = \frac{1}{(x - 1)(x + 1)}, \; x \ne 0$

has nonremovable discontinuities at $x = \pm 1$ because $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ do not exist, and has a removable discontinuity at $x = 0$ because

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{(x - 1)(x + 1)} = -1.$

58. $f(x) = \frac{x + 3}{x^2 - 3x - 18} = \frac{x + 3}{(x + 3)(x - 6)} = \frac{1}{x - 6},$
$x \ne -3$

has a nonremovable discontinuity at $x = 6$ because $\lim_{x\to 6} f(x)$ does not exist, and has a removable discontinuity at $x = -3$ because

$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{1}{x - 6} = -\frac{1}{9}.$

59. $f(2) = 5$

Find c so that $\lim_{x\to 2^+} (cx + 6) = 5$.

$c(2) + 6 = 5$
$2c = -1$
$c = -\frac{1}{2}$

60. $f(-1) = e^{-1+1} + 3 = 1 + 3 = 4$

So, find c such that $\lim_{x\to -1^-} \left(cx^2 - 2x\right) = 4$.

$c(-1)^2 - 2(-1) = 4$
$c + 2 = 4$
$c = 2$

61. $f(x) = -3x^2 + 7$

Continuous on $(-\infty, \infty)$

62. $f(x) = \frac{4x^2 + 7x - 2}{x + 2} = \frac{(4x - 1)(x + 2)}{x + 2}$

Continuous on $(-\infty, -2) \cup (-2, \infty)$. There is a removable discontinuity at $x = -2$.

63. $f(x) = \sqrt{x} + \cos x$ is continuous on $[0, \infty)$.

64. $f(x) = [\![x + 3]\!]$

$\lim_{x\to k^+} [\![x + 3]\!] = k + 3$ where k is an integer.

$\lim_{x\to k^-} [\![x + 3]\!] = k + 2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k + 1)$ for all integers k

65. $g(x) = 2e^{[\![x]\!]/4}$ is continuous on all intervals $(n, n + 1)$, where n is an integer. g has nonremovable discontinuities at each n.

66. $h(x) = -2 \ln |5 - x|$

Because $|5 - x| > 0$ except for $x = 5$, h is continuous on $(-\infty, 5) \cup (5, \infty)$.

67. $f(x) = \dfrac{3x^2 - x - 2}{x - 1} = \dfrac{(3x + 2)(x - 1)}{x - 1}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} (3x + 2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

68. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-} (5 - x) = 3$

$\lim_{x\to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

69. $f(x) = 2x^3 - 3$

f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

70. $f(x) = x^2 + x - 2$

Consider the intervals $[-3, 0]$ and $[0, 3]$.

$f(-3) = (-3)^2 - 3 - 2 = 4 > 0$

$f(0) = -2 < 0$

By the Intermediate Value Theorem, there is at least one zero in $[-3, 0]$.

$f(0) = -2 < 0$

$f(3) = (3)^2 + 3 - 2 = 10 > 0$

Again, there is at least one zero in $[0, 3]$.

So, there are at least two zeros in $[-3, 3]$.

71. $f(x) = x^2 + 5x - 4$

f is continuous on $[-1, 2]$.

$f(-1) = (-1)^2 + 5(-1) - 4 = -8 < 2$

$f(2) = 2^2 + 5(2) - 4 = 10 > 2$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^2 + 5x - 4 &= 2 \\ x^2 + 5x - 6 &= 0 \\ (x + 6)(x - 1) &= 0 \\ x &= 1 \ (x = -6 \text{ lies outside the interval.}) \\ c &= 1 \end{aligned}$$

So, $f(1) = 2$.

72. $f(x) = (x - 6)^3 + 4$

f is continuous on $[4, 7]$.

$f(4) = (4 - 6)^3 + 4 = -8 + 4 = -4 < 3$

$f(7) = (7 - 6)^3 + 4 = 1 + 4 = 5 > 3$

The Intermediate Value Theorem applies.

$$\begin{aligned} (x - 6)^3 + 4 &= 3 \\ (x - 6)^3 &= -1 \\ x - 6 &= -1 \\ x &= 5 \\ c &= 5 \end{aligned}$$

So, $f(5) = 3$.

73. $\lim_{x\to 6^-} \dfrac{1}{x - 6} = -\infty$

$\lim_{x\to 6^+} \dfrac{1}{x - 6} = \infty$

74. $\lim_{x\to 6^-} \dfrac{-1}{(x - 6)^2} = -\infty$

$\lim_{x\to 6^+} \dfrac{-1}{(x - 6)^2} = -\infty$

75. $f(x) = \dfrac{3}{x}$

$\lim_{x\to 0^-} \dfrac{3}{x} = -\infty$

$\lim_{x\to 0^+} \dfrac{3}{x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

76. $f(x) = \dfrac{5}{(x - 2)^4}$

$\lim_{x\to 2^-} \dfrac{5}{(x - 2)^4} = \infty = \lim_{x\to 2^+} \dfrac{5}{(x - 2)^4}$

Therefore, $x = 2$ is a vertical asymptote.

