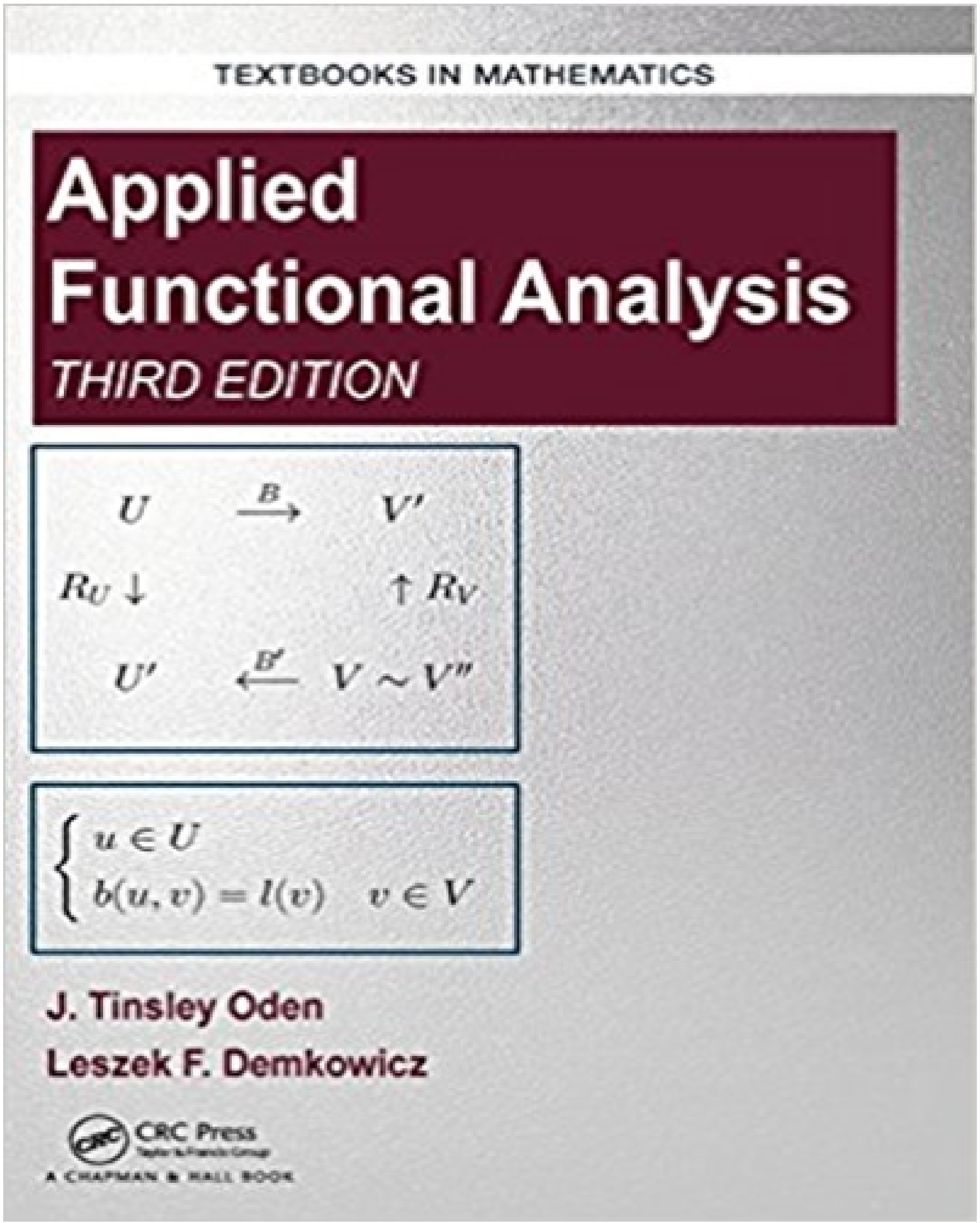
Solutions for Applied Functional Analysis 3rd Edition by Oden



Solutions

2

Linear Algebra

Vector Spaces—The Basic Concepts

2.1 Concept of a Vector Space

Exercises

Exercise 2.1.1 Let V be an abstract vector space over a field $\mathbb{F}$. Denote by 0 and 1 the identity elements with respect to addition and multiplication of scalars, respectively. Let $-1 \in \mathbb{F}$ be the element* opposite to 1 (with respect to scalar addition). Prove the identities

(i) $\mathbf{0} = 0\,\boldsymbol{x}, \quad \forall \boldsymbol{x} \in V$

(ii) $-\boldsymbol{x} = (-1)\,\boldsymbol{x}, \quad \forall \boldsymbol{x} \in V$

where $\mathbf{0} \in V$ is the zero vector, i.e., the identity element with respect to vector addition, and $-\boldsymbol{x}$ denotes the opposite vector to $\boldsymbol{x}$.

(i) Let $\boldsymbol{x}$ be an arbitrary vector. By the axioms of a vector space, we have

$$\boldsymbol{x} + 0\,\boldsymbol{x} = 1\,\boldsymbol{x} + 0\,\boldsymbol{x} = (1+0)\,\boldsymbol{x} = 1\,\boldsymbol{x} = \boldsymbol{x}$$

Adding to both sides the inverse element $-\boldsymbol{x}$, we obtain that

$$\mathbf{0} + 0\,\boldsymbol{x} = 0\,\boldsymbol{x} = \mathbf{0}$$

(ii) Using the first result, we obtain

$$\boldsymbol{x} + (-1)\,\boldsymbol{x} = (1 + (-1))\,\boldsymbol{x} = 0\,\boldsymbol{x} = \mathbf{0}$$

*It is unique.

