Solutions for Applied Probability and Stochastic Processes 2nd Edition by Beichelt

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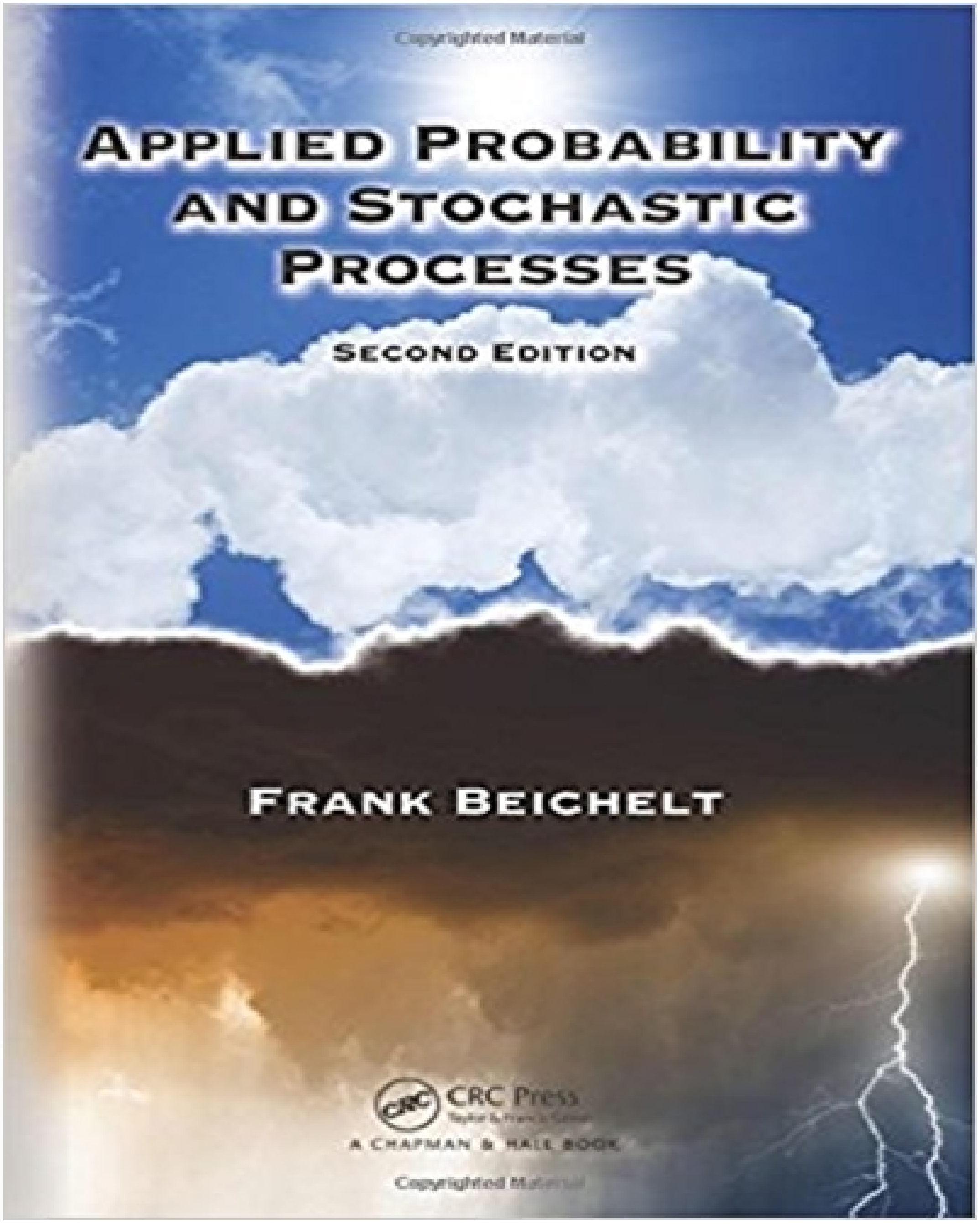


Solutions

CHAPTER 2

One-Dimensional Random Variables

Sections 2.1 and 2.2

2.1) An ornithologist measured the weight of 132 eggs of helmeted guinea fowls [*gram*]:

number i	1	2	3	4	5	6	7	8	9	10	
weight x_i	38	41	42	43	44	45	46	47	48	50	(*)
number of eggs n_i	4	6	7	10	13	26	33	16	10	7	

There are no eggs weighing less than 38 or more than 50. Let X be the weight of a randomly picked egg from this sample.

(1) Determine the probability distribution of X.

(2) Draw the distribution function of X.

(3) Determine the probabilities $P(43 \le X \le 48)$ and $P(X > 45)$.

(4) Determine $E(X)$, $\sqrt{Var(X)}$, and $E(|X - E(X)|)$.

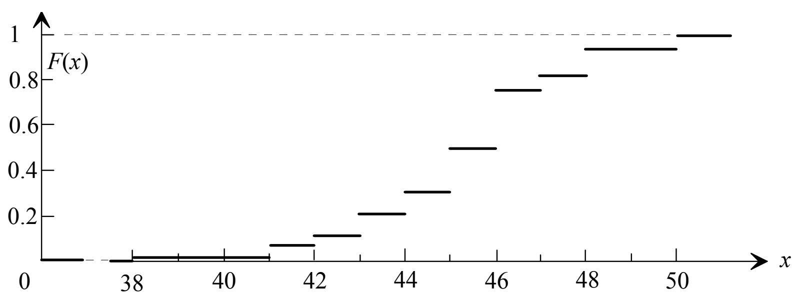
Solution

(1) The probability distribution of the discrete random variable X is given by

$$\{p_i = P(X = x_i) = \tfrac{n_i}{132};\ i = 1, 2, ..., 10\}.$$

i	1	2	3	4	5	6	7	8	9	10
p_i	0.0303	0.0455	0.0530	0.0758	0.0985	0.1970	0.2500	0.1212	0.0758	0.0530
x_i	38	41	42	43	44	45	46	47	48	50
$F(x_i) = P(X \le x_i)$	0.0303	0.0758	0.1288	0.2046	0.3031	0.5001	0.7500	0.8712	0.9470	1

(2)



(3) $P(43 \le X \le 48) = \Sigma_{i=4}^{9} p_i = 0.8183$, $\quad P(X > 45) = \Sigma_{i=7}^{10} p_i = 0.5$.

(4) $E(X) = \Sigma_{i=1}^{10} p_i x_i = 45.182$, $\quad \sqrt{Var(X)} = \sqrt{\Sigma_{i=1}^{10} p_i (x_i - E(X))^2} = 2.4056$.

$E(|X - E(X)|) = \Sigma_{i=1}^{10} p_i |x_i - E(X)| = 1.7879$.

These numerical values have been directly calculated from table (*).

2.2) 114 nails are classified by length:

number i		1	2	3	4	5	6	7	
length x_i (*mm*)	< 15	15.0	15.1	15.2	15.3	15.4	15.5	15.6	> 15.6
number of nails n_i	0	3	10	25	40	18	16	2	0

Let X denote the length of a nail selected randomly from this set.

(1) Determine the probability distribution of X and draw its histogram.

(2) Determine the probabilities $P(X \le 5)$ and $P(15.0 < X \le 15.5)$.

(3) Determine

$$E(X),\ m_3 = E(X - E(X))^3,\ \sqrt{Var(X)}\ ,\ x_m,\ \gamma_C, \text{ and } \gamma_P.$$

Interprete the skewness measures.

Solution

(1) The probability distribution of X is given by

$$\{p_i = P(X = x_i) = \tfrac{n_i}{114};\ i = 1, 2, ..., 7\}.$$

(2) $P(X \le 5) = 0,\quad P(15.0 < X \le 15.5) = \frac{1}{114}(n_2 + n_3 + n_4 + n_5 + n_6) = \frac{109}{114} \approx 0.9561.$

(3) $E(X) = \frac{1}{114}(3 \cdot 15 + 10 \cdot 15.1 + 25 \cdot 15.2 + 40 \cdot 15.3 + 18 \cdot 15.4 + 16 \cdot 15.5 + 2 \cdot 15.6) = 15.3018.$

$m_3 = \frac{1}{114} 3 \cdot (15 - 15.3018)^3 + 10 \cdot (15.1 - 15.3018)^3 + \cdots + 2 \cdot (15.6 - 15.3018)^3) = 0.0000319.$

$\sigma^2 = Var(X) = \frac{1}{114}\left[3 \cdot (15 - 15.3018)^2 + 10 \cdot (15.1 - 15.3018)^2 + \cdots + 2 \cdot (15.6 - 15.3018)^2\right]$

$= 1.91965/114 = 0.01684.$

$\sigma = \sqrt{Var(X)}\ = 0.12977$

$x_m = 15.3,\quad \gamma_C = \frac{m_3}{\sigma^3} = 0.0146,\quad \gamma_P = \frac{E(X) - x_m}{\sigma} = \frac{15.3018 - 15.3}{0.12977} = 0.0139.$

2.3) A set of 100 coins from an ongoing production process had been sampled and their diameters measured. The measurement procedure allows for a degree of accuracy of $\pm 0.04\,mm$. The table shows the measured values and their numbers:

i	1	2	3	4	5	6	7
x_i	24.88	24.92	24.96	25.00	25.04	25.08	25.12
n_i	2	6	20	40	22	8	2

Let X be the diameter of a randomly from this set picked coin. Determine

$$E(X),\ E(|X - E(X)|),\ \sqrt{Var(X)}\ ,\ V(X).$$

Solution

$E(X) = \sum_{i=1}^{7} p_i x_i = \frac{1}{100} \sum_{i=1}^{7} n_i x_i = 25.0024,$

$E(|X - E(X)|) = \frac{1}{100} \sum_{i=1}^{7} n_i |x_i - 25.0024| = 0.033664,$

$Var(X) = \frac{1}{100} \sum_{i=1}^{7} n_i (x_i - 25.0024)^2 = 0.00214,\quad \sqrt{Var(X)}\ = 0.0463,\quad V(X) = 0.0018.$

2.4) 83 specimen copies of soft coal, sampled from the ongoing production in a colliery over a period of 7 days, had been analyzed with regard to ash and water content, respectively [in %]. Both ash and water content have been partitioned into 6 classes. The table shows the results.