77. $f(x) = \dfrac{x^3}{x^2 - 9} = \dfrac{x^3}{(x + 3)(x - 3)}$

$\lim_{x\to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to -3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = -3$ is a vertical asymptote.

$\lim_{x\to 3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to 3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

78. $f(x) = \dfrac{6x}{36 - x^2} = -\dfrac{6x}{(x+6)(x-6)}$

$\lim\limits_{x \to -6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x \to -6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = -6$ is a vertical asymptote.

$\lim\limits_{x \to 6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x \to 6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = 6$ is a vertical asymptote.

79. $f(x) = \sec \dfrac{\pi x}{2} = \dfrac{1}{\cos \dfrac{\pi x}{2}}$

$\cos \dfrac{\pi x}{2} = 0$ when $x = \pm 1, \pm 3, \ldots$

Therefore, the graph has vertical asymptotes at $x = 2n + 1$, where n is an integer.

80. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

$\sin \pi x = 0$ for $x = n$, where n is an integer.

$\lim\limits_{x \to n} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = n$.

81. $g(x) = \ln(25 - x^2) = \ln[(5 + x)(5 - x)]$

$\lim\limits_{x \to 5} \ln(25 - x^2) = 0$

$\lim\limits_{x \to -5} \ln(25 - x^2) = 0$

Therefore, the graph has holes at $x = \pm 5$. The graph does not have any vertical asymptotes.

82. $f(x) = 7e^{-3/x}$

$\lim\limits_{x \to 0^-} 7e^{-3/x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

83. $\lim\limits_{x \to 1^-} \dfrac{x^2 + 2x + 1}{x - 1} = -\infty$

84. $\lim\limits_{x \to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

85. $\lim\limits_{x \to -1^+} \dfrac{x + 1}{x^3 + 1} = \lim\limits_{x \to -1^+} \dfrac{1}{x^2 - x + 1} = \dfrac{1}{3}$

86. $\lim\limits_{x \to -1^-} \dfrac{x + 1}{x^4 - 1} = \lim\limits_{x \to -1^-} \dfrac{1}{(x^2 + 1)(x - 1)} = -\dfrac{1}{4}$

87. $\lim\limits_{x \to 0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

88. $\lim\limits_{x \to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

89. $\lim\limits_{x \to 0^+} \dfrac{\sin 4x}{5x} = \lim\limits_{x \to 0^+} \left[\dfrac{4}{5}\left(\dfrac{\sin 4x}{4x}\right)\right] = \dfrac{4}{5}$

90. $\lim\limits_{x \to 0^-} \dfrac{\sec x^3}{2x} = -\infty$

(**Note:** $\sec x^3 \approx 1$ for x near 0.)

91. $\lim\limits_{x \to 0^+} \dfrac{\csc 2x}{x} = \lim\limits_{x \to 0^+} \dfrac{1}{x \sin 2x} = \infty$

92. $\lim\limits_{x \to 0^-} \dfrac{\cos^2 x}{x} = -\infty$

93. $\lim\limits_{x \to 0^+} \ln(\sin x) = -\infty$

94. $\lim\limits_{x \to 0^-} 12e^{-2/x} = \infty$

95. $C = \dfrac{80{,}000p}{100 - p}, \ 0 \le p < 0$

(a) $C(50) = \dfrac{80{,}000(50)}{100 - 50} = \$80{,}000$

(b) $C(90) = \dfrac{80{,}000(90)}{100 - 90} = \$720{,}000$

(c) $\lim\limits_{p \to 100^-} C(p) = \infty$

It would be financially impossible to remove 100% of the pollutants.

Problem Solving for Chapter 2

1. (a) Perimeter $\Delta PAO = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\Delta PBO = \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

x	4	2	1	0.1	0.01
Perimeter ΔPAO	33.02	9.08	3.41	2.10	2.01
Perimeter ΔPBO	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

(c) $\lim_{x \to 0^+} r(x) = \dfrac{1 + 0 + 1}{1 + 0 + 1} = \dfrac{2}{2} = 1$

2. (a) Area $\Delta PAO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(x) = \dfrac{x}{2}$

Area $\Delta PBO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(y) = \dfrac{y}{2} = \dfrac{x^2}{2}$

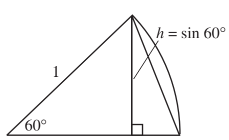
(b) $a(x) = \dfrac{\text{Area } \Delta PBO}{\text{Area } \Delta PAO} = \dfrac{x^2/2}{x/2} = x$

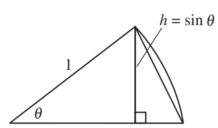
x	4	2	1	0.1	0.01
Area ΔPAO	2	1	1/2	1/20	1/200
Area ΔPBO	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\lim_{x \to 0^+} a(x) = \lim_{x \to 0^+} x = 0$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. So,

Area hexagon $= 6\left[\dfrac{1}{2}bh\right] = 6\left[\dfrac{1}{2}(1)\sin\dfrac{\pi}{3}\right] = \dfrac{3\sqrt{3}}{2} \approx 2.598.$