Exercise 2.1.2 Let $\mathbb{C}$ denote the field of complex numbers. Prove that $\mathbb{C}^n$ satisfies the axioms of a vector space with analogous operations to those in $\mathbb{R}^n$, i.e.,

$$\boldsymbol{x} + \boldsymbol{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \stackrel{\text{def}}{=} (x_1 + y_1, \ldots, x_n + y_n)$$

$$\alpha\, \boldsymbol{x} = \alpha\,(x_1, \ldots, x_n) \stackrel{\text{def}}{=} (\alpha\, x_1, \ldots, \alpha\, x_n)$$

This is really a trivial exercise. One by one, one has to verify the axioms. For instance, the associative law for vector addition is a direct consequence of the definition of the vector addition, and the associative law for the addition of complex numbers, etc.

Exercise 2.1.3 Prove Euler's theorem on rigid rotations. Consider a rigid body fixed at a point A in an initial configuration Ω. Suppose the body is carried from the configuration Ω to a new configuration Ω_1, by a rotation about an axis l_1, and next, from Ω_1 to a configuration Ω_2, by a rotation about another axis l_2. Show that there exists a unique axis l, and a corresponding rotation carrying the rigid body from the initial configuration Ω to the final one, Ω_2, directly. Consult any textbook on rigid body dynamics, if necessary.

This seemingly obvious result is far from trivial. We offer a proof based on the use of matrices, and you may want to postpone studying the solution after Section 2.7 or even later.

A matrix $\boldsymbol{A} = A_{ij}$ is called *orthonormal* if its transpose coincides with its inverse, i.e.,

$$\boldsymbol{A}\,\boldsymbol{A}^T = \boldsymbol{A}^T\,\boldsymbol{A} = \boldsymbol{I}$$

or, in terms of its components,

$$\sum_k A_{ik}A_{jk} = \sum_k A_{ki}A_{kj} = \delta_{ij} \tag{2.1}$$

The Cauchy theorem for determinants implies that

$$\det\boldsymbol{A}\,\det\boldsymbol{A}^T = \det{}^2\boldsymbol{A} = \det\boldsymbol{I} = 1$$

Consequently, for an orthonormal matrix $\boldsymbol{A}$, $\det\boldsymbol{A} = \pm 1$. It is easy to check that orthonormal matrices form a (noncommutative) group. Cauchy's theorem implies that orthonormal matrices with $\det\boldsymbol{A} = 1$ constitute a subgroup of this group.

We shall show now that, for $n = 2, 3$, orthonormal matrices with $\det\boldsymbol{A} = 1$ represent (rigid body) rotations.

Case: $n = 2$. Matrix representation of a rotation by angle θ has the form

$$\boldsymbol{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and it is easy to see that it is an orthonormal matrix with unit determinant. Conversely, let a_{ij} satisfy conditions (2.1). Identities

$$a_{11}^2 + a_{12}^2 = 1 \text{ and } a_{21}^2 + a_{22}^2 = 1$$

imply that there exist angles $\theta_1, \theta_2 \in [0, 2\pi)$ such that

$$a_{11} = \cos\theta_1, \quad a_{12} = \sin\theta_1, \quad a_{21} = -\sin\theta_2, \quad a_{22} = \cos\theta_2$$

But

$$a_{11}a_{21} + a_{12}a_{22} = \sin(\theta_1 - \theta_2) = 0$$

and

$$a_{11}a_{22} - a_{12}a_{21} = \cos(\theta_1 - \theta_2) = 1$$

The two equations admit only one solution: $\theta_1 - \theta_2 = 0$.

Case: $n = 3$. Linear transformation represented by an orthonormal matrix preserves the length of vectors (it is an isometry). Indeed,

$$\begin{aligned}\|\boldsymbol{Ax}\|^2 &= \sum_i (\sum_j A_{ij}x_j)(\sum_k A_{ik}x_k) \\ &= \sum_j \sum_k (\sum_i A_{ik}A_{jk})x_j x_k = \sum_j \sum_k \delta_{jk} x_j x_k = \sum_k x_k x_k = \|\boldsymbol{x}\|^2\end{aligned}$$

Consequently, the transformation maps unit ball into itself. By the Schauder Fixed Point Theorem (a heavy but very convenient argument), there exists a vector $\boldsymbol{x}$ that is mapped into itself. Selecting a system of coordinates in such a way that vector $\boldsymbol{x}$ coincides with the third axis, we deduce that $\boldsymbol{A}$ has the following representation

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}$$

Orthogonality conditions (2.1) imply that $a_{31} = a_{32} = 0$ and that $a_{ij}, i, j = 1, 2$, is a two-dimensional orthonormal matrix with unit determinant. Consequently, the transformation represents a rotation about the axis spanned by the vector $\boldsymbol{x}$.

Exercise 2.1.4 Construct an example showing that the sum of two finite rotation "vectors" does not need to lie in a plane generated by those vectors.

Use your textbook to verify that the composition of rotations represented by "vectors" $(\pi, 0, 0)$ and $(0, -\pi, 0)$ is represented with the "vector" $(0, 0, -\pi)$.

Exercise 2.1.5 Let $\mathcal{P}^k(\Omega)$ denote the set of all real- or complex-valued polynomials defined on a set $\Omega \subset \mathbb{R}^n (\mathbb{C}^n)$ with degree less or equal to k. Show that $\mathcal{P}^k(\Omega)$ with the standard operations for functions is a vector space.

It is sufficient only to show that the set is closed with respect to the vector space operations. But this is immediate: sum of two polynomials of degree $\leq k$ is a polynomial with degree $\leq k$ and, upon multiplying a polynomial from $\mathcal{P}^k(\Omega)$ with a number, we obtain a polynomial from $\mathcal{P}^k(\Omega)$ as well.

Exercise 2.1.6 Let $\mathcal{G}_k(\Omega)$ denote the set of all polynomials of order greater or equal to k. Is $\mathcal{G}_k(\Omega)$ a vector space? Why?

No, it is not. The set is not closed with respect to function addition. Adding polynomial $f(x)$ and $-f(x)$, we obtain a zero function, i.e., a polynomial of zero degree which is outside of the set.

Exercise 2.1.7 The extension f_1 in the definition of a function f from class $C^k(\bar{\Omega})$ need not be unique. The boundary values of f_1, however, do not depend upon a particular extension. Explain why.

