Solutions for Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd Edition by Pepper

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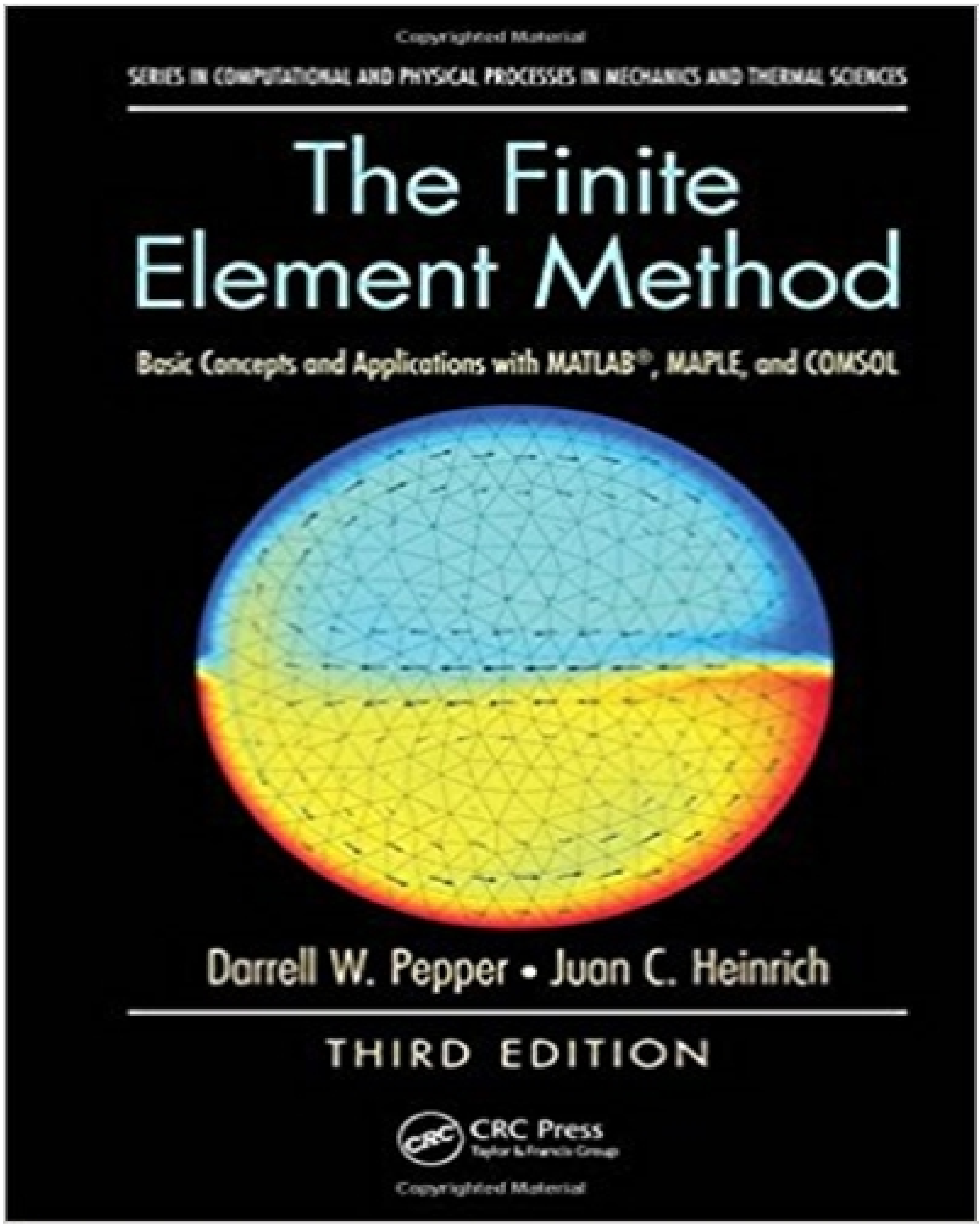

Solutions

2.1 Fill in the details leading to Eq. (2.17) and use it to find the weighted residuals formulation of Example 2.1

Solution:

a)

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=0 \qquad (Eq.2.13)$$

Integration by parts: $\int_0^L u\,dv = uv\,|_0^L - \int_0^L v\,du$. Set $u=\phi(x)$ and $dv=-K\frac{d^2T}{dx^2}$, then