Error $=$ Area (Circle) $-$ Area (Hexagon) $= \pi - \dfrac{3\sqrt{3}}{2} \approx 0.5435$

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. So,

$A_n = n\left[\dfrac{1}{2}bh\right] = n\left[\dfrac{1}{2}(1)\sin\dfrac{2\pi}{n}\right] = \dfrac{n\sin(2\pi/n)}{2}.$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

As n gets larger and larger, $2\pi/n$ approaches 0. Letting $x = 2\pi/n$, $A_n = \dfrac{\sin(2\pi/n)}{2/n} = \dfrac{\sin(2\pi/n)}{(2\pi/n)}\pi = \dfrac{\sin x}{x}\pi$

which approaches $(1)\pi = \pi$, which is the area of the circle.

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$

Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$y = -\dfrac{3}{4}x + \dfrac{25}{4}$

(c) Let $Q = (x, y) = \left(x, \sqrt{25-x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2}-4}{x-3}$$

(d)
$$\lim_{x\to3} m_x = \lim_{x\to3}\frac{\sqrt{25-x^2}-4}{x-3}\cdot\frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}$$
$$= \lim_{x\to3}\frac{25-x^2-16}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to3}\frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to3}\frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P.

5. (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$y + 12 = \dfrac{5}{12}(x-5)$

$y = \dfrac{5}{12}x - \dfrac{169}{12}$ Tangent line

(c) $Q = (x, y) = \left(x, -\sqrt{169-x^2}\right)$

$$m_x = \frac{-\sqrt{169-x^2}+12}{x-5}$$

(d)
$$\lim_{x\to5} m_x = \lim_{x\to5}\frac{12-\sqrt{169-x^2}}{x-5}\cdot\frac{12+\sqrt{169-x^2}}{12+\sqrt{169-x^2}}$$
$$= \lim_{x\to5}\frac{144-\left(169-x^2\right)}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to5}\frac{x^2-25}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to5}\frac{(x+5)}{12+\sqrt{169-x^2}} = \frac{10}{12+12} = \frac{5}{12}$$

This is the same slope as part (b).

6.
$$\frac{\sqrt{a+bx}-\sqrt{3}}{x} = \frac{\sqrt{a+bx}-\sqrt{3}}{x}\cdot\frac{\sqrt{a+bx}+\sqrt{3}}{\sqrt{a+bx}+\sqrt{3}} = \frac{(a+bx)-3}{x\left(\sqrt{a+bx}+\sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

So, $\displaystyle\lim_{x\to0}\frac{\sqrt{3+bx}-\sqrt{3}}{x} = \lim_{x\to0}\frac{bx}{x\left(\sqrt{3+bx}+\sqrt{3}\right)} = \lim_{x\to0}\frac{b}{\sqrt{3+bx}+\sqrt{3}}$

Setting $\dfrac{b}{\sqrt{3}+\sqrt{3}} = \sqrt{3}$, you obtain $b = 6$. So, $a = 3$ and $b = 6$.

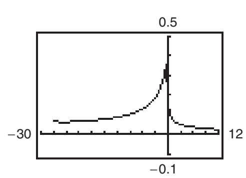
7. (a) $3 + x^{1/3} \ge 0$

$x^{1/3} \ge -3$

$x \ge -27$

Domain: $x \ge -27, x \ne 1$ or $[-27, 1) \cup (1, \infty)$

(b)



(c) $\displaystyle\lim_{x\to-27^+} f(x) = \frac{\sqrt{3+(-27)^{1/3}}-2}{-27-1} = \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$

(d) $$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2} = \lim_{x\to 1} \frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to 1} \frac{x^{1/3} - 1}{\left(x^{1/3} - 1\right)\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)} = \lim_{x\to 1} \frac{1}{\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

8. $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(a^2 - 2\right) = a^2 - 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{ax}{\tan x} = a\left(\text{because } \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

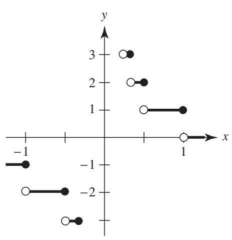
Thus, $$a^2 - 2 = a$$
$$a^2 - a - 2 = 0$$
$$(a - 2)(a + 1) = 0$$
$$a = -1, 2$$

9. (a) $\lim_{x\to 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x\to 2^-} f(x) = 3$: g_1, g_3, g_4

10.



(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x\to 1^-} f(x) = 1$

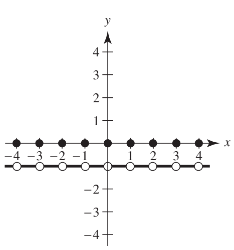
$\lim_{x\to 1^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = -\infty$

$\lim_{x\to 0^+} f(x) = \infty$

(c) f is continuous for all real numbers except $x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x\to 1^-} f(x) = -1$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1/2} f(x) = -1$

(c) f is continuous for all real numbers except $x = 0, \pm 1, \pm 2, \pm 3, \ldots$

12. (a) $$v^2 = \frac{192{,}000}{r} + v_0^2 - 48$$

$$\frac{192{,}000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192{,}000}{v^2 - v_0^2 + 48}$$

$$\lim_{v\to 0} r = \frac{192{,}000}{48 - v_0^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ mi/sec.