By definition, function f_1 is continuous in the larger set Ω_1. Let $x \in \partial\Omega \in \Omega_1$, and let $\Omega \ni x_n \to x$. By continuity, $f_1(x) = \lim_{n\to\infty} f_1(x_n) = \lim_{n\to\infty} f(x_n)$ since $f_1 = f$ in Ω. The same argument applies to the derivatives of f_1.

Exercise 2.1.8 Show that $C^k(\Omega),\ k = 0, 1, \dots, \infty$, is a vector space.

It is sufficient to notice that all these sets are closed with respect to the function addition and multiplication with a number.

2.2 Subspaces

2.3 Equivalence Relations and Quotient Spaces

Exercises

Exercise 2.3.1 Prove that the operations in the quotient space V/M are well defined, i.e., the equivalence classes $[\boldsymbol{x} + \boldsymbol{y}]$ and $[\alpha\boldsymbol{x}]$ do not depend upon the choice of elements $\boldsymbol{x} \in [\boldsymbol{x}]$ and $\boldsymbol{y} \in [\boldsymbol{y}]$.

Let $\boldsymbol{x}_i \in [\boldsymbol{x}] = \boldsymbol{x} + M,\ \boldsymbol{y}_i \in [\boldsymbol{y}] = \boldsymbol{y} + M,\ i = 1, 2$. We need to show that

$$\boldsymbol{x}_1 + \boldsymbol{y}_1 + M = \boldsymbol{x}_2 + \boldsymbol{y}_2 + M$$

Let $\boldsymbol{z} \in \boldsymbol{x}_1 + \boldsymbol{y}_1 + M$, i.e., $\boldsymbol{z} = \boldsymbol{x}_1 + \boldsymbol{y}_1 + \boldsymbol{m},\ \boldsymbol{m} \in M$. Then

$$\boldsymbol{z} = \boldsymbol{x}_2 + \boldsymbol{y}_2 + (\boldsymbol{x}_1 - \boldsymbol{x}_2) + (\boldsymbol{y}_1 - \boldsymbol{y}_2) + \boldsymbol{m} \in \boldsymbol{x}_2 + \boldsymbol{y}_2 + M$$

since each of vectors $\boldsymbol{x}_1 - \boldsymbol{x}_2,\ \boldsymbol{y}_1 - \boldsymbol{y}_2,\ \boldsymbol{m}$ is an element of subspace M, and M is closed with respect to the vector addition. By the same argument $\boldsymbol{x}_2 + \boldsymbol{y}_2 + M \subset \boldsymbol{x}_1 + \boldsymbol{y}_1 + M$.

Similarly, let $\boldsymbol{x}_i \in [\boldsymbol{x}] = \boldsymbol{x} + M,\ i = 1, 2$. We need to show that

$$\alpha\boldsymbol{x}_1 + M = \alpha\boldsymbol{x}_2 + M$$

This is equivalent to show that $\alpha\boldsymbol{x}_1 - \alpha\boldsymbol{x}_2 = \alpha(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ is an element of subspace M. But this follows from the fact that $\boldsymbol{x}_1 - \boldsymbol{x}_2 \in M$ and that M is closed with respect to the multiplication by a scalar.

Exercise 2.3.2 Let M be a subspace of a real space V and R_M the corresponding equivalence relation. Together with three equivalence axioms (i) - (iii), relation R_M satisfies two extra conditions:

$$\text{(iv)} \quad \boldsymbol{x}R\boldsymbol{y}, \boldsymbol{u}R\boldsymbol{v} \quad \Leftrightarrow \quad (\boldsymbol{x}+\boldsymbol{u})R(\boldsymbol{y}+\boldsymbol{v})$$

$$\text{(v)} \quad \boldsymbol{x}R\boldsymbol{y} \Leftrightarrow (\alpha\boldsymbol{x})R(\alpha\boldsymbol{y}) \qquad \forall\alpha \in \mathbb{R}$$

We say that R_M is *consistent* with linear structure on V. Let R be an arbitrary relation satisfying conditions (i)–(v), i.e., an equivalence relation consistent with linear structure on V. Show that there exists a unique subspace M of V such that $R = R_M$, i.e., R is generated by the subspace M.

Define M to be the equivalence class of zero vector, $M = [\mathbf{0}]$. Axioms (iv) and (v) imply that M is closed with respect to vector space operations and, therefore, is a vector subspace of V. Let $\boldsymbol{y}R\boldsymbol{x}$. Since $\boldsymbol{x}R\boldsymbol{x}$, axiom (v) implies $-\boldsymbol{x}R-\boldsymbol{x}$ and, by axiom (iv), $(\boldsymbol{y}-\boldsymbol{x})R\mathbf{0}$. By definition of M, $(\boldsymbol{y}-\boldsymbol{x}) \in M$. But this is equivalent to $\boldsymbol{y} \in \boldsymbol{x} + M = [\boldsymbol{x}]_{R_M}$.

Exercise 2.3.3 Another way to see the difference between two equivalence relations discussed in Example 2.3.3 is to discuss the equations of rigid body motions. For the sake of simplicity let us consider the two-dimensional case.

(i) Prove that, under the assumption that the Jacobian of the deformation gradient $\boldsymbol{F}$ is positive, $\boldsymbol{E}(\boldsymbol{u}) = \mathbf{0}$ if and only if $\boldsymbol{u}$ takes the form

$$u_1 = c_1 + \cos\theta x_1 + \sin\theta x_2 - x_1$$
$$u_2 = c_2 - \sin\theta x_1 + \cos\theta x_2 - x_2$$

where $\theta \in [0, 2\pi)$ is the angle of rotation.

(ii) Prove that $\varepsilon_{ij}(\boldsymbol{u}) = 0$ if and only if $\boldsymbol{u}$ has the following form

$$u_1 = c_1 + \theta x_2$$
$$u_2 = c_2 - \theta x_1$$

One can see that for small values of angle θ $(\cos\theta \approx 1, \sin\theta \approx \theta)$ the second set of equations can be obtained by linearizing the first.