$v=-K\frac{dT}{dx}$ and $du=\frac{d\phi}{dx}dx$ and we have

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=\phi(x)\left(-K\frac{dT}{dx}\right)_0^L-\int_o^L\left(-K\frac{dT}{dx}\right)\frac{d\phi}{dx}dx$$

$$=\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-K\phi\frac{dT}{dx}\bigg|_0^L$$

Substitute into Eq. 2.13 and we get

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx-K\phi\frac{dT}{dx}\bigg|_0^L=0 \qquad (Eq.2.17)$$

b)

$$T(0)=T_0 \qquad and \qquad -K\frac{dT}{dx}\bigg|_{x=L}=h(T-T_\infty)$$

Replacing in Eq. 2.17

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx-\phi(L)h(T(L)-T_\infty)=0$$

2.2 Use Eq. (2.17) to find the weighted residuals formulation of Eqs. (2.1) – (2.3) in terms of the heat flux q.

Solution:

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx - \int_o^L \phi Q\, dx + \phi\left(-K\frac{dT}{dx}\right)\Bigg|_0^L = 0$$

The heat flux term at x = L must be zero, and $-K\frac{dT}{dx}\Big|_{x=0} = q$ Hence

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx = \int_o^L \phi Q\, dx + \phi(0)\, q$$

2.3 Obtain expression (2.30).

Solution:

The weighted residuals form for the second term is

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=0$$

After integration by parts on the first term

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx+\phi\left(-K\frac{dT}{dx}\right)\Bigg|_{L/2}^{L}=0$$

Note: The boundary terms are set to zero if no fluxes are prescribed.

The functions $\phi_i(x)$ for element 2 are

$$\phi_2(x)=\frac{2}{L}(L-x)$$

$$\phi_3(x)=\frac{2}{L}(x-L/2)$$

Let $$T(x)=\frac{2}{L}(L-x)a_2+\frac{2}{L}(x-L/2)a_3 \quad (Eq.\ 2.19)$$

Using Eq. 2.28

$$\int_{L/2}^{L}\begin{bmatrix}-\frac{2}{L}\\ \frac{2}{L}\end{bmatrix}\begin{bmatrix}-\frac{2}{L} & \frac{2}{L}\end{bmatrix}dx\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-Q\int_{L/2}^{L}\begin{Bmatrix}\frac{2}{L}(L-x)\\ \frac{2}{L}(x-L/2)\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Or

$$\frac{4K}{L^2}\int_{L/2}^{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-\frac{2Q}{L}\int_{L/2}^{L}\begin{Bmatrix}(L-x)\\ (x-L/2)\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Integrating we have

$$\frac{2K}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-\frac{QL}{4}\int_{L/2}^{L}\begin{Bmatrix}1\\1\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

2.4 Consider the equation

$$\frac{d^2u}{dx^2}+u+x=0, \qquad 0<x<1$$

with $u(0) = u(1) = 0$. Assume an approximation for $u(x) = a_1\phi_1(x)$ with $\phi_1(x) = x(1 - x)$.

The residual is

$$R(u,x)=-2a_1+a_1[x(1-x)]+x$$

and the weight $W_1(x) = \phi_1(x) = x(1 - x)$. Find the solution from the following integral

$$\int_0^1 W(x)R(u,x)dx=0$$

.Solution:

$$\int_0^1 W(x)R(u,x)dx=\int_0^1 x(1-x)(-2a_1+a_1x(1-x)+x)dx=0$$

Or

$$\int_0^1 W(x)R(u,x)dx=\int_0^1(-2a_1x+(3a_1+1)x^2-2a_1+1)x^3+a_1x^4)dx=0$$

$$-a_1+\frac{1}{3}(3a_1+1)-\frac{1}{4}(2a_1+1)+\frac{1}{5}a_1=-\frac{3}{10}a_1+\frac{1}{12}=0$$

$a_1=\dfrac{5}{18}$ therefore $u(x)\approx\dfrac{5}{18}(x-x^2)$

Compare with the exact solution given by $u^*(x)=\dfrac{\sin x}{\sin 1}-x$

Evaluating at x=1/2 $u(1/2)=0.069444$ less than 5% error.