Let X be the water content and Y be the ash content of a randomly chosen specimen out of the 84 ones. Since the originally measured values are not given, it is assumed that the values, which X and Y can take on, are the centres of the given classes: 16.5, 17.5, $\cdots$, 21.5.

(1) Determine $E(X),\ Var(X),\ E(Y)$, and $Var(Y)$.

(2) Compare the variation of X and Y.

water

ash		[16, 17)	[17, 18)	[18, 19)	[19, 20)	[20, 21)	[21, 22]	sum
	[23, 24)	0	0	1	1	2	4	8
	[24, 25)	0	1	3	4	3	3	14
	[25, 26)	0	2	8	7	2	1	20
	[26, 27)	1	4	10	8	1	0	24
	[27, 28)	0	5	4	4	0	0	13
	[28, 29)	2	0	1	0	1	0	4
	sum	3	12	27	24	9	8	83

Solution

(1) $E(X) = \frac{1}{83}[3 \cdot 16.5 + 12 \cdot 17.5 + 27 \cdot 18.5 + 24 \cdot 19.5 + 9 \cdot 20.5 + 8 \cdot 21.5] = 19.08,$

$Var(X) = \frac{1}{83}[3 \cdot (16.5 - 19.08)^2 + 12 \cdot (17.5 - 19.08)^2 + \cdots + 8 \cdot (21.5 - 19.08)^2] = 1.545,$

$E(Y) = \frac{1}{83}[8 \cdot 23.5 + 14 \cdot 24.5 + 20 \cdot 25.5 + 24 \cdot 26.5 + 13 \cdot 27.5 + 4 \cdot 28.5] = 25.886,$

$Var(Y) = \frac{1}{83}[8 \cdot (23.5 - 25.886)^2 + 14 \cdot (24.5 - 25.886)^2 + \cdots + 4 \cdot (28.5 - 25.886)^2 = 1.594.$

(2) $V(X) = \sqrt{Var(X)}\,/E(X) = 0.0662, \quad V(Y) = \sqrt{Var(Y)}\,/\,E(Y) = 0.0488.$

X has a higher variability than Y.

2.5) It costs \$ 50 to find out whether a spare part required for repairing a failed device is faulty or not. Installing a faulty spare part causes damage of \$1000.
Is it on average more profitable to use a spare part without checking if
(1) 1% of all spare parts of that type,
(2) 3% of all spare parts of that type, and
(3) 10 % of all spare parts of that type are faulty?

Solution

Let X be the random damage (in \$) when not checking.
(1) $E(X) = 0.01 \times 1000 = 10.$
(2) $E(X) = 0.03 \times 1000 = 30.$
(3) $E(X) = 0.10 \times 1000 = 100.$
Since $E(X) = 100 > 50,$ only in this case checking is cost efficient.

2.6) Market analysts predict that a newly developed product in design 1 will bring in a profit of \$500 000, whereas in design 2 it will bring in a profit of \$200 000 with a probability of 0.4, and a profit of \$800 000 with probability of 0.6. What design should the producer prefer?

Solution

With design 2, the mean profit is

$$200\,000 \times 0.4 + 800\,000 \times 0.6 = 560\,000.$$

Hence, the producer should opt for design 2.

2.7) Let X be the random number one has to throw a die, till for the first time a 6 occurs. Determine $E(X)$ and $Var(X)$.

Solution

X has the geometric distribution (2.26) with parameter $p = 1/6$ and state space $Z = \{1, 2, ...\}$. Thus,

$$E(X) = 1/p = 6 \text{ and } Var(X) = (1-p)/p^2 = 30.$$

2.8) 2% of the citizens of a country are HIV-positive. Test persons are selected at random from the population and checked for their HIV-status. What is the mean number of persons which have to be checked till for the first time an HIV-positive person is found?

Solution

The random number X of persons which have to be checked till an HIV-positive person is found has a geometric distribution with parameter $p = 0.02$ and state space $Z = \{1, 2, ..\}$. Hence,

$$E(X) = 1/p = 50.$$

2.9) Let X be the difference between the number of *head* and the number of *tail* if a coin is flipped 10 times.

(1) What is the range of X?

(2) Determine the probability distribution of X.

Solution

(1) $Z = \{-10, -8, \cdots -2, 0, 2, \cdots, 8, 10\}$.

(2) $X = N_H - N_T$, where $N_H\,(N_T)$ is the number of heads (tails) in a series of 10 flippings. Hence, $N_H + N_T = 10$ so that the distribution of X is fully determined by the distribution of N_H or N_T.

$$P(X=10) = \binom{10}{10}\left(\frac{1}{2}\right)^{10}, \; P(X=8) = \binom{10}{9}\left(\frac{1}{2}\right)^{10}, \cdots, P(X=0) = \binom{10}{5}\left(\frac{1}{2}\right)^{10}, \cdots,$$

$$P(X=-8) = \binom{10}{1}\left(\frac{1}{2}\right)^{10}, \; P(X=-10) = \binom{10}{0}\left(\frac{1}{2}\right)^{10}.$$

2.10) A locksmith stands in front of a locked door. He has 9 keys and knows that only one of them fits, but he has otherwise no a priori knowledge. He tries the keys one after the other. What is the mean value of the random number X of trials till the door opens?

Solution

The door will open at the 1st, 2nd, 3rd, ... , 8th, 9th trial with respective probabilities

$$p_1 = \frac{1}{9}, \; p_2 = \frac{8}{9}\cdot\frac{1}{8}, \; p_3 = \frac{8}{9}\cdot\frac{7}{8}\cdot\frac{1}{7}, \cdots, \; p_8 = \frac{8}{9}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdots\frac{2}{3}\cdot\frac{1}{2}, \; p_9 = \frac{8}{9}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdots\frac{2}{3}\cdot\frac{1}{2}\cdot 1.$$

Thus, $p_i = P(X = i) = 1/9$ for all $i = 1, 2, ..., 9$. Hence, X has a discrete uniform distribution with state space $Z = \{1, 2, ..., 9\}$ and $E(X) = 5$.

2.11) A submarine attacks a frigate with 3 torpedoes. The torpedoes hit the frigate independently of each other with probability 0.9. Any successful torpedo hits any of the 4 submerged chambers of the frigate independently of each other successful ones with probability 1/4. The chambers are isolated from each other. In case of one or more hits, a chamber fills up with water. The ship will sink if at least 2 chambers are hit by one or more torpedos. What is the probability of the event A that the attack sinks the frigate?

Solution

Let N be the random number of torpedoes which hit the frigate. Then N has a binomial distribution with parameters $p = 0.9$ and $n = 3$:

$$P(N = k) = \binom{3}{k}(0.9)^k(0.1)^{3-k}; \; k = 0, 1, ..., 3.$$

Obviously,

$$P(A|N=0) = P(A|N=1) = 0.$$

If $N=2$, then there are 6 elementary events out of a total of 10, which are favorable for the occurrence of event A. Hence, $P(A|N=2) = 3/5$.

If $N=3$, then there are 16 elementary events out of 20, which are favorable for the occurrence of event A. Hence, $P(A|N=3) = 4/5$.

By the total probability rule,

$$P(A) = \binom{3}{2}(0.9)^2(0.1)\cdot\tfrac{3}{5} + \binom{3}{3}(0.9)^3(0.1)^0\cdot\tfrac{4}{5} = 0.729.$$

2.12) Three hunters shoot at a flock of three partridges. Every hunter, independently of the others, takes aim at a randomly selected partridge and hits his/her target with probability 1. Thus, a partridge may be hit by up to 3 pellets, whereas a lucky one escapes a hit.

Determine the mean of the random number X of hit partridges.

Solution

The state space is

$$Z = \{(i_1, i_2, i_3);\ i_k = 0, 1, 2, 3;\ \Sigma_{k=1}^3 i_k = 3\}$$

with meaning that partridge k is hit by i_k pellets; $k = 1, 2, 3$. This space has ten elementary events:

(3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1), (1, 0, 2), (0, 1, 2), (1, 1, 1).