(b) $$v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v\to 0} r = \frac{1920}{2.17 - v_0^2}$$

Let $v_0 = \sqrt{2.17}$ mi/sec $\quad(\approx 1.47$ mi/sec$)$.

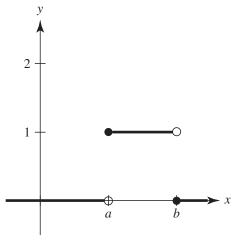
(c) $$r = \frac{10{,}600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v\to 0} r = \frac{10{,}600}{6.99 - v_0^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Because this is smaller than the escape velocity for Earth, the mass is less.

13. (a)



(b) (i) $\lim_{x\to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x\to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x\to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of U, and above the x-axis, is 1.

14. Let $a \neq 0$ and let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta_1$ then $|f(x) - L| < \varepsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \varepsilon.$$

As a counterexample, let $a = 0$ and

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x) = 1 = L$, but

$$\lim_{x\to 0} f(ax) = \lim_{x\to 0} f(0) = \lim_{x\to 0} 2 = 2.$$

Chapter 2 Limits and Their Properties

Chapter Comments

Section 2.1 gives a preview of calculus. On pages 67 and 68 of the textbook are examples of some of the concepts from precalculus extended to ideas that require the use of calculus. Review these ideas with your students to give them a feel for where the course is heading.

The idea of a limit is central to calculus. So, you should take the time to discuss the tangent line problem and/or the area problem in this section. Exercise 11 of Section 2.1 is yet another example of how limits will be used in calculus. A review of the formula for the distance between two points can be found in Appendix C.

The discussion of limits is difficult for most students the first time that they see it. For this reason, you should carefully go over the examples and the informal definition of a limit presented in Section 2.2. Stress to your students that a limit exists only if $f(x)$ moves arbitrarily close to a single real number as x approaches c from either side (the function does not have to exist at $x = c$). Otherwise, the limit fails to exist, as shown in Examples 3, 4, and 5 of this section. You might want to work Exercise 71 of Section 2.2 with your students in preparation for the definition of the number e coming up in Section 2.3. You may choose to omit the formal definition of a limit.

Carefully go over the properties of limits found in Section 2.3 to ensure that your students are comfortable with the idea of a limit and also with the notation used for limits. Suggest that students write these properties of limits on a sheet of paper for reference. By the time you get to Theorem 2.6, it should be obvious to your students that all of these properties amount to direct substitution.

When direct substitution for the limit of a quotient yields the indeterminate form $\frac{0}{0}$, remind your students that they must rewrite the fraction using legitimate algebra. Then do at least one problem using dividing out techniques and another using rationalizing techniques. Exercises 58 and 60 of Section 2.3 are examples of other algebraic techniques needed for the limit problems. It is important to go over the Squeeze Theorem, Theorem 2.8, with your students so that you can use it to prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$. The proof of Theorem 2.8 and many other theorems can be found in Appendix A. Your students need to memorize all three of the special limits in Theorem 2.9 as they will need these facts to do problems throughout the textbook. Most of your students will need help with Exercises 63–85 in this section.

Continuity, which is discussed in Section 2.4, is another idea that often puzzles students. However, if you describe a continuous function as one in which you can draw the entire graph without lifting your pencil, the idea seems to stay with them. Distinguishing between removable and nonremovable discontinuities will help students determine vertical asymptotes.

To discuss infinite limits, Section 2.5, remind your students of the graph of the function $f(x) = 1/x$ studied in Section 1.3. Be sure to make your students write a vertical asymptote as an equation, not just a number. For example, for the function $y = 1/x$, the vertical asymptote is $x = 0$.

Section 2.1 A Preview of Calculus

Section Comments

2.1 **A Preview of Calculus**—Understand what calculus is and how it compares with precalculus. Understand that the tangent line problem is basic to calculus. Understand that the area problem is also basic to calculus.

Teaching Tips

In this first section, students see how limits arise when attempting to find a tangent line to a curve. Be sure to say to students that they can think about the word *tangent* as meaning “touching” a curve at one particular point. Drawing a circle with a line that touches the circle at a specific point can illustrate tangency. Drawing a curve where a line intersects a curve twice illustrates a line that is not tangent to a curve as it crosses the curve more than once.

How Do You See It? Exercise

Page 71, Exercise 10 How would you describe the instantaneous rate of change of an automobile’s position on a highway?

Solution

Answers will vary. *Sample answer:* The instantaneous rate of change of an automobile’s position is the velocity of the automobile, and can be determined by the speedometer.

Suggested Homework Assignment

Page 71: 1–9 odd.