$u^*(1/2)=0.069747$

2.5 Use the weak statement formulation to find solutions to the equation

$$\frac{d^2u}{dx^2} = 1, \qquad 0 < x < 1$$

$$u(0) = 0, \qquad \left.\frac{du}{dx}\right|_{x=1} = 1$$

(a) Using linear interpolation functions, as explained in Section 2.3, use (i) one element and (ii) two elements.

(b) Using simple polynomial functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1x$, hence $\phi_1(x) = W_1(x) = x$

(ii) $u(x) = a_1x + a_2x^2$, hence $\phi_1(x) = W_1(x) = x$,

$\phi_2(x) = W_2(x) = x^2$.

(c) Using circular functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1 \sin(\pi/2Lx)$

(ii) $u(x) = a_1 \sin(\pi/2Lx) + aa_2 \sin(3\pi/2Lx)$

(d) Find the analytical solution and compare with results from (a), (b) and (c).

Solution

a) The residual function is $R(u,x) = \frac{d^2u}{dx^2} + 1$ The weak form after integration by parts is

$$\int_0^1 \left(\frac{dW}{dx}\frac{du}{dx} + W \right) dx - W\left.\frac{du}{dx}\right|_0^1 = 0$$

i)

One element solution $\phi_1 = 1 - x \qquad \phi_2 = x$

$$u(x) = (1-x)a_1 + x\,a_2$$

$$\int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} dx \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \int_0^1 \begin{bmatrix} 1-x \\ x \end{bmatrix} dx - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Integrating

$$a_1 - a_2 = -1/2$$
$$-a_1 + a_2 = 1/2$$

since $a_1 = 0$ then $a_2 = 1/2$ *or* $u(x) = x/2$

ii)

Two element solution. Element 1:

$$\phi_1 = 1-2x \qquad \phi_2 = 2x$$

$$\int_0^{1/2}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_1\\a_2\end{Bmatrix}+\int_0^{1/2}\begin{bmatrix}1-2x\\2x\end{bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/4\end{Bmatrix}$$

Element 2

$$\phi_2 = 2(1-x) \qquad \phi_3 = 2(x-1/2)$$

$$\int_{1/2}^{1}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_2\\a_3\end{Bmatrix}+\int_{1/2}^{1}\begin{bmatrix}2(1-x)\\2(x-1/2)\end{bmatrix}dx-\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\3/4\end{Bmatrix}$$

Assemble the elements

$$\begin{bmatrix}2 & -2 & 0\\-2 & 4 & -2\\0 & -2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/2\\3/4\end{Bmatrix} \quad \text{since} \quad a_1=0 \;\; we\;get \;\; a_2=1/8 \;\; and \;\; a_3=1/2$$

$$u(x)=\begin{cases} x/4 & 0\le x\le 1/2 \\ \dfrac{3x-1}{4} & 1/2\le x\le 1\end{cases}$$

b)

Using polynomials

i)

$$\phi_1 = x \qquad u(x)=a_1x$$

$\int_0^1 (1\cdot a_1 + x)\,dx - 1 = 0$, $\quad a_1 = 1/2 \quad$, $\quad u(x)=x/2$, same as using one element.

ii)

$$\phi_1 = x \;,\; \phi_2 = x^2 \;,\quad u(x)=a_1x+a_2x^2$$

For W=ϕ_1 $\quad \int_0^1\left[1\cdot(a_1+2a_2x)+x\right]dx-1=0$

For W=ϕ_2 $\quad \int_0^1\left[2x\cdot(a_1+2a_2x)+x^2\right]dx-1=0$

The system becomes

$$a_1+a_2=1/2$$

$$a_1+\frac{4}{3}a_2=2/3 \qquad or \qquad a_1=0 \;,\; a_2=1/2 \;\; and \;\; u(x)=x^2/2$$

c)

Using circular functions

i)

$$\phi_1 = \sin\left(\frac{\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) \quad , \quad \phi_1' = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$$

$$\int_0^1 \left[\frac{\pi^2}{4} a_1 \cos^2\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\right] dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0$$