Each elementary event has the same probability p to occur: $p = 1/10$ (Laplace experiment). Hence, the probabilities that 1, 2, or 3 partridges are hit are in this order 3/10, 6/10, and 1/10 and the mean mean number of hit partridges is

$$E(X) = \tfrac{3}{10}\cdot 1 + \tfrac{6}{10}\cdot 2 + 3\cdot\tfrac{1}{10} = 1.8\,.$$

2.13) A lecturer, for having otherwise no merits, claims to be equipped with extrasensory powers. His students have some doubt about it and ask him to predict the outcomes of ten flippings of a fair coin. The lecturer is five times successful. Do you believe that, based on this test, the claim of the lecturer is justified?

Solution

No. The result is the most likely outcome when predicting the results purely randomly, i.e., both head and tail are predicted with probability 1/2.

2.14) Let X have a binomial distribution with parameters $p = 0.4$ and $n = 5$.

(1) Determine the probabilities

$$P(X > 6),\ P(X < 2),\ P(3 \le X < 7),\ P(X > 3|X \le 2),\ \text{and}\ P(X \le 3|X \ge 4).$$

(2) Draw the histogram of the probability distribution of X.

Solution

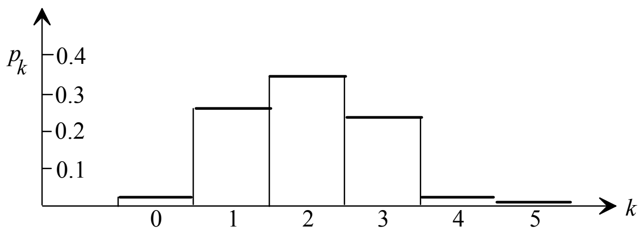
(1) $$p_k = P(X = k) = \binom{5}{k}(0.4)^k(0.6)^{5-k};\ \ k = 0, 1, ..., 5.$$

$$p_0 = 0.07776,\ p_1 = 0.2592,\ p_2 = 0.3456,\ p_3 = 0.2304,\ p_4 = 0.0768,\ p_5 = 0.01024$$

$$P(X > 6) = 0,\ P(X < 2) = 0.33696,\ P(3 \le X < 7) = 0.31744,$$

$$P(X > 3|X \le 2) = 0,\ \ P(X \le 3|X \le 4) = \frac{P(X \le 3)}{P(X \le 4)} = \frac{0.91296}{0.98976} = 0.9224.$$

(2)



Probability histogram of a binomial distribution

2.15) Let X have a binomial distribution with parameters $n = 5$ and p. Determine an interval **I** so that $P(X = 2) \le P(X = 3)$ for all $p \in \mathbf{I}$.

Solution

$$P(X = 2) = \binom{5}{2} p^2 (1-p)^3 \le \binom{5}{3} p^3 (1-p)^2 = P(X = 3)$$

This inequality is equivalent to $10(1-p) \le 10p$ so that the desired interval is $\mathbf{I} = [0.5, 1]$.

2.16) The stop sign at an intersection is on average ignored by 4% of all cars. A car, which ignores the stop sign, causes an accident with probability 0.01. Assuming independent behavior of the car drivers:

(1) What is the probability $p_{(1)}$ that from 100 cars at least 3 ignore the stop sign?

(2) What is the probability $p_{(2)}$ that at least one of the 100 cars causes an accident due to ignoring the stop sign?

Solution

(1) The probability that from 100 cars exactly k ignore the stop sign is

$$p_k = \binom{100}{k} (0.04)^k (0.96)^{100-k}; \quad k = 0, 1, ..., 100.$$

The desired probability is

$$p_{(1)} = 1 - p_0 - p_1 - p_2 = 1 - 0.01687 - 0.07029 - 0.14498 = 0.76786.$$

(2) The probability that from 100 cars exactly k cause an accident due to ignoring the stop sign is

$$\pi_k = \binom{100}{k} (0.0004)^k (0.9996)^{100-k}; \quad k = 0, 1, ..., 100.$$

The desired probability is

$$p_{(2)} = 1 - \pi_0 = 0.03922.$$

2.17) Tessa bought a dozen claimed to be fresh-laid farm eggs in a supermarket. There are 2 rotten eggs amongst them. For breakfast she randomly picks two eggs one after the other.

What is the probability that her breakfast is spoilt if already one bad egg will have this effect?

Solution

Let B be the event that the first taken egg from the dozen is rotten, and A be the event that there is at least one rotten egg amongst the two chosen ones. Then

$$P(A|B) = 1, \quad P(A|\overline{B}) = 2/11.$$

Since $P(B) = 2/12$ and $P(\overline{B}) = 10/12$, by the total probability rule,

$$P(A) = P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B}) = 1 \cdot \tfrac{2}{12} + \tfrac{2}{11} \cdot \tfrac{10}{12} = 0.31818.$$

(Otherwise, apply the hypergeometric distribution.)

2.18) A smart baker mixes 20 stale breads from previous days with 100 freshly baked ones and offers this mixture for sale. Tessa randomly chooses 3 breads one after the other from the 120, i.e., she does not feel and smell them. What is the probability $p_{\geq 1}$ that she has bought at least one stale bread?

Solution

$$p_{\geq 1} = 1 - \frac{100}{120} \cdot \frac{99}{119} \cdot \frac{98}{118} = 0.42423.$$

2.19) Some of the 270 spruces of a small forest stand are infested with rot (a fungus affecting first the core of the stems). Samples are taken from the stems of 30 randomly selected trees.

(1) If 45 trees from the 270 are infested, what is the probability $p_{(1)}$ that there are less than 4 infested trees in the sample? Determine this probability both by the binomial approximation and by the Poisson approximation to the hypergeometric distribution.

(2) If the sample contains six infested trees, what is the most likely number of infested trees in the forest stand (see example 2.7)?

Solution

(1) Hypergeometric distribution with $N = 270,\ M = 45,$ and $n = 30$. The probability p_m that there are m infested trees in the sample, is

$$p_m = \frac{\binom{N-M}{n-m}\binom{M}{m}}{\binom{N}{n}}.$$

Binomial approximation:

$$p_m \approx \binom{30}{m}\left(\frac{45}{270}\right)^m\left(\frac{225}{270}\right)^{30-m} = \binom{30}{m}\left(\frac{45}{270}\right)^m\left(\frac{225}{270}\right)^{30-m} = \binom{30}{m}\left(\frac{1}{6}\right)^m\left(\frac{5}{6}\right)^{30-m}.$$

The desired probability is approximately

$$p_{(1)} \approx p_0 + p_1 + p_2 + p_3 = \left(\frac{5}{6}\right)^{30} + 30 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{29} + 435 \cdot \left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{28} + 4060 \cdot \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{27}$$
$$= 0.0042 + 0.0253 + 0.0733 + 0.1368 = 0.2396.$$

Poisson approximation:

$$p_m \approx \frac{\lambda^m}{m!}e^{-\lambda};\ \ \lambda = n \cdot \frac{M}{N}.$$

Hence, $\lambda = 5$ and

$$p_{(1)} \approx p_0 + p_1 + p_2 + p_3 = 0.00674 + 0.0337 + 0.0842 + 0.1404 = 0.2650.$$

(2) Let M be the unknown number of infested trees in the forest stand. Then the probability that 6 trees in a sample of 30 are infested is

$$p_6 = \frac{\binom{270-M}{30-6}\binom{M}{6}}{\binom{270}{6}}.$$

If M maximizes p_6, then the following inequalities must be true:

$$\frac{\binom{270-M+1}{30-6}\binom{M-1}{6}}{\binom{270}{6}} \leq \frac{\binom{270-M}{30-6}\binom{M}{6}}{\binom{270}{6}} \text{ and } \frac{\binom{270-M-1}{30-6}\binom{M+1}{6}}{\binom{270}{6}} \leq \frac{\binom{270-M}{30-6}\binom{M}{6}}{\binom{270}{6}}.$$

The left inequality implies $M \leq 54.2$, and the right inequality implies $M \geq 54.2$. Hence, $M_{opt} = 54$.

2.20) Because it happens that one or more airline passengers do not show up for their reserved seats, an airline would sell 602 tickets for a flight that holds only 600 passengers. The probability that, for some reason or other, a passenger does not show up is 0.008. What is the probability that every passenger who shows up will have a seat?

Solution

The random number X of passengers who show up for a flight has a binomial distribution with parameters $n = 602$ and $p = 0.992$. Hence, the probability that 601 or 602 passengers show up is

$$\binom{602}{602}(0.992)^{602} + \binom{602}{601}(0.992)^{601}(0.008) = (0.992)^{602} + 602\,(0.992)^{601}(0.008) = 0.0465.$$

Hence, the desired probability is $1 - 0.04651 = 0.95349$.

2.21) Flaws are randomly located along the length of a thin copper wire. The number of flaws follows a Poisson distribution with a mean of 0.15 flaws per *cm*. What is the probability $p_{\geq 2}$ of at least 2 flaws in a section of length $10\,cm$?

Solution

The random number X of flaws, which occur in a section of length 10 *cm*, has a Poisson distribution with parameter $\lambda = 0.15 \cdot 10 = 1.5$. Hence,

$$p_{\geq 2} = P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-1.5} - 1.5e^{-1.5} = 0.4422.$$

2.22) The random number of crackle sounds produced per hour by Karel's old radio has a Poisson distribution with parameter $\lambda = 12$. What is the probability that there is no crackle sound during the 4 minutes transmission of a Karel's favorite hit?