Section 2.2 Finding Limits Graphically and Numerically

Section Comments

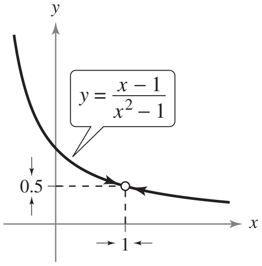
2.2 **Finding Limits Graphically and Numerically**—Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

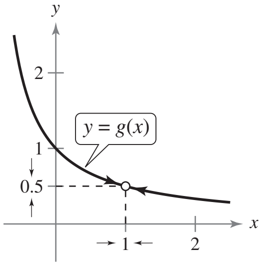
Teaching Tips

In this section, we turn our focus to limits in general, and numerical and graphical ways we can find them. Ask students to consider $f(x) = \dfrac{x^2 - 4}{x - 2}$ and what is happening to $f(x)$ as x approaches 2. This will lead into the definition of a limit.

Make sure that students find limits analytically instead of using their graping utilities, as they can be misleading. For example, present to the class, $\lim\limits_{x \to 0} \sin \dfrac{\pi}{x}$. Constructing a table of values of $x = 1$, $\dfrac{1}{3}$, 0.1, $\dfrac{1}{2}$, $\dfrac{1}{4}$, and 0.01 leads to the assumption that $\lim\limits_{x \to 0} \sin \dfrac{\pi}{x} = 0$. However, this limit does not exist.

When introducing the definition of a limit, using graphs will help students understand the meaning of the definition. Suggested graphs are below:

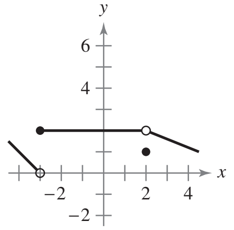




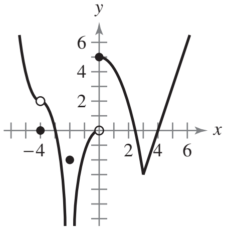
How Do You See It? Exercise

Page 82, Exercise 74 Use the graph of f to identify the values of c for which $\lim_{x \to c} f(x)$ exists.

(a)



(b)



Solution

(a) $\lim_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

Suggested Homework Assignment

Pages 79–82: 1, 5, 7, 9, 15, and 19–35 odd.

Section 2.3 Evaluating Limits Analytically

Section Comments

2.3 **Evaluating Limits Analytically**—Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using the dividing out technique. Evaluate a limit using the rationalizing technique. Evaluate a limit using the Squeeze Theorem.

Teaching Tips

When starting this section, review how to factor using differences of two squares and two cubes, sum of two cubes, and rationalizing numerators. If you do not have time to review these concepts, encourage students to study the following material in *Precalculus*, 10th edition, by Larson.

- Factoring the difference of two squares: *Precalculus* Appendix A.3
- Factoring the difference of two cubes: *Precalculus* Appendix A.3
- Factoring the sum of two cubes: *Precalculus* Appendix A.3
- Rationalizing numerators: *Precalculus* Appendix A.2

State all limit properties as presented in this section and discuss the proper ways to write solutions when finding limits analytically. Give examples of direct substitution; some suggested examples include $\lim_{x \to 2} \dfrac{x^2 - 3x + 4}{x + 2}$ and $\lim_{x \to -3} 4x^3 - 5x + 1$.

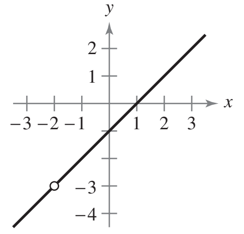
When rationalizing the numerator, start with an example without limits. A suggestion is $\dfrac{\sqrt{x+2} - \sqrt{2}}{x}$. After rationalizing the numerator, find the limit as x approaches 0.

When evaluating trigonometric limits, show students examples directly related to Theorem 2.9 in the text. Some examples are $\lim_{x \to 0} \dfrac{\sin 7x}{4x}$ and $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{\sin \theta}$.

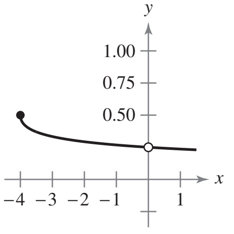
How Do You See It? Exercise

Page 92, Exercise 104 Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim_{x \to -2} \frac{x^2 + x - 2}{x + 2}$



(b) $\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$



Solution

(a) Use the dividing out technique because the numerator and denominator have a common factor.

(b) Use the rationalizing technique because the numerator involves a radical expression.

Suggested Homework Assignment

Pages 91–93: 1–75 odd, 89, 99, 105, 107, and 119–123 odd.

Section 2.4 Continuity and One-Sided Limits

Section Comments

2.4 **Continuity and One-Sided Limits**—Determine continuity at a point and continuity on an open interval. Determine one-sided limits and continuity on a closed interval. Use properties of continuity. Understand and use the Intermediate Value Theorem.

Teaching Tips

Students have trouble with limits involving trigonometric functions and rationalizing denominators and numerators. Students also have trouble with limits involving rational functions. Encourage students to sharpen their algebra skills by studying the following material in the text or in *Precalculus*, 10th edition, by Larson.