Use the identities $\int \cos^2 \alpha x\, dx = \frac{1}{2\alpha}(\sin\alpha x \cos\alpha x - \alpha x)$ and $\int \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$ .Then

$$\frac{\pi^2 a_1}{4}\left(\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right) - \frac{x}{2}\right)\bigg|_0^1 + \left(-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\right)\bigg|_0^1 - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0 \quad or$$

$$\frac{\pi^2 a_1}{8} + \frac{2}{\pi} - 1 = 0 \qquad \text{hence} \quad a_1 \approx 0.295 \quad \text{and} \quad u(x) = 0.295\sin\left(\frac{\pi}{2}x\right)$$

ii)

$$\phi_2 = \sin\left(\frac{3\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) + a_2 \sin\left(\frac{3\pi}{2}x\right).$$

For $W = \phi_1$: $$\int_0^1 \left[\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)\left(a_1\cos\left(\frac{\pi}{2}x\right) + a_2\cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{\pi}{2}x\right)\right] dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 = 0$$

For $W = \phi_2$: $$\int_0^1 \left[3\frac{\pi^2}{4}\cos\left(3\frac{\pi}{2}x\right)\left(a_1\cos\left(\frac{\pi}{2}x\right) + a_2\cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{3\pi}{2}x\right)\right] dx - \sin\left(\frac{3\pi}{2}x\right)\frac{du}{dx}\bigg|_0^1 =$$

$$\frac{\pi^2}{8}a_1 + 0\cdot a_2 + \frac{2}{\pi} - 1 = 0 \quad (a_1 \text{ is unchanged})$$

$$0\cdot a_1 + \frac{9\pi^2}{8}a_2 + \frac{2}{3\pi} + 1 = 0 \quad , \quad a_2 \approx -0.109 \quad and$$

$$u(x) \approx 0.295\sin\left(\frac{\pi}{2}x\right) - 0.109\sin\left(\frac{3\pi}{2}x\right)$$

d) Analytical solution

$\frac{d^2u}{dx^2} = 1$,integrating $u(x) = \frac{x^2}{2} + c_1 x + c_2$ and applying boundary conditions $u^*(x) = \frac{x^2}{2}$.

Notice that all the approximations are exact at x = 1/2, and the Galerkin approximations using x and x^2 both yield the exact solution. Also notice that the boundary condition at x = 1 is satisfied only approximately. In case a), the first solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{1}{2}$ and the second solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{3}{4}$ which is getting better.This point is discussed in more detail in problem 2.6.

2.6 Subdivide the interval $0 \le x \le L$ into 10 linear elements, construct the element equations for four consecutive elements starting with element four, and show that none of the parameters a_i is related to more than two additional parameters in the equations. The only ones involved being a_{i-1} and a_{i+1} (as a consequence of this, the final system of linear equations will involve a tri-diagonal matrix which is easy to solve).

Solution:

For an element $e_k = \{x / x_k \le x \le x_{k+1})$ the element stiffness matrix is obtained from $\int_{x_k}^{x_{k+1}} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$. All the stiffness matrices must be equal, so let's calculate the first one.

$$\phi_1 = 1 - \frac{10}{L}x \qquad \phi_2 = \frac{10}{L}x \quad thus \ \int_0^{L/10} K \begin{bmatrix} -\frac{10}{L} \\ \frac{10}{L} \end{bmatrix} \begin{bmatrix} -\frac{10}{L} & \frac{10}{L} \end{bmatrix} dx = \frac{10K}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and after assembling
$$\frac{10K}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that each equation i involves only three unknowns $a_{i-1}, a_i \ and \ a_{i+1}$, so the stiffness matrix is tri-diagonal.

2.7 The Rayleigh-Ritz formulation for the problem of solving Eq. (2.1) with boundary conditions (2.2) and (2.3) can be stated as: Minimize the functional

$$F(T)=\int_0^L\left\{\frac{K}{2}\left[\frac{dT}{dx}\right]^2-QT\right\}dx-Tq\,\big|_{x=0}$$

over all functions $T(x)$ with square integrable first derivatives that satisfy Eq. (2.3) at $x = L$. Approximate $T(x)$ using two linear elements as we did before and replace into Eq. (2.1) to obtain $F(T) \cong F(a_1, a_2, a_3)$. Now minimize $F(a_1, a_2, a_3)$ as a function of three variables. Show that the final system of equations is identical to Eq. (2.31), thus, in this case, the Galerkin method and the Rayleigh-Ritz method are equivalent.