Solution

The random number X of crackle sounds within a time interval of four minutes has Poisson distribution with parameter 0.8. Hence, the desired probability is $P(X = 0) = e^{-0.8} = 0.4493$.

2.23) The random number of tickets car driver Odundo receives a year has a Poisson distribution with parameter $\lambda = 2$. In the current year, Odundo had received his first ticket on the 31st of March. What is the probability p that he will receive another ticket in that year?

Solution

The number of tickets Odundo has received in the first quarter of a year has no influence on the random number of tickets X he receives till the end of that year. Hence, X has a Poisson distribution with parameter $2 \cdot \frac{3}{4} = 1.5$ so that

$$p = P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1.5} = 0.7769.$$

2.24) Let X have a Poisson distribution with parameter λ. For which nonnegative integer n is the probability $p_n = P(X = n)$ maximal?

Solution

The optimal $n = n_{opt}$ is the largest integer n with property

$$\frac{p_n}{p_{n-1}} = \frac{\frac{\lambda^n}{n!}e^{-\lambda t}}{\frac{\lambda^{n-1}}{(n-1)!}e^{-\lambda t}} \geq 1.$$

Hence, n_{opt} is the largest integer n satisfying $\lambda \geq n$. In other words: n_{opt} is the largest integer being equal or less than the mean value of this distribution.

2.25) In 100 kg of a low-grade molten steel tapping there are on average 120 impurities. Castings weighing 1 kg are manufactured from this raw material. What is the probability $p_{\geq 2}$ that there are at least 2 impurities in a casting if the spacial distribution of the impurities in the raw material is Poisson?

Solution

The intensity of impurities in a casting of $1kg$ is $\lambda = 1.2$. Hence,

$$p_{\geq 2} = 1 - e^{-1.2} - 1.2\, e^{-1.2} = 0.3374.$$

2.26) In a piece of fabric of length $100m$ there are on average 10 flaws. These flaws are assumed to be Poisson distributed over the length. The $100m$ of fabric are cut in pieces of length $4m$. What percentage of the $4m$ cuts can be expected to be without flaws?

Solution

The probability that a $4m$ cut contains no flaw is $p_0 = e^{-0.4} = 0.6703$, i.e., it can be expected that about 67% of the $4m$ cuts have no flaws.

2.27) X have a binomial distribution with parameters n and p. Compare the following exact probabilities with the corresponding Poisson approximations and give reasons for possible larger deviations:

(1) $P(X=2)$ for $n = 20, p = 0.1$, (2) $P(X=2)$ for $n = 20, p = 0.9$,

(3) $P(X=4)$ for $n = 10, p = 0.1$, (4) $P(X=4)$ for $n = 60, p = 0.1$.

Solution

(1) $P(X=2) = \binom{20}{2}(0.1)^2(0.9)^{18} = 0.28518, \quad P(X=2) \approx \frac{2^2}{2!}e^{-2} = 0.27067.$

The approximative value is satisfactory.

(2) $P(X=2) = \binom{20}{2}(0.9)^2(0.1)^{18} \approx 1.5 \times 10^{-16}, \quad P(X=2) \approx \frac{18^2}{2!}e^{-18} \approx 2.5 \times 10^{-6}.$

The condition $np < 10$ is not satisfied ($np = 18$).

(3) $P(X=4) = \binom{10}{4}(0.1)^4(0.9)^6 = 0.01116, \quad P(X=4) \approx \frac{1}{4!}e^{-1} = 0.01533.$

n is not large enough

(4) $P(X=4) = \binom{60}{4}(0.1)^4(0.9)^{56} = 0.13356, \quad P(X=4) \approx \frac{6^4}{4!}e^{-6} = 0.13385.$

The approximative value is satisfactory.

2.28) A random variable X has range $R = \{x_1, x_2, \cdots, x_m\}$ and probability distribution

$$\{p_k = P(X = x_k);\ k = 1, 2, ..., m\},\ \Sigma_{k=1}^m p_k = 1.$$

A random experiment with outcome X is repeated independently of each other n times.

Show: The probability of the event

$$\{x_1 \text{ occurs } n_1 \text{ times, } x_2 \text{ occurs } n_2 \text{ times, } \cdots, x_m \text{ occurs } n_m \text{ times}\}$$

is given by

$$\frac{n!}{n_1!\, n_2! \cdots n_m!}\, p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \quad \text{with} \quad \Sigma_{k=1}^m n_k = n.$$

This probability distribution is called the *multinomial distribution*. It contains as a special case the binomial distribution ($n = 2$).

Solution

The *multinomial coefficient*

$$\frac{n!}{n_1!\,n_2!\cdots n_m!}$$

is equal to the number of ways to subdivide a set consisting of m different elements into disjoint subsets comprising each $n_1, n_2, \cdots, n_m$ elements with property

$$\Sigma_{k=1}^{m} n_k = n.$$

This is easily verified by repeated application of the following representation of the binomial coefficient:

$$\binom{n}{k} = \left(\frac{n!}{k!(n-k)!}\right).$$

2.29) A branch of the PROFIT-Bank has found that on average 68% of its customers visit the branch for routine money matters (type 1-visitors), 14% are there for investment matters (type 2-visitors), 9% need a credit (type 3-visitors), 8% need foreign exchange service (type 4-visitors), and 1% only make a suspicious impression or even carry out a robbery (type 5-visitors).

(1) What is the probability $p_{(1)}$ that amongst 10 randomly chosen visitors 5, 3, 1, 1, and 0 are of type 1, 2, 3, 4, or 5, respectively.

(2) What is the probability $p_{(2)}$ that amongst 12 randomly chosen visitors 4, 3, 3, 1, and 1 are of type 1, 2, 3, 4, or 5, respectively?

Solution

Application of the multinomial distribution with

$$m = 5,\ p_1 = 0.68,\ p_2 = 0.14,\ p_3 = 0.09,\ p_4 = 0.08, \text{ and } p_5 = 0.01.$$

(1) With $n = 10$, the desired probability is

$$p_{(1)} = \frac{10!}{5!\,3!\,1!\,1!\,0!}\,0.68 \cdot 0.14 \cdot 0.09 \cdot 0.08 \cdot 0.01 = 0.03455.$$

With $n - 12$, the desired probability is

$$p_{(2)} = \frac{10!}{4!\,3!\,3!\,1!\,1!}\,0.68 \cdot 0.14 \cdot 0.09 \cdot 0.08 \cdot 0.01 = 0.02879.$$

Section 2.3

2.30) Let $F(x)$ and $f(x)$ be the respective distribution function and the probability density of a random variable X. Answer with *yes* or *no* the following questions:

(1)	$F(x)$ and $f(x)$ can be arbitrary real functions.	no
(2)	$f(x)$ is a nondecreasing function in $(-\infty, +\infty)$	no
(3)	$f(x)$ cannot have jumps.	no
(4)	$f(x)$ cannot be negative.	yes
(5)	$F(x)$ is always a continuous function.	no
(6)	$F(x)$ can assume values between in $[-1, 0)$	no
(7)	The area between the abscissa and the graph of $F(x)$ is always equal to 1.	no
(8)	$f(x)$ must always be smaller than 1.	no
(9)	The area between the abscissa and the graph of $f(x)$ is always equal to 1.	yes
(10)	The properties of $F(x)$ and $f(x)$ are all the same to me.	

2.31) Check whether by suitable choice of the parameter a the following functions are densities of random variables. If the answer is *yes*, determine the respective distribution functions, mean values, medians, and modes.

(1) $f(x) = a|x|, \ -3 \le x \le +3.$

(2) $f(x) = a\,x\,e^{-x^2}, \ x \ge 0.$

(3) $f(x) = a\,\sin x, \ 0 \le x \le \pi.$

(4) $f(x) = a\,\cos x, \ 0 \le x \le \pi.$

Solution

(1) $\int_{-3}^{+3} a|x|dx = 4.5\,a + 4.5a = 9a$. Hence,

$$f(x) = \frac{1}{9}\,|x|, \ -3 \le x \le +3,$$

is a density, and X has distribution function

$$F(x) = \begin{cases} 0.5 - x^2/18\,, & -3 \le x < 0, \\ 0.5 + x^2/18, & 0 \le x \le +3. \end{cases}$$

Since the density is symmetric with symmetry center $x = 0$, $E(X) = x_{0.5} = 0$. The modes are

$$x_{m_1} = -3 \text{ and } x_{m_2} = +3.$$

(2) $\int_0^{\infty} a\,x\,e^{-x^2} dx = a \cdot \frac{1}{2}$ so that $f(x)$ is a density for $a = 2$. Hence,

$$F(x) = 1 - e^{-x^2}, \ x \ge 0.$$

Thus, X has a Rayleigh distribution with

$$E(X) = \sqrt{\pi/4}\,, \ x_{0.5} = 0.8326, \ \text{ and } x_m = 0.71.$$

(3) $\int_0^{\pi} a\,\sin x\,dx = a\,[\cos x]_0^{\pi} = 2a$ so that $f(x)$ is a density for $a = 0.5$. The mean value is

$$E(X) = 0.5\int_0^{\pi} x\,\sin x\,dx = 0.5\,[\sin x - x\cos x]_0^{\pi} = 0.5\pi\,.$$

Since the density has symmetry center $\pi/2$, $x_{05} = \pi/2$ and $x_m = \pi/2$.