(i) Using the notation from Example 2.3.3, we need to show that the right Cauchy-Green tensor $C_{ij} = x^k_{,i} x^k_{,j} = \delta_{ij}$ if and only if

$$x^1 = c_1 + \cos\theta X_1 + \sin\theta X_2$$
$$x^2 = c_2 - \sin\theta X_1 + \cos\theta X_2$$

A direct computation shows that $C_{ij} = \delta_{ij}$ for the (relative) configuration above. Conversely,

$$(x^1_{,1})^2 + (x^2_{,1})^2 = 1$$

implies that there exists an angle θ_1 such that

$$x^1_{,1} = \cos\theta_1, \quad x^2_{,1} = \sin\theta_1$$

Similarly,

$$(x^1_{,2})^2 + (x^2_{,2})^2 = 1$$

implies that there exists an angle θ_2 such that

$$x^1_{,2} = -\sin\theta_2, \quad x^2_{,2} = \cos\theta_2$$

Finally, condition

$$x^1_{,1}x^1_{,2} + x^2_{,1}x^2_{,2} = 0$$

implies that $\sin(\theta_1 - \theta_2) = 0$. Restricting ourselves to angles in $[0, 2\pi)$, we see that either $\theta_1 = \theta_2 + \pi$ or $\theta_1 = \theta_2$. In the first case, $\sin\theta_1 = -\sin\theta_2$ and $\cos\theta_1 = -\cos\theta_2$ which results in a deformation gradient with negative Jacobian. Thus $\theta_1 = \theta_2 =: \theta$. A direct integration results then in the final formula. Angle θ is the angle of rotation, and integration constants c_1, c_2 are the components of the rigid displacement.

(ii) Integrating $u_{1,1} = 0$, we obtain

$$u_1 = c_1 + \theta_1 x_2$$

Similarly, $u_{2,2} = 0$ implies

$$u_2 = c_2 - \theta_2 x_2$$

Finally, $u_{1,2} + u_{2,1} = 0$ results in $\theta_1 = \theta_2 =: \theta$.

2.4 Linear Dependence and Independence, Hamel Basis, Dimension

Linear Transformations

2.5 Linear Transformations—The Fundamental Facts

Exercises

Exercise 2.5.1 Find the matrix representation of rotation R about angle θ in $\mathbb{R}^2$ with respect to basis $\boldsymbol{a}_1 = (1, 0),\ \boldsymbol{a}_2 = (1, 1)$.

Start by representing basis $\boldsymbol{a}_1, \boldsymbol{a}_2$ in terms of canonical basis $\boldsymbol{e}_1 = (1, 0),\ \boldsymbol{e}_2 = (0, 1)$,

$$\begin{array}{l} \boldsymbol{a}_1 = \boldsymbol{e}_1 \\ \boldsymbol{a}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} \boldsymbol{e}_1 = \boldsymbol{a}_1 \\ \boldsymbol{e}_2 = \boldsymbol{a}_2 - \boldsymbol{a}_1 \end{array}$$

Then,

$$R\boldsymbol{a}_1 = R\boldsymbol{e}_1 = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2 = \cos\theta\boldsymbol{a}_1 + \sin\theta(\boldsymbol{a}_2 - \boldsymbol{a}_1) = (\cos\theta - \sin\theta)\boldsymbol{a}_1 + \sin\theta\boldsymbol{e}_2$$

Similarly,

$$R\boldsymbol{a}_2 = \sqrt{2}\cos(\theta + \frac{\pi}{4})\boldsymbol{e}_1 + \sqrt{2}\sin(\theta + \frac{\pi}{4})\boldsymbol{e}_2 = \sqrt{2}(\cos(\theta + \frac{\pi}{4}) - \sin(\theta + \frac{\pi}{4}))\boldsymbol{a}_1 + \sqrt{2}\sin(\theta + \frac{\pi}{4})\boldsymbol{a}_2$$

or,

$$R\boldsymbol{a}_2 = R(\boldsymbol{e}_1 + \boldsymbol{e}_2) = (\cos\theta - \sin\theta)\boldsymbol{e}_1 + (\sin\theta + \cos\theta)\boldsymbol{e}_2 = -2\sin\theta\boldsymbol{a}_1 + (\sin\theta + \cos\theta)\boldsymbol{a}_2$$

Therefore, the matrix representation is:

$$\begin{bmatrix} (\cos\theta - \sin\theta) & \sqrt{2}(\cos(\theta + \frac{\pi}{4}) - \sin(\theta + \frac{\pi}{4})) \\ \sin\theta & \sqrt{2}\sin(\theta + \frac{\pi}{4}) \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \cos\theta - \sin\theta & -2\sin\theta \\ \sin\theta & \cos\theta + \sin\theta \end{bmatrix}$$

Exercise 2.5.2 Let $V = X \oplus Y$, and $\dim X = n,\ \dim Y = m$. Prove that $\dim V = n + m$.

Let $e_1, \ldots, e_n$ be a basis for X, and let $g_1, \ldots, g_m$ be a basis for Y. It is sufficient to show that $e_1, \ldots, e_n, g_1, \ldots, g_m$ is a basis for V. Let $v \in V$. Then $v = x + y$ with $x \in X$, $y \in Y$, and $x = \sum_i x_i e_i,\ y = \sum_j y_j g_j$, so $v = \sum_i x_i e_i + \sum_j y_j g_j$ which proves the span condition. To prove linear independence, assume that

$$\sum_i \alpha_i e_i + \sum_j \beta_j g_j = 0$$

From the fact that $X \cap Y = \{0\}$ follows that

$$\sum_i \alpha_i e_i = 0 \text{ and } \sum_j \beta_j g_j = 0$$

which in turn implies that $\alpha_1 = \ldots = \alpha_n = 0$, and $\beta_1 = \ldots = \beta_m = 0$, since each of the two sets of vectors is separately linearly independent.

2.6 Isomorphic Vector Spaces

2.7 More About Linear Transformations

Exercises

Exercise 2.7.1 Let V be a vector space and id_V the identity transformation on V. Prove that a linear transformation $T\colon V \to V$ is a projection if and only if $\mathrm{id}_V - T$ is a projection.