Solution:

Element1:

$$\phi_1=1-\frac{2}{L}x \qquad \phi_2=\frac{2}{L}x$$

$$\int_0^{L/2}\frac{K}{2}\left(-\frac{2}{L}a_1+\frac{2}{L}a_2\right)^2-Q\left[\left(1-\frac{2}{L}x\right)a_1+\frac{2}{L}xa_2\right]dx-a_1q=\frac{K}{L}(a_2-a_1)^2-\frac{QL}{4}(a_1+a_2)-a_1q$$

(1)

Element2:

$$\phi_2=2-\frac{2}{L}x \qquad \phi_3=\frac{2}{L}x-1$$

$$\int_{L/2}^{L}\frac{K}{2}\left(-\frac{2}{L}a_2+\frac{2}{L}a_3\right)^2-Q\left[\left(2-\frac{2}{L}x\right)a_2+\left(\frac{2}{L}x-1\right)a_3\right]dx-a_1q=\frac{K}{L}(a_3-a_2)^2-\frac{QL}{4}(a_2+a_3)$$

(2)

$$F(T)=(1)+(2)=\frac{K}{L}(a_2-a_1)^2+\frac{K}{L}(a_3-a_2)^2-\frac{QL}{4}(a_1+2a_2+a_3)-a_1q$$

$$F(T)=F(a_1,a_2,a_3)=\frac{K}{L}\left(a_1^2-2a_1a_2+2a_2^2-2a_2a_3+a_3^2\right)$$

The conditions for a minimum are $\frac{\partial F}{\partial a_1}=\frac{\partial F}{\partial a_2}=\frac{\partial F}{\partial a_3}=0$ thus

$$\frac{\partial F}{\partial a_1}=\frac{2K}{L}(a_1-a_2)-\frac{QL}{4}-q=0$$

$$\frac{\partial F}{\partial a_2}=\frac{2K}{L}(-a_1+2a_2-a_2)-\frac{QL}{2}=0$$

$$\frac{\partial F}{\partial a_3}=\frac{2K}{L}(a_3-a_2)-\frac{QL}{4}=0$$

In Matrix form $\frac{2K}{L}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\frac{QL}{4}\begin{Bmatrix}1\\2\\1\end{Bmatrix}+\begin{Bmatrix}q\\0\\0\end{Bmatrix}$ which is identical to Eq. 2.31.

2.8 Let us consider once more Problem 2.3 above. Assume that $u(x) = a_1x^2$ and $W_1(x) = x^2$, but this time use Eq. (2.11) and proceed as in Problem 2.2 in order to obtain a solution. Show that this solution is identical to that obtained in Problem 2.3(b)(ii), and hence, the process of integration by parts does not introduce any changes.

Solution:

$$\int_0^1 \left[W_1 \left(\frac{d^2u}{dx^2} - 1 \right) \right] dx = \int_0^1 x^2 (2a - 1) dx = 0 \ , \ \left(\frac{2}{3} x^3 a_1 - \frac{x^3}{3} \right) \Bigg|_0^1 = 0 \ , \ \frac{2}{3} a - \frac{1}{3} = 0 \ , \ a_1 = 1/2 \ , \ u(x) = \frac{x^2}{2}$$

2.9 In example 2.2, first find the exact analytical solution, then calculate two more Galerkin approximations using

Solution:

The analytical solution is $u(x)=\frac{1}{2}\left(x-x^2\right)\cdot$ The Galerkin approximation is

$$\int_0^1\left(\frac{dW}{dx}\frac{du}{dx}-W\right)dx=0\cdot$$

$$u(x)=a_1\sin\pi x+a_2\sin 2\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x$$

a)

$$W_1:\ \int_0^1\left[\pi^2\cos\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin\pi x\right]dx=0$$

$$W_2:\ \int_0^1\left[2\pi^2\cos 2\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin 2\pi x\right]dx=0 \quad or$$

$$\frac{\pi^2}{2}a_1+0\cdot a_2-\frac{2}{\pi}=0\ ,\ a_1=\frac{4}{\pi^3}\quad ,\ and\quad 0\cdot a_1+2\pi^2a_2+0=0\ ,\ a_2=0$$

$$u(x)=\frac{4}{\pi^3}\sin\pi x$$

b) $u(x)=a_1\sin\pi x+a_2\sin 2\pi x+a_3\sin 3\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x\ ,\ W_3=\sin 3\pi x$

We only need to calculate a_3 , a_1 *and* a_2 remain the same due to the orthogonality of the sine functions.