- Trigonometric functions: Section 1.4
- Rationalizing numerators: *Precalculus* Appendix A.2
- Simplifying rational expressions: *Precalculus* Appendix A.4

Address the following for this section:

- Describe the difference between a removable discontinuity and a nonremovable discontinuity.
- Write a function that has a removable discontinuity.
- Write a function that has a nonremovable discontinuity.
- Write a function that has a removable discontinuity and a nonremovable discontinuity.

Students will need to know these concepts when they study vertical asymptotes (Section 2.5), improper integrals (Section 8.8), and functions of several variables (Section 13.2).

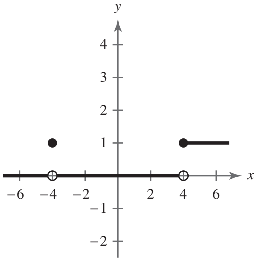
Discuss the difference between a removable discontinuity and a nonremovable discontinuity. Use the function $f(x) = \frac{|x - 4|}{x - 4}$ to illustrate a function with a nonremovable discontinuity.

Note that it is not possible to remove the discontinuity by redefining the function. When you illustrate a removable discontinuity, as in the function $f(x) = \dfrac{\sin(x + 4)}{x + 4}$, note that the function can be redefined to remove the discontinuity. Ask students how they would do this. (Let $f(-4) = 1$.) Finally, illustrate the example below. Show students the function and the graph, and then ask them if there are any removable or nonremovable discontinuities. Once they have properly identified $x = -4$ as removable and $x = 4$ as nonremovable, ask them to redefine f to remove the discontinuity at $x = -4$. (Let $f(-4) = 0$.)

Ask students to consider the following function with its graph below:

$$f(x) = \begin{cases} 0, & x < -4 \\ 1, & x = -4 \\ 0, & -4 < x < 4 \\ 1, & x \geq 4 \end{cases}$$

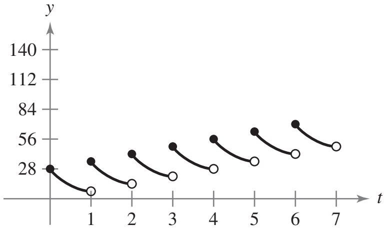
$x = 4$ is nonremovable. $x = -4$ is removable.



Use this example to show which are removable and nonremovable discontinuities. In addition, use a rational function with both a vertical asymptote and a hole to show the differences between removable and nonremovable discontinuities. A suggested example is $f(x) = \dfrac{8x^2 + 26x + 15}{2x^2 - x - 15}$.

How Do You See It? Exercise

Page 105, Exercise 116 Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days. Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.



Solution

$\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 ounces. At the beginning of day 4, more chlorine was added, and the amount is now about 56 ounces.

Suggested Homework Assignment

Pages 103–105: 1, 3, 7–19 odd, 21, 23, 25, 29, 31, 33, 37, 39, 47, 49, 51, 55, 59, 61, 65, 67, 75–81 odd, 83, 99, 101, 105, and 109–113 odd.

Section 2.5 Infinite Limits

Section Comments

2.5 **Infinite Limits**—Determine infinite limits from the left and from the right. Find and sketch the vertical asymptotes of the graph of a function.

Teaching Tips

It is vital for students to know asymptotes of a rational function for this section. Encourage students to review material on asymptotes by studying Section 2.6 in *Precalculus*, 10th edition, by Larson.

Students need to be able to decipher when vertical asymptotes versus holes will occur in a rational function. A suggested problem for students to consider is to determine if $x = 1$ is a vertical asymptote of $f(x) = \dfrac{p(x)}{x - 1}$.

In class, you can ask students who believe the graph of f has a vertical asymptote at $x = 1$ to prove it. For those who think this is not always true, ask them to provide a counterexample. A sample answer is given in the solution. If desired, you could also ask students the following:

(a) At which x-values (if any) is f not continuous?

(b) Which of the discontinuities are removable?

Solution

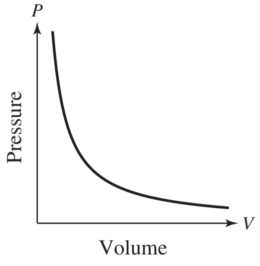
No, it is not always true. Consider $p(x) = x^2 - 1$. The function

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{p(x)}{x - 1}$$

has a hole at $(1, 2)$, not a vertical asymptote.

How Do You See It? Exercise

Page 113, Exercise 62 For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. What is the limit of P as V approaches 0 from the right? Explain what this means in the context of the problem.



Solution

$$\lim_{V \to 0^+} P = \infty$$

As the volume of the gas decreases, the pressure increases.

Suggested Homework Assignment

Pages 112–114: 1, 3, 7, 13, 17–31 odd, 37–51 odd, 55, 57, and 67–71 odd.

Chapter 2 Project

Medicine in the Bloodstream

A patient's kidneys purify 25% of the blood in her body in 4 hours.

Exercises

In Exercises 1–3, a patient takes one 16-milliliter dose of a medication.