Assume T is a projection, i.e., $T^2 = T$. Then

$$\begin{aligned}(\mathrm{id}_V - T)^2 &= (\mathrm{id}_V - T)\,(\mathrm{id}_V - T)\\ &= \mathrm{id}_V - T - T + T^2\\ &= \mathrm{id}_V - T - T + T\\ &= \mathrm{id}_V - T\end{aligned}$$

The converse follows from the first step and $T = \mathrm{id}_V - (\mathrm{id}_V - T)$.

2.8 Linear Transformations and Matrices

2.9 Solvability of Linear Equations

Exercises

Exercise 2.9.1 Equivalent and Similar Matrices. Given matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, when nonsingular matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ exist such that

$$\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{Q}$$

we say that the *matrices* $\boldsymbol{A}$ and $\boldsymbol{B}$ are *equivalent*. If $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$, we say $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar*.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar $n \times n$ matrices. Prove that $\det \boldsymbol{A} = \det \boldsymbol{B}$, $r(\boldsymbol{A}) = r(\boldsymbol{B})$, $n(\boldsymbol{A}) = n(\boldsymbol{B})$.

The first assertion follows immediately from the Cauchy's Theorem for Determinants. Indeed, $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{P}$ implies

$$\det \boldsymbol{P} \, \det \boldsymbol{A} = \det \boldsymbol{B} \, \det \boldsymbol{P}$$

and, consequently, $\det \boldsymbol{A} = \det \boldsymbol{B}$.

Let $A : X \to X$ be a linear map. Recall that the rank of A equals the maximum number of linearly independent vectors $A\boldsymbol{e}_j$ where $\boldsymbol{e}_j, j = 1, \ldots, n$ is an arbitrary basis in X. Let $P : X \to X$ be now an isomorphism. Consider a basis $\boldsymbol{e}_j, j = 1, \ldots, n$ in space X. Then $P\boldsymbol{e}_j$ is another basis in X, and the rank of A equals the maximum number of linearly independent vectors $AP\boldsymbol{e}_j$ which is also the rank of AP. The Rank and Nullity Theorem implies then that nullity of AP equals nullity of A.

Similarly, nullity of A is the maximum number of linearly independent vectors $\boldsymbol{e}_j$ such that $A\boldsymbol{e}_j = \boldsymbol{0}$. But

$$A\boldsymbol{e}_j = \boldsymbol{0} \quad \Leftrightarrow \quad PA\boldsymbol{e}_j = \boldsymbol{0}$$

so the nullity of A is equal to the nullity of PA. The Rank and Nullity Theorem implies then that rank of PA equals nullity of A.

Consequently, for similar transformations (matrices) rank and nullity are the same.

Exercise 2.9.2 Let T_1 and T_2 be two different linear transformations from an n-dimensional linear vector space V into itself. Prove that T_1 and T_2 are represented relative to two different bases by the *same* matrix if and only if there exists a nonsingular transformation Q on V such that $T_2 = Q^{-1}T_1Q$.

Let $T_1\boldsymbol{g}_j = \sum_i T_{ij}\boldsymbol{g}_i$ and $T_2\boldsymbol{e}_j = \sum_i T_{ij}\boldsymbol{e}_i$ where $\boldsymbol{g}_j, \boldsymbol{e}_j$ are two bases in V. Define a nonsingular transformation Q mapping basis $\boldsymbol{e}_j$ into basis $\boldsymbol{g}_j$. Then

$$T_1Q\boldsymbol{e}_j = \sum_i T_{ij}Q\boldsymbol{e}_i = Q\sum_i T_{ij}\boldsymbol{e}_i$$

which implies

$$Q^{-1}T_1Q\boldsymbol{e}_j = \sum_i T_{ij}\boldsymbol{e}_i = T_2\boldsymbol{e}_j$$

Conversely, if $T_2 = Q^{-1}T_1Q$ and Q maps basis $\boldsymbol{e}_j$ into basis $\boldsymbol{g}_j$, then matrix representation of T_2 with respect to $\boldsymbol{e}_j$ equals the matrix representation of T_1 with respect basis $\boldsymbol{g}_j$.

Exercise 2.9.3 Let T be a linear transformation represented by the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & 2 \end{bmatrix}$$

relative to bases $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ of $\mathbb{R}^2$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ of $\mathbb{R}^3$. Compute the matrix representing T relative to the new bases:

$$\begin{aligned} \boldsymbol{\alpha}_1 &= 4\boldsymbol{a}_1 - \boldsymbol{a}_2 & \boldsymbol{\beta}_1 &= 2\boldsymbol{b}_1 - \boldsymbol{b}_2 + \boldsymbol{b}_3 \\ \boldsymbol{\alpha}_2 &= \boldsymbol{a}_1 + \boldsymbol{a}_2 & \boldsymbol{\beta}_2 &= \boldsymbol{b}_1 - \boldsymbol{b}_3 \\ & & \boldsymbol{\beta}_3 &= \boldsymbol{b}_1 + 2\boldsymbol{b}_2 \end{aligned}$$

We have

$$\begin{aligned} T\boldsymbol{b}_1 &= \boldsymbol{a}_1 \\ T\boldsymbol{b}_2 &= -\boldsymbol{a}_1 + 3\boldsymbol{a}_2 \\ T\boldsymbol{b}_3 &= 4\boldsymbol{a}_1 + 2\boldsymbol{a}_2 \end{aligned}$$

Inverting the formulas for $\boldsymbol{a}_i$, we get

$$\begin{aligned} \boldsymbol{a}_1 &= \tfrac{1}{5}\boldsymbol{\alpha}_1 + \tfrac{1}{5}\boldsymbol{\alpha}_2 \\ \boldsymbol{a}_2 &= -\tfrac{1}{5}\boldsymbol{\alpha}_1 + \tfrac{4}{5}\boldsymbol{\alpha}_2 \end{aligned}$$