(4) This $f(x)$ is not a density since $\cos x$ is negative in $(\pi/2, \pi)$.

2.32) (1) Show that $f(x) = \frac{1}{2\sqrt{x}}, \ 0 < x \le 1,$ is the probability density of a random variable X.

(2) Determine the corresponding mean value and the ε–percentile x_ε.

Solution

(1) $$\int_0^1 f(y)\,dy = \frac{1}{2}\int_0^1 y^{-1/2}\,dy = \frac{1}{2}[2y^{+1/2}]_0^1 = 1, \ 0 \le x \le 1.$$

Hence, $f(x)$ is a probability density with distribution function

$$F(x) = \int_0^x f(y)\,dy = \sqrt{x}\,, \ 0 \le x \le 1.$$

(2) $$E(X) = \frac{1}{2}\int_0^1 x^{1/2}\,dy = \frac{1}{2}\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3} \text{ and } x_\varepsilon = \varepsilon^2.$$

2.33) Let X be a continuous random variable. Confirm or deny the following statements:

(1) The probability $P(X = E(X))$ is always positive. no

(2) There is always $Var(X) \le 1$. no

(3) $Var(X)$ can be negative if X assumes negative values with positive probability. no

(4) $E(X)$ is never negative. no

2.34) The current which flows through a thin copper wire is uniformly distributed in the interval $[0,\ 10\,mA]$. For safety reasons, the current should not fall below the crucial level of $4mA$.

What is the probability $p_{\le 4}$ that at any randomly chosen time point the current is below $4mA$?

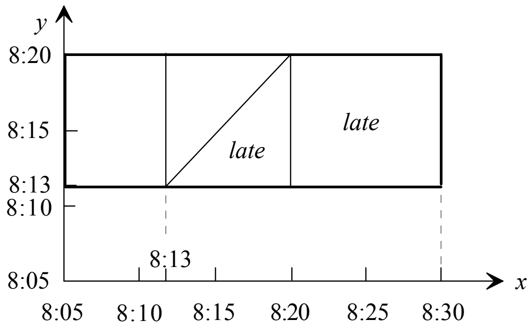
Solution

$$p_{\le 4} = 4/10 = 0.4.$$

2.35) According to the timetable, a lecture begins at 8:15a.m. The arrival time of Professor *Wisdom* in the venue is uniformly distributed between 8:13 and 8:20, whereas the arrival time of student *Sluggish* is uniformly distributed over the time interval from 8:05 to 8:30.

What is the probability that *Sluggish* arrives after *Wisdom* in the venue?

Solution



Let X be the arrival time of Sluggish and Y be the arrival time of Wisdom. Then the random vector (X, Y) has a two-dimensional uniform distribution in the rectangular

$$R = \{8:05 \le x \le 8:30,\quad 8:13 \le y \le 8:20\}.$$

This rectangular covers an area of

$$\mu(R) = 25 \cdot 7 = 175\,[\min^2].$$

To determine that subarea of R given by $R_{late} = \{(x,y)\ \text{with}\ x > y\}$ consider the figure:

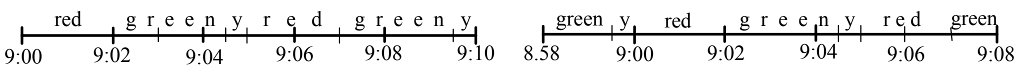
R_{late} has area

$$\mu(R_{late}) = 10 \cdot 7 + \tfrac{1}{2}\,7^2 = 94.5\,[\min^2].$$

Thus, the desired probability is

$$p_{late} = \frac{\mu(R_{late})}{\mu(R)} = 0.54.$$

2.36) A road traffic light is switched on every day at 5:00a.m. It always begins with *red* and holds this colour for two minutes. Then it changes to *green* and holds this colour for 2.5 minutes before it switches to *yellow* to hold this colour for 30 seconds. This cycle continues till midnight.



(1) A car driver arrives at this traffic light at a time point which is uniformly distributed between 9:00 and 9:10a.m. What is the probability that the driver catches the green light period?

(2) Determine the same probability on condition that the driver's arrival time point has a uniform distribution over the interval [8:58, 9:08].

Solution

The figures show the red, yellow, and green periods in the time intervals [9:00 to 9:10] and from [8:58 to 9:08], respectively.

Hence, in either case (1) and (2) the traffic light holds colour green for a total of 5 minutes during the car driver's arrival time interval of length 10 minutes so that the desired probabilities for both (1) and (2) are 0.5.

2.37) A continuous random variable X has the probability density

$$f(x) = \begin{cases} 1/4 & \text{for } 0 \le x \le 2, \\ 1/2 & \text{for } 2 < x \le 3. \end{cases}$$

Determine and compare the measures of variability (1) $\sqrt{Var(X)}$ and (2) $E(|X - E(X)|)$.

Solution

$$E(X) = \int_0^2 x\,\tfrac{1}{4}\,dx + \int_2^3 x\,\tfrac{1}{2}\,dx = \tfrac{7}{4} = 1.75.$$

$$Var(X) = \int_0^2 (x-1.75)^2\,\tfrac{1}{4}\,dx + \int_2^3 (x-1.75)^2\,\tfrac{1}{2}\,dx$$

$$= \tfrac{1}{4}\int_0^2 (x^2 - 3.5x + 1.75^2)\,dx + \tfrac{1}{2}\int_2^3 (x^2 - 3.5x + 1.75^2)\,dx$$

$$= \tfrac{1}{4}\left[\tfrac{x^3}{3} - 1.75\,x^2 + 1.75^2 x\right]_0^2 + \tfrac{1}{2}\left[\tfrac{x^3}{3} - 1.75\,x^2 + 1.75^2 x\right]_2^3$$

$$= \tfrac{1}{4}\left(\tfrac{8}{3} - 7 + 1.75^2 \cdot 2\right) + \tfrac{1}{2}\left(9 - 1.75 \cdot 9 + 1.75^2 \cdot 3 - \tfrac{8}{3} + 7 - 1.75^2 \cdot 2\right)$$

$$= 0.44792 + 0.32292 = 0.77084.$$

$$\sqrt{Var(X)} = 0.87797.$$

(2)

$$E(|X - E(X)|) = E(|X - 1.75|)$$

$$= \int_0^{1.75} (1.75 - x)\tfrac{1}{4}dx + \int_{1.75}^2 (x - 1.75)\tfrac{1}{4}dx + \int_2^3 (x - 1.75)\tfrac{1}{2}dx$$

$$= \tfrac{1}{4}\left[1.75x - x^2/2\right]_0^{1.75} + \tfrac{1}{4}\left[x^2/2 - 1.75x\right]_{1.75}^2 + \tfrac{1}{2}\left[x^2/2 - 1.75x\right]_2^3$$

$$\tfrac{1}{4}\left[1.75^2 - 1.75^2/2\right] + \tfrac{1}{4}\left[2 - 3.5 - 1.75^2/2 + 1.75^2\right] + \tfrac{1}{2}[4.5 - 1.75 \cdot 3 - 2 + 3.5]$$

$$= 0.38281 + 0.01563 + 0.37500.$$

Thus,

$$E(|X - E(X)|) = 0.77344.$$

2.38) A continuous random variable X has the probability density

$$f(x) = 2x, \quad 0 \le x \le 1.$$

(1) Determine the distribution function $F(x)$ and by means of it $E(X)$.

(2) Determine and compare the measures variability a) $\sqrt{Var(X)}$ and b) $E(|X - E(X)|)$.

Solution

(1)

$$F(x) = \int_0^x 2y\,dy = \left[y^2\right]_0^x = x^2, \quad 0 \le x \le 1.$$

$$E(X) = \int_0^1 (1 - F(x))\,dx = \int_0^1 (1 - x^2)\,dx = \left[x - x^3/3\right] = 2/3.$$

(2) a) $$Var(X) = E(X-2/3)^2 = \int_0^1 (x-2/3)^2\, 2x\, dx = 2\int_0^1 \left(x^3 - \tfrac{4}{3}x^2 + \tfrac{4}{9}x\right) dx$$
$$= 2\left[\tfrac{x^4}{4} - \tfrac{4}{9}x^3 + \tfrac{4}{18}x^2\right]_0^1 = \tfrac{1}{18}.$$

Thus,

$$\sqrt{Var(X)} = 0.23570.$$

b) $$E(|X-E(X)|) = E(|X-2/3|) = \int_0^{2/3}(2/3-x)\,2x\,dx + \int_{2/3}^1 (x-2/3)\,2x\,dx$$
$$= 2\left[\tfrac{2}{6}x^2 - \tfrac{x^3}{3}\right]_0^{2/3} + 2\left[\tfrac{x^3}{3} - \tfrac{2}{6}x^2\right]_{2/3}^1.$$

Hence,

$$E(|X-E(X)|) = 0.19753.$$

2.39) The lifetime X of a bulb has an exponential distribution with a mean value of $E(X) = 8000$ [time unit: *hours*]. Calculate the probabilities

$$P(X \le 4000),\ P(X > 12000),\ P(7000 \le X < 9000), \text{ and } P(X < 4000).$$

Solution

X has an exponential distribution with parameter $\lambda = 8000^{-1} = 0.000125$ and distribution function

$$F(x) = 1 - e^{-\frac{1}{8000}x},\ x \ge 0.$$

$$P(X \le 4000) = 1 - e^{-0.5} = 0.3935,\ P(X > 12000) = e^{-1.5} = 0.2231,$$

$$P(7000 \le X < 9000) = e^{-7/8} - e^{-9/8} = 0.0922,\ P(7000 \le X < 9000),\ P(X < 4000) = P(X \le 4000).$$

2.40) The lifetimes of 5 identical bulbs are exponentially distributed with parameter

$$\lambda = 1.25 \cdot 10^{-4}\,[h^{-1}],\ \text{i.e.,}\ E(X) = 1/\lambda = 8000\,[h].$$

All of them are switched on at time $t = 0$ and will fail independently of each other.