1. Determine the amount of medication left in the patient's body after 4, 8, 12, and 16 hours.

2. Notice that after the first 4-hour period, $\frac{3}{4}$ of the 16 milliliters of medication is left in the body, after the second 4-hour period, $\frac{9}{16}$ of the 16 milliliters of medication is left in the body, and so on. Use this information to write an equation that represents the amount a of medication left in the patient's body after n 4-hour periods.

3. Can you find a value of n for which a equals 0? Explain.

In Exercises 4–9, the patient takes an additional 16-milliliter dose every 4 hours.

4. Determine the amount of medication in the patient's body immediately after taking the second dose.

5. Determine the amount of medication in the patient's body immediately after taking the third and fourth doses. What is happening to the amount of medication in the patient's body over time?

6. The medication is eliminated from the patient's body at a constant rate. Sketch a graph that shows the amount of medication in the patient's body during the first 16 hours. Let x represent the number of hours and y represent the amount of medication in the patient's body in milliliters.

7. Use the graph in Exercise 6 to find each limit.

(a) $\lim_{x \to 4^-} f(x)$

(b) $\lim_{x \to 4^+} f(x)$

(c) $\lim_{x \to 12^-} f(x)$

(d) $\lim_{x \to 12^+} f(x)$

8. Discuss the continuity of the function represented by the graph in Exercise 6. Interpret any discontinuities in the context of the problem.

9. The amount of medication in the patient's body remains constant when the amount eliminated in 4 hours is equal to the additional dose taken at the end of the 4-hour period. Write and solve an equation to find this amount.

Chapter 2, Section 1, page 69

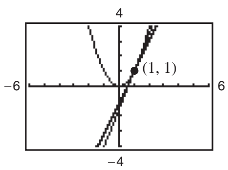
$P(1, 1)$ and $Q_1(1.5, f(1.5))$: $m = \dfrac{2.25 - 1}{1.5 - 1} = \dfrac{1.25}{0.5} = 2.5$

$P(1, 1)$ and $Q_2(1.1, f(1.1))$: $m = \dfrac{1.21 - 1}{1.1 - 1} = \dfrac{0.21}{0.1} = 2.1$

$P(1, 1)$ and $Q_3(1.01, f(1.01))$: $m = \dfrac{1.0201 - 1}{1.01 - 1} = \dfrac{0.0201}{0.01} = 2.01$

$P(1, 1)$ and $Q_4(1.001, f(1.001))$: $m = \dfrac{1.002001}{1.001 - 1} = \dfrac{0.002001}{0.001} = 2.001$

$P(1, 1)$ and $Q_5(1.0001, f(1.0001))$: $m = \dfrac{1.00020001}{1.0001 - 1} = \dfrac{0.00020001}{0.0001} = 2.0001$



$m = 2$

Chapter 2, Section 1, page 70

The width of each rectangle is $\dfrac{1}{5}$.

The area of the inscribed set of rectangles is:

$$\begin{aligned} A &= \frac{1}{5} \cdot f(0) + \frac{1}{5} \cdot f\left(\frac{1}{5}\right) + \frac{1}{5} \cdot f\left(\frac{2}{5}\right) + \frac{1}{5} \cdot f\left(\frac{3}{5}\right) + \frac{1}{5} \cdot f\left(\frac{4}{5}\right) \\ &= \frac{1}{5}\left[0 + \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right] \\ &= \frac{1}{5}\left[\frac{30}{25}\right] \\ &= \frac{6}{25} = 0.24. \end{aligned}$$

The area of the circumscribed set of rectangles is:

$$\begin{aligned} A &= \frac{1}{5} \cdot f\left(\frac{1}{5}\right) + \frac{1}{5} \cdot f\left(\frac{2}{5}\right) + \frac{1}{5} \cdot f\left(\frac{3}{5}\right) + \frac{1}{5} \cdot f\left(\frac{4}{5}\right) + \frac{1}{5} \cdot f(1) \\ &= \frac{1}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + 1\right] \\ &= \frac{1}{5}\left[\frac{55}{25}\right] \\ &= \frac{11}{25} = 0.44. \end{aligned}$$

The area of the region is approximately $\dfrac{1}{2}\left(\dfrac{6}{25} + \dfrac{11}{25}\right) = \dfrac{17}{50} = 0.34.$

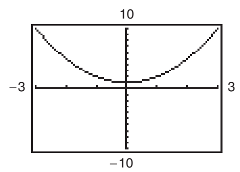
Chapter 2, Section 2, page 72

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25

The graph of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ agrees with the graph of $g(x) = x - 1$ at all points but one, where $x = 2$. If you trace along f getting closer and closer to $x = 2$, the value of y will get closer and closer to 1.

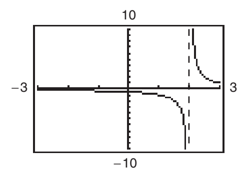
Chapter 2, Section 4, page 94

a.



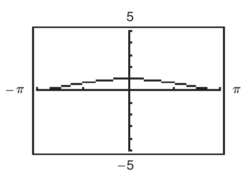
$y = x^2 + 1$ looks continuous on $(-3, 3)$.

b.