We have now,

$$\begin{aligned} T\boldsymbol{\beta}_1 &= T(2\boldsymbol{b}_1 - \boldsymbol{b}_2 + \boldsymbol{b}_3) \\ &= 2T\boldsymbol{b}_1 - T\boldsymbol{b}_2 + T\boldsymbol{b}_3 \\ &= 2\boldsymbol{a}_1 + \boldsymbol{a}_1 - 3\boldsymbol{a}_2 + 4\boldsymbol{a}_1 + 2\boldsymbol{a}_2 \\ &= 7\boldsymbol{a}_1 - \boldsymbol{a}_2 \\ &= 7(\tfrac{1}{5}\boldsymbol{\alpha}_1 + \tfrac{1}{5}\boldsymbol{\alpha}_2) - (-\tfrac{1}{5}\boldsymbol{\alpha}_1 + \tfrac{4}{5}\boldsymbol{\alpha}_2) \\ &= \tfrac{8}{5}\boldsymbol{\alpha}_1 + \tfrac{3}{5}\boldsymbol{\alpha}_2 \end{aligned}$$

Similarly,

$$\begin{aligned} T\boldsymbol{\beta}_2 &= T\boldsymbol{b}_1 - T\boldsymbol{b}_3 \\ &= -3\boldsymbol{a}_1 - 2\boldsymbol{a}_2 \\ &= -\tfrac{1}{5}\boldsymbol{\alpha}_1 - \tfrac{11}{5}\boldsymbol{\alpha}_2 \end{aligned}$$

and

$$\begin{aligned} T\boldsymbol{\beta}_3 &= T\boldsymbol{b}_1 + 2T\boldsymbol{b}_2 \\ &= -\boldsymbol{a}_1 + 6\boldsymbol{a}_2 \\ &= -\tfrac{7}{5}\boldsymbol{\alpha}_1 + \tfrac{23}{5}\boldsymbol{\alpha}_2 \end{aligned}$$

Thus, the matrix representation of transformation T wrt to new bases is

$$\begin{bmatrix} \frac{8}{5} & -\frac{1}{5} & -\frac{7}{5} \\ \frac{3}{5} & -\frac{4}{5} & \frac{23}{5} \end{bmatrix}$$

Exercise 2.9.4 Let $\boldsymbol{A}$ be an $n \times n$ matrix. Show that transformations which

(a) interchange rows or columns of $\boldsymbol{A}$

(b) multiply any row or column of $\boldsymbol{A}$ by a scalar $\neq 0$

(c) add any multiple of a row or column to a parallel row or column

produce a matrix with the same rank as $\boldsymbol{A}$.

Recall that j-th column represents value $A\boldsymbol{e}_j$. All discussed operations on columns redefine the map but do not change its range. Indeed,

$$\operatorname{span}\{A\boldsymbol{e}_1,\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_i,\ldots,A\boldsymbol{e}_n\} = \operatorname{span}\{A\boldsymbol{e}_1,\ldots,A\boldsymbol{e}_i,\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_n\}$$
$$\operatorname{span}\{A\boldsymbol{e}_1,\ldots,A(\alpha\boldsymbol{e}_i),\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_n\} = \operatorname{span}\{A\boldsymbol{e}_1,\ldots,A\boldsymbol{e}_i,\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_n\}$$
$$\operatorname{span}\{A\boldsymbol{e}_1,\ldots,A(\boldsymbol{e}_i+\beta\boldsymbol{e}_j),\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_n\} = \operatorname{span}\{A\boldsymbol{e}_1,\ldots,A\boldsymbol{e}_i,\ldots,A\boldsymbol{e}_j,\ldots,A\boldsymbol{e}_n\}$$

The same conclusions apply to the rows of matrix $\boldsymbol{A}$ as they represent vectors $A^T\boldsymbol{e}_i^*$, and rank A^T = rank A.

Exercise 2.9.5 Let $\{\boldsymbol{a}_1,\boldsymbol{a}_2\}$ and $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ be two bases for $\mathbb{R}^2$, where $\boldsymbol{a}_1 = (-1,2), \boldsymbol{a}_2 = (0,3)$, and $\boldsymbol{e}_1 = (1,0), \boldsymbol{e}_2 = (0,1)$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x,y) = (3x-4y, x+y)$. Find the matrices for T for each choice of basis and show that these matrices are similar.

Matrix representation of T in the canonical basis $\boldsymbol{e}_1,\boldsymbol{e}_2$ is:

$$\boldsymbol{T} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$$

We have

$$\boldsymbol{a}_j = \sum_{k=1}^{2} P_{jk}\boldsymbol{e}_k \quad \Rightarrow \quad \boldsymbol{e}_l = \sum_{i=1}^{2} P_{li}^{-1}\boldsymbol{e}_i$$

where

$$\boldsymbol{P} = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

Linearity of map T implies the following relations.

$$\begin{aligned} T\boldsymbol{a}_j &= T(\sum_{k=1}^{2} P_{jk}\boldsymbol{e}_k) \\ &= \sum_{k=1}^{2} P_{jk}T\boldsymbol{e}_k \\ &= \sum_{k=1}^{2} P_{jk}\sum_{l=1}^{2} T_{lk}\boldsymbol{e}_l \\ &= \sum_{k=1}^{2} P_{jk}\sum_{l=1}^{2} T_{lk}\sum_{i=1}^{2} P_{li}^{-1}\boldsymbol{e}_i \\ &= \sum_{i=1}^{2}\left(\sum_{k=1}^{2}\sum_{l=1}^{2} P_{li}^{-1}T_{lk}P_{jk}\right) \end{aligned}$$

Consequently, matrix representation $\tilde{T}_{ij}$ in basis $\boldsymbol{a}_1,\boldsymbol{a}_2$ is:

$$\tilde{T}_{ij} = \sum_{k=1}^{2}\sum_{l=1}^{2} P_{li}^{-T}T_{lk}P_{jk}$$

or, in the matrix form,

$$\tilde{\boldsymbol{T}} = \boldsymbol{P}^{-T}\boldsymbol{T}\boldsymbol{P}^T$$

which shows that matrices $\tilde{\boldsymbol{T}}$ and $\boldsymbol{T}$ are similar. Finally, computing the products above, we get

$$\tilde{\boldsymbol{T}} = \begin{bmatrix} -1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ -7 & -7 \end{bmatrix}$$