(1) What is the probability that at time $t = 8000\,[h]$ a) all 5 bulbs and b) at least 3 bulbs have failed?

(2) What is the probability that at least one bulb survives 12 000 hours?

Solution

The distribution function of the lifetimes of the bulbs is the same as in the previous example. Hence, the probability that a bulb fails in [0, 8000] is $p = 1 - 1/e \approx 0.63212$.

(1) a) All 5 bulbs fail in [0, 8000] with probability $(1 - 1/e)^5 = 0.10093$.

b) The random number N of bulbs, which fail in [0, 8000], has a binomial distribution with parameters $n = 5$ and $p = 1 - 1/e$. Hence, the desired probability is

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + p^5(1-p) = 0.7364.$$

(2) All bulbs fail in [0, 12 000] with probability

$$\left(1 - e^{-\frac{12\,000}{8000}}\right)^5 = \left(1 - e^{-1.5}\right)^5 = 0.28297.$$

Hence, at least one bulb survives this interval with probability

$$1 - (1 - e^{-1.5})^5 = 0.71703.$$

2.41) The random length X of employment of staff in a certain company has an exponential distribution with property that 92% of staff leave the company after only 16 months.

What is the mean time an employee is with this company and the corresponding standard deviation?

Solution

The 0.92-percentile is $x_{0.92} = 16$ months. Hence, the parameter λ of this distribution satisfies the equation $1 - e^{-\lambda \cdot 16} = 0.92$. It follows

$$\lambda = 0.15768 \text{ and } E(X) = \sqrt{Var(X)} = 1/\lambda = 6.335 \text{ [months]}.$$

2.42) The times between the arrivals of taxis at a rank are independent and have an exponential distribution with mean value $10\, min$. An arriving customer does not find an available taxi and the previous one left 3 minutes earlier. No other customers are waiting.

What is the probability p_w that the customer has to wait at least $5min$ for the next free taxi?

Solution

In view of the memoryless property of the exponential distribution,

$$p_w = e^{-5/10} = 0.60653.$$

2.43) A small branch of the *Profit Bank* has the two tellers 1 and 2. The service times at these tellers are independent and exponentially distributed with parameter $\lambda = 0.4\,[\text{min}^{-1}]$. When Pumeza arrives, the tellers are occupied by a customer each. So she has to wait. Teller 1 is the first to become free, and the service of Pumeza starts immediately. What is the probability p that the service of Pumeza is finished sooner than the service of the customer at teller 2?

Solution

In view of the memoryless property of the exponential distribution, the residual service time of the customer at teller 2 has the same probability distribution as the one of Pumeza, namely an exponential distribution with parameter $\lambda = 0.4\,[\text{min}^{-1}]$. Hence, the desired probability is 0.5. Analytically, this probability is given by the integral

$$p = \int_0^\infty (1 - e^{-\lambda x})\,\lambda e^{-\lambda x} dx, \text{ which is 0.5 for all positive } \lambda.$$

2.44) Four weeks later Pumeza visits the same branch as in the previous exercise. Now the service times at tellers 1 and 2 are again independent, but exponentially distributed with respective parameters $\lambda_1 = 0.4\,[\text{min}^{-1}]$ and $\lambda_2 = 0.2\,[\text{min}^{-1}]$.

(1) When Pumeza enters the branch, both tellers are occupied and no customer is waiting. What is the mean time Pumeza spends in the branch till the end of her service?

(2) When Pumeza enters the branch, both tellers are occupied, and another customer is waiting for service. What is the mean time Pumeza spends in the branch till the end of her service? (Pumeza does not get preferential service.)

Solution

(1) Let T be the total time Pumeza spends in the branch. Then T can be represented as $T = W + S$, where W is the time Pumeza is waiting for service, and S is her actual service time. If X_1 and X_2 denote the respective random service times at teller 1 and 2, then W is given by $W = \min\,(X_1, X_2)$ with survival function

$$\overline{F}_W(x) = e^{-(\lambda_1+\lambda_2)x},\ x \geq 0,$$

so that

$$E(W) = \int_0^\infty e^{-(\lambda_1+\lambda_2)x} dx = 1/(\lambda_1 + \lambda_2).$$

Her mean service time has structure

$$E(S) = E(S|X_1 \le X_2)\,P(X_1 \le X_2) + E(S|X_2 \le X_1)\,P(X_2 \le X_1).$$

Since $P(X_1 \le X_2) = \int_0^\infty (1 - e^{-\lambda_1 x})\,\lambda_2 e^{-\lambda_2 x} dx = \frac{\lambda_1}{\lambda_1+\lambda_2}, \quad P(X_2 \le X_1) = \frac{\lambda_2}{\lambda_1+\lambda_2},$

Pumeza's mean service time is

$$E(S) = \frac{1}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{2}{\lambda_1+\lambda_2}.$$

Hence,

$$E(T) = \frac{3}{\lambda_1+\lambda_2} = \frac{3}{0.4+0.2} = 5\,[\text{min}].$$

(2) The customer before Pumeza faces the same situation as Pumeza under (1). Thus, his mean sojourn time in the branch is $3/(\lambda_1 + \lambda_2)$. Since the service time of Pumeza in the branch, when not knowing which teller becomes available first, is $2/(\lambda_1 + \lambda_2)$, her mean total sojourn time in the branch is

$$5/(\lambda_1 + \lambda_2) = 5/0.6 = 8.\underline{3}\,[\text{min}].$$

2.45) An insurance company offers policies for fire insurance. Achmed holds a policy according to which he gets full refund for that part of the claim which exceeds \$3000. He gets nothing for a claim size less than or equal to \$3000. The company knows that the average claim size is \$5642.

(1) What is the mean refund Achmed gets from the company for a claim if the claim size is exponentially distributed?

(2) What is the mean refund Achmed gets from the company for a claim if the claim size is Rayleigh distributed?

Solution

(1) The random claim size C has distribution function

$$F(x) = P(C \le x) = 1 - e^{-\lambda x} = 1 - e^{-\frac{x}{5642}}, \quad x \ge 0.$$

Let R be the refund Achmed gets from the insurance company when submitting a claim. Then

$$E(R) = 0 \cdot P(C \le 3000) + E(C - 3000|C > 3000) \cdot P(C > 3000) = \int_{3000}^\infty (x - 3000)\lambda e^{-\lambda x} dx$$

$$= \left[-\frac{e^{-\lambda x}}{\lambda}(\lambda x + 1)\right]_{3000}^\infty - 3000\left[1 - e^{-\lambda x}\right]_{3000}^\infty$$

$$= 5642 \cdot e^{-\frac{3000}{5642}} = \$3315.$$

(2) In this case, for a positive parameter Θ, the claim size C has distribution function

$$F(x) = 1 - e^{-(x/\theta)^2} \text{ and density } f(x) = \frac{2x}{\theta^2} e^{-(x/\theta)^2}, \quad x \ge 0,$$

where the parameter Θ is determined by

$$E(C) = \Theta\sqrt{\pi/4}\,, \text{ i.e., } \theta = \frac{5642}{\sqrt{\pi/4}} = 6366.3.$$

The corresponding mean refund is

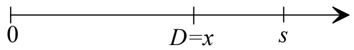
$$E(R) = 2\int_{3000}^\infty (x - 3000)\,\frac{x}{6366.3^2}\,e^{-(x/6366.3)^2} dx = \$\,2850.0.$$

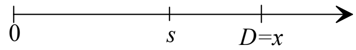
Thus, the claim size distribution has considerable influence on the mean refund, even under invariant mean claim size.

2.46) Pedro runs a fruit shop. Mondays he opens his shop with a fresh supply of strawberries of s pounds, which is supposed to satisfy the demand for three days. He knows that for this time span the demand X is exponentially distributed with a mean value of 200 pounds. Pedro pays \$ 2 for a pound and sells it for \$ 5. So he will lose \$ 2 for each pound he cannot sell, and he will make a profit of \$ 3 out of each pound he sells. What amount of strawberries Pedro should stock for a period of three days to maximize his mean profit?