$$W_3:\ \int_0^1\left[3\pi^2\cos 3\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x+3a_3\cos 3\pi x\right)-\sin 3\pi x\right]dx=0$$

$$\frac{9\pi^2}{2}a_3-\frac{2}{3\pi}=0\ ,\ a_3=\frac{4}{27\pi^3}\ ,\ u_3(x)=\frac{4}{3\pi^3}\left(\sin\pi x+\frac{1}{27}\sin 3\pi x\right)$$

2.10 Derive Eq. (2.32) from Eq. (2.31).

Eqs. 2.32 $a_1 - a_2 = \frac{qL}{2K} + \frac{QL^2}{8K}$, $-a_1 + 2a_2 = \frac{QL^2}{4K} + T_L$ adding $a_2 = \frac{qL}{2K} + \frac{3QL^2}{8K} + T_L$ and

substituting into the first equation $a_1 = \frac{qL}{K} + \frac{QL^2}{2K} + T_L$

2.11 Fill in the details in Example 2.3 and verify the solution.

Solution:

The weak form is $\int_0^1 \left(\frac{d\phi}{dx}\frac{dT}{dx} + W \right) dx - \int_0^1 \phi \, dx = 0$ Use the Galerkin approximation $u(x) = a_1 \sin \pi$, $\phi(x) = \sin \pi x$, $du = a_1 \pi \cos \pi x \, dx$, $d\phi = \pi \cos \pi x \, dx$. Replace in the integrals to get

$\int_0^1 (\pi \cos \pi x)(a_1 \pi \cos \pi x) dx = \int_0^1 \sin \pi x \; dx$ and using the trigonometric identity

$\int_0^1 \cos^2 \pi x \, dx = \int_0^1 \frac{1}{2}(1 + \cos 2\pi x) \; dx$ we get $a_1 = \frac{4}{\pi^3}$.

2.12 Show that if the expressions for ϕ_i given by Eq. (2.20) are used in Eq. (2.19), then $a_i = T_i$.

Solution:

Eq. 2.19 is $T(x) = \phi_i(x)a_i + \phi_{i+1}(x)a_{i+1} \quad x_i \le x \le x_{i+1}$ where

$$\phi_i(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} \text{ , } and \quad \phi_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i}$$

Substituting $T(x) = \left[\frac{x_{i+1} - x}{x_{i+1} - x_i}\right] a_i + \left[\frac{x - x_i}{x_{i+1} - x_i}\right] a_{i+1}$ and evaluating at the nodes

$T(x_i) = a_i = T_i \; and \; T(x_{i+1}) = a_{i+1} = T_{i+1}$.

2.13 Find the weak formulation of Eq. (2.1) with the boundary condition

$$-K\left.\frac{dT}{dx}\right|_{x=0} = h\left(T - T_\infty\right) \text{ and } -K\left.\frac{dT}{dx}\right|_{x=1} = q.$$

Solution:

The general form is given in Eq. 2.17 $\int_0^L \frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi Q\,dx - K\,\phi\left.\frac{dT}{dx}\right|_o^L = 0$

Substituting the flux boundary conditions we get $\int_0^L \frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi Q\,dx + \phi(L)q - \phi(0)\,h(T(0) - T_\infty) = 0$

2.14 Using two linear elements solve the equations in problem (2.13) with $L = 1\ m$, $K = 200$ w/mK, $Q = 100\ W/m^3$, $h = 150\ W/m^2\ K$, $T_\infty = 100C$ and $q = 2{,}000\ W/m^2$

Solution:

Using Eqs. 2.19 and 2.30 $\frac{2K}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} - \frac{QL}{4}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - h\begin{Bmatrix} T_1 - T_\infty \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ Substitute the parameters values 1) $\begin{bmatrix} 250 & -400 \\ -400 & 400 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -14750 \\ 25 \end{Bmatrix}$ 2) $\begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 - q \end{Bmatrix}$

assembling

1) $\begin{bmatrix} 250 & -400 & 0 \\ -400 & 800 & -400 \\ 0 & -400 & 400 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -14750 \\ 50 \\ 25 - q \end{Bmatrix}$ With solution

$$T_1 = 97.833 + 0.0066667q$$
$$T_2 = 98.021 + 0.0041667q$$
$$T_3 = 98.084 + 0.0016667q$$