Algebraic Duals

2.10 The Algebraic Dual Space, Dual Basis

Exercises

Exercise 2.10.1 Consider the canonical basis $\boldsymbol{e}_1 = (1,0), \boldsymbol{e}_2 = (0,1)$ for $\mathbb{R}^2$. For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, x_1, x_2 are the components of $\boldsymbol{x}$ with respect to the canonical basis. The dual basis functional $\boldsymbol{e}_j^*$ returns the j-th component:

$$\boldsymbol{e}_j^* \ : \ \mathbb{R}^2 \ni (x_1, x_2) \to x_j \in \mathbb{R}$$

Consider now a different basis for $\mathbb{R}^2$, say $\boldsymbol{a}_1 = (1,1), \boldsymbol{a}_2 = (-1,1)$. Write down the explicit formulas for the dual basis.

We follow the same reasoning. Expanding $\boldsymbol{x}$ in the new basis, $\boldsymbol{x} = \xi_1 \boldsymbol{a}_1 + \xi_2 \boldsymbol{a}_2$, we apply $\boldsymbol{a}_j^*$ to both sides to learn that the dual basis functionals $\boldsymbol{a}_j^*$ return the components with respect to basis $\boldsymbol{a}_j$,

$$\boldsymbol{a}_j^* \ : \ \mathbb{R}^2 \ni (x_1, x_2) \to \xi_j \in \mathbb{R}$$

The whole issue is thus simply in computing the components ξ_j. This is done by representing the canonical basis vectors $\boldsymbol{e}_i$ in terms of vectors $\boldsymbol{a}_j$,

$$\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \boldsymbol{a}_2 &= -\boldsymbol{e}_1 + \boldsymbol{e}_2 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \boldsymbol{e}_1 &= \tfrac{1}{2}\boldsymbol{a}_1 - \tfrac{1}{2}\boldsymbol{a}_2 \\ \boldsymbol{e}_2 &= \tfrac{1}{2}\boldsymbol{a}_1 + \tfrac{1}{2}\boldsymbol{a}_2 \end{aligned}$$

Then,

$$\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 = x_1 (\frac{1}{2}\boldsymbol{a}_1 - \frac{1}{2}\boldsymbol{a}_2) + x_2 (\frac{1}{2}\boldsymbol{a}_1 + \frac{1}{2}\boldsymbol{a}_2) = \frac{1}{2}(x_1 + x_2)\boldsymbol{a}_1 + \frac{1}{2}(x_2 - x_1)\boldsymbol{a}_2$$

Therefore, $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_2 - x_1)$ are the dual basis functionals.

Exercise 2.10.2 Let V be a finite-dimensional vector space, and V^* denote its algebraic dual. Let $\boldsymbol{e}_i, i = 1, \ldots, n$ be a basis in V, and $\boldsymbol{e}_j^*, j = 1, \ldots, n$ denote its dual basis. What is the matrix representation of the duality pairing with respect to these two bases? Does it depend upon whether we define the dual space as linear or antilinear functionals?

It follows from the definition of the dual basis that the matrix representation of the duality pairing is the Kronecker's delta δ_{ij}. This is true for both definitions of the dual space.

Exercise 2.10.3 Let V be a complex vector space. Let $L(V, \mathbb{C})$ denote the space of linear functionals defined on V, and let $\bar{L}(V, \mathbb{C})$ denote the space of antilinear functionals defined on V. Define the (complex conjugate) map C as

$$C : L(V, \mathbb{C}) \ni f \to \bar{f} \in \bar{L}(V, \mathbb{C}), \quad \bar{f}(v) \stackrel{\text{def}}{=} \overline{f(v)}$$

Show that operator C is well defined, bijective, and antilinear. What is the inverse of C?

Let f be a linear functional defined on V. Then,

$$\overline{f(\alpha_1 v_1 + \alpha_2 v_2)} = \overline{\alpha_1 f(v_1) + \alpha_2 f(v_2)} = \overline{\alpha_1}\,\overline{f(v_1)} + \overline{\alpha_2}\,\overline{f(v_2)}$$

so $\bar{f}$ is antilinear. Similarly,

$$\overline{(\alpha_1 f_1 + \alpha_2 f_2)(v)} = \overline{\alpha_1 f_1(v) + \alpha_2 f_2(v)} = \overline{\alpha_1}\,\overline{f_1(v)} + \overline{\alpha_2}\,\overline{f_2(v)}$$

so the map C is itself antilinear. Similarly, map

$$D : \bar{L}(V, \mathbb{C}) \ni f \to \bar{f} \in L(V, \mathbb{C}), \quad \bar{f}(v) \stackrel{\text{def}}{=} \overline{f(v)}$$

is well defined and antilinear. Notice that C and D are defined on different space so you cannot say that $C = D$. Obviously, both compositions $D \circ C$ and $C \circ D$ are identities, so D is the inverse of C, and both maps are bijective.

Exercise 2.10.4 Let V be a finite-dimensional vector space. Consider the map ι from V into its bidual space V^{**}, prescribing for each $\boldsymbol{v} \in V$ the evaluation at $\boldsymbol{v}$, and establishing the canonical isomorphism between space V and its bidual V^{**}. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis for V, and let $\boldsymbol{e}_1^*, \ldots, \boldsymbol{e}_n^*$ be the corresponding dual basis. Consider the bidual basis, i.e., the basis $\boldsymbol{e}_i^{**}, i = 1, \ldots, n$ in the bidual space, dual to the dual basis, and prove that

$$\iota(\boldsymbol{e}_i) = \boldsymbol{e}_i^{**}$$

This is simple. Definition of map ι implies that

$$< \iota(v), f >_{V^{**} \times V^*} = < f, v >_{V^* \times V}$$

Thus,

$$\delta_{ij} = < \boldsymbol{e}_i^{**}, \boldsymbol{e}_j^* >_{V^{**} \times V^*} \quad \text{and} \quad < \iota(\boldsymbol{e}_i), \boldsymbol{e}_j^* >_{V^{**} \times V^*} = < \boldsymbol{e}_j^*, \boldsymbol{e}_i >_{V^* \times V} = \delta_{ji} = \delta_{ij}$$

The relation follows then form the uniqueness of the (bi)dual basis.