Solution

There are two possibilities: The random demand D is less than s, and D exceeds s:





This leads to the net profit of Pedro:

$$G(s) = \int_0^s [3x - 2(s-x)] \frac{1}{200} e^{-\frac{1}{200}x} dx + \int_s^\infty [3s - 3(x-s)] \frac{1}{200} e^{-\frac{1}{200}x} dx.$$

In the second integral, $3s$ is the profit Pedro made by selling his entire stock, whereas $3(x-s)$ is the loss Pedro suffered for being not in the position to meet the full demand. The optimum value for s is $s^* = 246$ pounds, and the corresponding maximum net profit of Pedro is \$40.33.

2.47) The probability density function of the random annual energy consumption X of an enterprise [in $10^8 kwh$] is

$$f(x) = 30(x-2)^2[1 - 2(x-2) + (x-2)^2], \quad 2 \le x \le 3.$$

(1) Determine the distribution function of X and by means of this functionthe probability that the annual energy consumption exceeds 2.8.

(2) What is the mean annual energy consumption?

Solution

(1)
$$F(x) = 30 \int_2^x (y-2)^2[1 - 2(y-2) + (y-2)^2]\, dy$$
$$= (x-2)^3[10 - 15(x-2) + 6(x-2)^2], \quad 2 \le x \le 3.$$

Hence,

$$F(x) = \begin{cases} 0, & x < 2 \\ (x-2)^3[10 - 15(x-2) + 6(x-2)^2], & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}.$$

(2) $P(X > 2.8) = 1 - F(2.8) \approx 0.0579.$

(3) $E(X) = 30 \int_2^3 x(x-2)^2[1 - 2(x-2) + (x-2)^2]\, dx = 2.5.$

2.48) The random variable X is normally distributed with mean $\mu = 5$ and standard deviation $\sigma = 4$: $X = N(\mu, \sigma^2) = N(5, 16)$.

Determine the respective values of x which satisfy

$$P(X \le x) = 0.5,\ P(X > x) = 0.95,\ P(x < X < 9) = 0.2,\ P(3 < X < x) = 0.95,\ P(-x < X < +x) = 0.99.$$

Solution

Since

$$P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right),$$

the equations for x are in this order equivalent to

$$P(N(0,1) \le \tfrac{x-5}{4}) = 0.5, \quad P(N(0,1) > \tfrac{x-5}{4}) = 0.95, \quad P(\tfrac{x-5}{4} < N(0,1) < \tfrac{9-5}{4}) = 0.2,$$

$$.P(\tfrac{3-5}{4} < N(0,1) < \tfrac{x-5}{4}) = 0.5, \quad P(-\tfrac{x+5}{4} < N(0,1) < +\tfrac{x-5}{4}) = 0.99$$

or

$$\Phi\left(\tfrac{x-5}{4}\right) = 0.5, \quad \Phi\left(\tfrac{x-5}{4}\right) = 0.05, \quad \Phi(1) - \Phi\left(\tfrac{x-5}{4}\right) = 0.2,$$

$$\Phi\left(\tfrac{x-5}{4}\right) - \Phi\left(-\tfrac{1}{2}\right) = 0.5, \quad \Phi\left(\tfrac{x-5}{4}\right) - \Phi\left(-\tfrac{x+5}{4}\right) = 0.99.$$

From the table of the standard normal distribution, the x-values satisfying these equations are in this order 5, -1.56, 6.45, 8.23, 14.3.

2.49) The response time of an average male car driver is normally distributed with mean value 0.5 and standard deviation 0.06 (in seconds).

(1) What is the probability that his response time is greater than 0.6 seconds?

(2) What is the probability that his response time is between 0.50 and 0.55 seconds?

Solution

(1) $P(X > 0.6) = P\left(N(0,1) > \frac{0.6-0.5}{0.06}\right) = 1 - \Phi(5/3) \approx 0.04746.$

(2) $P(0.5 \le X \le 0.55) = \Phi\left(\frac{0.55-0.5}{0.06}\right) - \Phi\left(\frac{0.5-0.5}{0.06}\right) = \Phi(5/6) - \frac{1}{2} = 0.2975.$

2.50) The tensile strength of a certain brand of cardboard has a normal distribution with mean 24*psi* and variance 9*psi*. What is the probability that the tensile strength X of a randomly chosen specimen does not fall below the critical level of 20*psi*?

Solution

Since $P(X < 20) = \Phi\left(\frac{20-24}{3}\right) = \Phi(-4/3) = 0.911$, the desired probability is

$$P(X \ge 20) = 0.089.$$

2.51) The total monthly sick leave time of employees of a small company has a normal distribution with mean 100 hours and standard deviation 20 hours.

(1) What is the probability that the total monthly sick leave time will be between 50 and 80 hours?

(2) How much time has to be budgeted for sick leave to make sure that the budgeted amount is exceeded with a probability of less than 0.1?

Solution

(1) The desired probability is

$$P(50 \le X \le 80) = \Phi\left(\frac{80-100}{20}\right) - \Phi\left(\frac{50-100}{20}\right) = \Phi(-1) - \Phi(-2.5) \approx 0.1524.$$

(2) The 0.9-percentile $x_{0.9}$ of the distribution defined by $P(X \le x_{0.9}) = 0.9$ or, equivalently,

$$\Phi\left(\frac{x_{0.9}-100}{20}\right) = 0.9$$

has to be determined. Since the 0.9-percentile of the standard normal distribution is 1.28, i.e., it is $\Phi(1.28) = 0.9$, the 0.9-percentile of X satisfies

$$\frac{x_{0.9}-100}{20} = +1.28.$$

Hence, $x_{0.9} = 125.6$.

2.52) The random variable X has a Weibull distribution with mean value 12 and variance 9.
(1) Calculate the parameters β and θ of this distribution.
(2) Determine the conditional probabilities $P(X > 10 | X > 8)$ and $P(X \le 6 | X > 8)$.

Solution
(1) Mean value and variance of X are

$$E(X) = \theta\Gamma(1 + 1/\beta), \quad Var(X) = \theta^2\left\{\Gamma(1+2/\beta) - [\Gamma(1+1/\beta)]^2\right\}.$$

Using tables of the gamma function or computer support gives the solutions of the equations $E(X) = 12$ and $Var(X) = 9$:

$$\beta = 4.542, \quad \theta = 13.1425.$$

(2) $$P(X > 10 | X > 4) = \frac{P(X > 10)}{P(X > 8)} = \frac{e^{-(10/\theta)^\beta}}{e^{-(8/\theta)^\beta}} = \frac{e^{-(10/13.1425)^{4.542}}}{e^{-(8/13.1425)^{4.542}}} = \frac{e^{-0.2890}}{e^{-0.1049}} = 0.8319.$$

2.53) The random measurement error X of a meter has a normal distribution with mean 0 and variance σ^2, i.e., $X = N(0, \sigma^2)$. It is known that the percentage of measurements which deviate from the 'true' value by more than 0.4 is 80%. Use this piece of information to determine σ.

Solution
It is given that $P(|X| > 0.4) = 1 - P(|X| \le 0.4) = 1 - P(-0.4 \le X \le +0.4) = 0.8$ or, equivalently, in terms of the standard normal distribution,

$$\Phi(+0.4/\sigma) - \Phi(-0.4/\sigma) = 2\Phi(+0.4/\sigma) - 1 = 0.2 \text{ or } \Phi(+0.4/\sigma) = 0.6.$$

The 0.6-percentile of the standard normal distribution is $x_{0.6} = 0.253$. Hence, $0.4/\sigma = 0.253$ so that $\sigma = 1.58$.

2.54) If sand from gravel pit 1 is used, then molten glass for producing armored glass has a random impurity content X which is $N(60, 16)$-distributed. But if sand from gravel pit 2 is used, then this content is $N(62, 9)$-distributed (μ and σ in 0.01%). The admissable degree of impurity should not exceed 0.64%. Sand from which gravel pit should be used?

Solution
Pit 1: $P(N(60, 16) \ge 64) = 1 - \Phi\left(\frac{64-60}{4}\right) = 1 - \Phi(1) = \Phi(-1) = 0.1587.$

Pit 2: $P(N(62, 9) \ge 64) = 1 - \Phi\left(\frac{64-62}{3}\right) = 1 - \Phi(2/3) = \Phi(-2/3) = 0.1587.$

Hence, sand from gravel pit 1 should be preferred.

2.55) Let X have a geometric distribution with

$$P(X = i) = (1-p)p^i; \quad i = 0, 1, ...; \quad 0 < p < 1.$$

By mixing this distribution with regard to a suitable structure distribution density $f(p)$ show that

$$\sum_{i=0}^{\infty} \frac{1}{(i+1)(i+2)} = 1. \tag{i}$$

Solution
Let the parameter p be the value of a random variable which has a uniform distribution over [0,1]:

$$f(p) = \begin{cases} 1 & \text{if } 0 \le p \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the mixture of geometric distributions with a structure distribution given by $f(p)$ yields the distribution of the corresponding mixed random variable Y:

$$P(Y=i)=\int_0^1(1-p)p^i dp=\frac{1}{(i+1)(i+2)};\ \ i=0,1,...$$

For $\{P(Y=i);\ i=0,1,...\}$ being a probability distribution, relation (i) must be true.