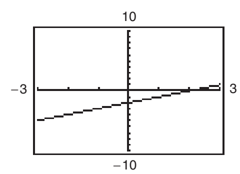
$y = \dfrac{1}{x - 2}$ does not look continuous on $(-3, 3)$.

c.



$y = \dfrac{\sin x}{x}$ looks continuous on $(-\pi, \pi)$.

d.

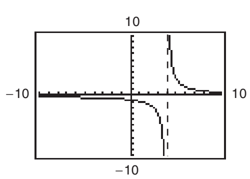


$y = \dfrac{x^2 - 4}{x + 2}$ looks continuous on $(-3, 3)$.

You cannot trust the results you obtain graphically. In these examples, only part **a.** is continuous. Part **b.** is discontinuous at $x = 2$; part **c.** is discontinuous at $x = 0$; part **d.** is discontinuous at $x = -2$.

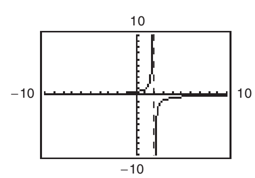
Chapter 2, Section 5, page 108

a.



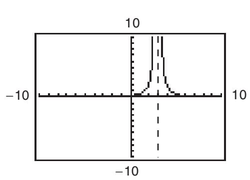
$c = 4;\ \lim\limits_{x \to 4^-} f(x) = -\infty;\ \lim\limits_{x \to 4^+} f(x) = +\infty$

b.



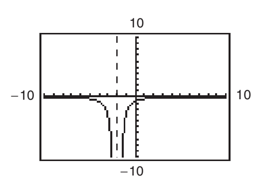
$c = 2;\ \lim\limits_{x \to 2^-} f(x) = +\infty;\ \lim\limits_{x \to 2^+} f(x) = -\infty$

c.



$c = 3;\ \lim\limits_{x \to 3^-} f(x) = +\infty;\ \lim\limits_{x \to 3^+} f(x) = +\infty$

d.

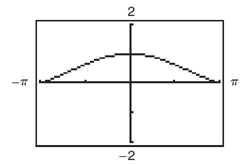


$c = -2;\ \lim\limits_{x \to 2^-} f(x) = -\infty;\ \lim\limits_{x \to -2^+} f(x) = -\infty$

SECTION PROJECTS

Chapter 2, Section 5, page 114 Graphs and Limits of Trigonometric Functions

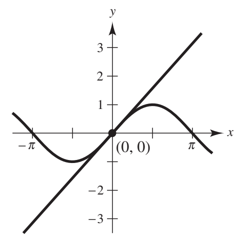
(a) On the graph of f, it appears that the y-coordinates of points lie as close to 1 as desired as long as you consider only those points with an x-coordinate near to but not equal to 0.



(b) Use a table of values of x and $f(x)$ that includes several values of x near 0. Check to see if the corresponding values of $f(x)$ are close to 1. In this case, because f is an even function, only positive values of x are needed.

x	0.5	0.1	0.01	0.001
$f(x)$	0.9589	0.9983	1.0000	1.0000

(c) The slope of the sine function at the origin appears to be 1. (It is necessary to use radian measure and have the same unit of length on both axes.)



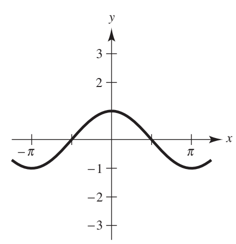
(d) In the notation of Section 2.1, $c = 0$ and $c + \Delta x = x$. Thus, $m_{\text{sec}} = \dfrac{\sin x - 0}{x - 0}$.

This formula has a value of 0.998334 if $x = 0.1$; $m_{\text{sec}} = 0.999983$ if $x = 0.01$.

The exact slope of the tangent line to g at $(0, 0)$ is $\lim\limits_{x\to 0} m_{\text{sec}} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

(e) The slope of the tangent line to the cosine function at the point $(0, 1)$ is 0. The analytical proof is as follows:

$$\lim_{\Delta x\to 0} m_{\text{sec}} = \lim_{\Delta x\to 0} \frac{\cos(0 + \Delta x) - 1}{\Delta x} = -\lim_{\Delta x\to 0} \frac{1 - \cos(\Delta x)}{\Delta x} = 0.$$



(f) The slope of the tangent line to the graph of the tangent function at $(0, 0)$ is:

$$\lim_{\Delta x\to \infty} m_{\text{sec}} = \lim_{\Delta x\to 0} \frac{\tan(0 + \Delta x) - 0}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sin(\Delta x)}{\Delta x} \cdot \frac{1}{\cos \Delta x} = 1 \cdot \frac{1}{1} = 1.$$

Chapter 3, Section 5, page 177 Optical Illusions

(a)
$$x^2 + y^2 = C^2$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

at the point $(3, 4)$, $y' = -\dfrac{3}{4}$

(b)
$$xy = C$$
$$xy' + y = 0$$
$$y' = -\frac{y}{x}$$

at the point $(1, 4)$, $y' = -4$