2.56) A random variable X has distribution function (*Frechét distribution*))

$$F_\alpha(x)=e^{-\alpha/x};\ \ \alpha>0,\ x>0.$$

What distribution type arises when mixing this distribution with regard to the exponential structure distribution density $f(\alpha)=\lambda\, e^{\lambda\,\alpha};\ \alpha>0,\ \lambda>0$?

Solution

The mixture of the distribution functions $F_\alpha(x)$ generates a random variable Y with distribution function (*Lomax distribution*, page 93)

$$G(x)=P(Y\le x)=\int_0^\infty e^{-\alpha/x}\lambda\, e^{-\lambda\,\alpha}\,d\alpha=\frac{\lambda\, x}{1+\lambda\, x};\ \lambda>0,\ x\ge 0.$$

2.57) The random variable X has distribution function (special Lomax distribution)

$$F(x)=\frac{x}{x+1};\ \ x\ge 0.$$

Check whether there is a subinterval of $[0,\infty)$ on which $F(x)$ is *DFR* or *IFR*.

Solution

The density of X is $f(x)=1/(x+1)^2,\ x\ge 0,$ so that it has the failure rate

$$\lambda(x)=f(x)/\overline{F}(x)=\frac{1}{x+1}\ .$$

This failure rate is decreasing with increasing x on $[0,\infty]$ so that $F(x)$ is *DFR* everywhere. (Note that $\lambda(x)=\overline{F}(x)$.)

2.58) Check the aging behavior of systems whose lifetime distributions have

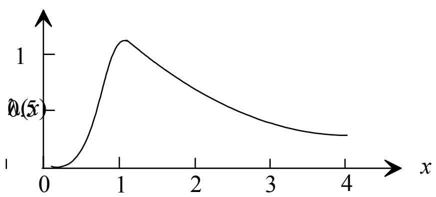
(1) a Frechét distribution with distribution function $F(x)=e^{-(1/x)^2}$ (sketch its failure rate), and

(2) a power distribution with distribution function $F(x)=1-(1/x)^2,\ x\ge 1.$

Solution

(1) Density $f(x)$ and failure rate $\lambda(x)$ of this distribution are

$$f(x)=2x^{-3}e^{-(1/x)^2},\ x>0,\ \text{ and }\ \lambda(x)=\frac{2\,x^{-3}}{e^{+(1/x)^2}-1},\ x>0.$$



This failure rate has an absolute maximum λ_m at $x_m=1.0695$, i.e, $\lambda_m=\lambda(x_m)=\lambda(1.0695)\approx 1.17.$ Hence, $F(x)$ is *IFR* in $[0,\ 1.0695]$ and *DFR* in $[1.0695,\ \infty)$, see Figure.

(2) Density $f(x)$ and failure rate $\lambda(x)$ of this distribution are

$$f(x)=2\,(1/x)^3,\ x\ge 1,\ \text{ and }\ \lambda(x)=2\,(1/x)^2,\ x\ge 1.$$

$F(x)$ is *DFR* on $[1,\ \infty)$.

2.59) Let $F(x)$ be the distribution function of a nonnegative random variable X with finite mean value μ.
(1) Show that the function $F_s(x)$ defined by

$$F_s(x) = \frac{1}{\mu} \int_0^x (1 - F(t))\, dt$$

is the distribution function of a nonnegative random variable X_s.
(2) Prove: If X is exponentially distributed with parameter $\lambda = 1/\mu$, then so is X_s and vice versa.
(3) Determine the failure rate $\lambda_s(x)$ of X_s.

Solution
(1) $F_s(x)$ is increasing from $x = 0$ to $x = \infty$. Moreover, $F_s(0) = 0$ and, in view of formula (2.52), it is $F_s(\infty) = 1$.
(2) Let $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$. Then $F_s(x)$ becomes

$$F_s(x) = \lambda \int_0^x e^{-\lambda t} dt = \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_0^x = 1 - e^{-\lambda x},\ x \geq 0.$$

Now let $F_s(x) = 1 - e^{-\lambda x}$, $x \geq 0$. Then,

$$1 - e^{-\lambda x} = \lambda \int_0^x (1 - F(t))\, dt.$$

Differentiation with regard to x on both sides of this equation yields

$$\lambda e^{-\lambda x} = \lambda (1 - F(x)) \quad \text{so that} \quad F(x) = 1 - e^{-\lambda x},\ x \geq 0.$$

(3) The density belonging to $F_s(x)$ is $f_s(x) = F_s'(x) = \frac{1}{\mu}(1 - F(x))$ and the corresponding survival probability is (again by formula (2.52)),

$$\overline{F}_s(x) = 1 - F_s(x) = \frac{1}{\mu} \int_x^\infty (1 - F(t))\, dt$$

Hence,

$$\lambda_s(x) = f_s(x)/\overline{F}_s(x) = \frac{1 - F(x)}{\int_x^\infty (1 - F(t))\, dt}.$$

2.60) Let X be a random variable with range $\{1, 2, ...\}$ and probability distribution

$$P(X = i) = \left(1 - \frac{1}{n^2}\right) \frac{1}{n^{2(i-1)}};\ i = 1, 2, \ldots.$$

Determine the z-transform of X and by means of it $E(X)$, $E(X^2)$, and $Var(X)$.

Solution
This is a geometrical distribution with $p = 1 - 1/n^2$ (see formula (2.26)) with generating function (see page 97)

$$M(z) = \sum_{i=1}^\infty \left(1 - \frac{1}{n^2}\right) \frac{1}{n^{2(i-1)}} z^{i-1} = \frac{n^2 - 1}{n^2 - z}.$$

Hence,

$$M'(z) = \frac{n^2 - 1}{(n^2 - z)^2}, \quad M''(z) = \frac{2(n^2 - 1)}{(n^2 - z)^3},$$

$$E(X) = M'(1) = \frac{1}{n^2 - 1}, \quad E(X^2) = M''(1) + M'(1) = \frac{n^2 + 1}{(n^2 - 1)^2}.$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{n^2}{(n^2 - 1)^2}.$$

2.61) Determine the Laplace transform $\hat{f}(s)$ of the density of the Laplace distribution with parameters λ and μ (page 66):

$$f(x) = \tfrac{1}{2}\lambda e^{-\lambda|x-\mu|}, \quad -\infty < x < +\infty.$$

By means of $\hat{f}(s)$ determine $E(X)$, $E(X^2)$, and $Var(X)$.

Solution

The Laplace transform of $f(x)$ is

$$\hat{f}(s) = \int_{-\infty}^{+\infty} e^{-sx}\frac{\lambda}{2}e^{-\lambda|x-\mu|}dx.$$

By partitioning the integration area, $\hat{f}(s)$ becomes

$$\hat{f}(s) = \frac{\lambda}{2}\int_{-\infty}^{\mu} e^{-sx}e^{-\lambda(\mu-x)}dx + \frac{\lambda}{2}\int_{\mu}^{+\infty} e^{-sx}e^{-\lambda(x-\mu)}dx$$

$$= \frac{\lambda}{2}e^{-\lambda\mu}\int_{-\infty}^{\mu} e^{-(s-\lambda)x}dx + \frac{\lambda}{2}e^{+\lambda\mu}\int_{\mu}^{+\infty} e^{-(s+\lambda)x}dx$$

$$= \frac{\lambda}{2}e^{-\lambda\mu}\left[-\frac{1}{s-\lambda}e^{-(s-\lambda)x}\right]_{-\infty}^{\mu} + \frac{\lambda}{2}e^{+\lambda\mu}\left[-\frac{1}{s+\lambda}e^{-(s+\lambda)x}\right]_{\mu}^{\infty}$$

$$= \frac{\lambda}{2}e^{-\lambda\mu}\frac{1}{\lambda-s}e^{-(s-\lambda)\mu} + \frac{\lambda}{2}e^{+\lambda\mu} + \frac{1}{\lambda+s}e^{-(s+\lambda)\mu}$$

$$= \frac{\lambda}{2}\left[\frac{1}{\lambda-s} + \frac{1}{\lambda+s}\right]e^{-s\mu}.$$

Thus,

$$\hat{f}(s) = \frac{\lambda^2}{\lambda^2 - s^2}e^{-\mu s}.$$

The first and second derivative of $\hat{f}(s)$ are

$$\hat{f}'(s) = \frac{\lambda^2}{s^2-\lambda^2}\left[\mu - \frac{2s}{s^2-\lambda^2}\right]e^{-\mu s}$$

and

$$.\hat{f}''(s) = \frac{2\lambda^2}{(s^2-\lambda^2)^2} - \frac{\lambda^2\mu^2}{s^2-\lambda^2} + o(s),$$

where in this case the Landau order function $o(s)$ represents all those terms which have factor s. (When s tends to 0, these terms will disappear.) Hence,

$$\hat{f}'(0) = -\mu \text{ and } \hat{f}''(0) = \mu^2 + 2/\lambda^2$$

so that

$$E(X) = \mu, \; E(X^2) = \mu^2 + 2/\lambda^2, \text{ and } Var(X) = 2/\lambda^2.$$