2.11 Transpose of a Linear Transformation

Exercises

Exercise 2.11.1 The following is a "sanity check" of your understanding of concepts discussed in the last two sections. Consider $\mathbb{R}^2$.

(a) Prove that $a_1 = (1,0), a_2 = (1,1)$ is a basis in $\mathbb{R}^2$.

It is sufficient to show linear independence. Any n linearly independent vectors in a n-dimensional vector space provide a basis for the space. The vectors are clearly not collinear, so they are linearly independent. Formally, $\alpha_1 a_1 + \alpha_2 a_2 = (\alpha_1 + \alpha_2, \alpha_2) = (0,0)$ implies $\alpha_1 = \alpha_2 = 0$, so the vectors are linearly independent.

(b) Consider a functional $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = 2x_1 + 3x_2$. Prove that the functional is linear, and determine its components in the dual basis a_1^*, a_2^*.

Linearity is trivial. Dual basis functionals return components with respect to the original basis,

$$a_j^*(\xi_1 \boldsymbol{a}_1 + \xi_2 \boldsymbol{a}_2) = \xi_j$$

It is, therefore, sufficient to determine ξ_1, ξ_2. We have,

$$\xi_1 \boldsymbol{a}_1 + \xi_2 \boldsymbol{a}_2 = \xi_1 \boldsymbol{e}_1 + \xi_2(\boldsymbol{e}_1 + \boldsymbol{e}_2) = (\xi_1 + \xi_2)\boldsymbol{e}_1 + \xi_2 \boldsymbol{e}_2$$

so $x_1 = \xi_1 + \xi_2$ and $x_2 = \xi_2$. Inverting, we get, $\xi_1 = x_1 - x_2$, $\xi_2 = x_2$. These are the dual basis functionals. Consequently,

$$f(x_1, x_2) = 2x_1 + 3x_2 = 2(\xi_1 + \xi_2) + 3\xi_2 = 2\xi_1 + 5\xi_2 = (2a_1^* + 5a_2^*)(x_1, x_2)$$

Using the argumentless notation,

$$f = 2a_1^* + 5a_2^*$$

If you are not interested in the form of the dual basis functionals, you get compute the components of f with respect to the dual basis faster. Assume $\alpha_1 a_1^* + \alpha_2 a_2^* = f$. Evaluating both sides at $x = a_1$ we get,

$$(\alpha_1 a_1^* + \alpha_2 a_2^*)(a_1) = \alpha_1 = f(a_1) = f(1,0) = 2$$

Similarly, evaluating at $x = a_2$, we get $\alpha_2 = 5$.

(c) Consider a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix representation in basis a_1, a_2 is

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

Compute the matrix representation of the transpose operator with respect to the dual basis.

Nothing to compute. Matrix representation of the transpose operator with respect to the dual basis is equal of the transpose of the original matrix,

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Exercise 2.11.2 Prove Proposition 2.11.3.

All five properties of the matrices are directly related to the properties of linear transformations discussed in Proposition 2.11.1 and Proposition 2.11.2. They can also be easily verified directly.

(i)

$$(\alpha A_{ij} + \beta B_{ij})^T = \alpha A_{ji} + \beta B_{ji} = \alpha (A_{ij})^T + \beta (\beta_{ij})^T$$

(ii)

$$\left(\sum_{l=1}^{n} B_{il} A_{lj} \right)^T = \sum_{l=1}^{n} B_{jl} A_{li} = \sum_{l=1}^{n} A_{li} B_{jl} = \sum_{l=1}^{n} (A_{il})^T (B_{lj})^T$$

(iii) $(\delta_{ij})^T = \delta_{ji} = \delta_{ij}$.

(iv) Follow the reasoning for linear transformations:

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \quad \Rightarrow \quad (\boldsymbol{A}^{-1})^T \boldsymbol{A}^T = \boldsymbol{I}^T = \boldsymbol{I}$$

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{A}^T (\boldsymbol{A}^{-1})^T = \boldsymbol{I}^T = \boldsymbol{I}$$

Consequently, matrix $\boldsymbol{A}^T$ is invertible, and $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$.

(v) Conclude this from Proposition 2.11.2. Given a matrix $A_{ij},\ ij, = 1, \ldots, n$, we can interpret it as the matrix representation of map $A : \mathbb{R}^n \to \mathbb{R}^n$ defined as:

$$\boldsymbol{y} = A\boldsymbol{x} \quad \text{where} \quad y_i = \sum_{j=1}^{n} A_{ij} x_j$$

with respect to the canonical basis $\boldsymbol{e}_i,\ i = 1, \ldots, n$. The transpose matrix $\boldsymbol{A}^T$ can then be interpreted as the matrix of the transpose transformation:

$$A^T : (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$$

The conclusion follows then from the facts that rank A = rank $\boldsymbol{A}$, rank $\boldsymbol{A}^T$ = rank A^T, and Proposition 2.11.2.

Exercise 2.11.3 Construct an example of square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

(a) $\boldsymbol{AB} \neq \boldsymbol{BA}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{BA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) $\boldsymbol{AB} = \boldsymbol{0}$, but neither $\boldsymbol{A} = \boldsymbol{0}$ nor $\boldsymbol{B} = \boldsymbol{0}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) $\boldsymbol{AB} = \boldsymbol{AC}$, but $\boldsymbol{B} \neq \boldsymbol{C}$

Take $\boldsymbol{A}, \boldsymbol{B}$ from (b) and $\boldsymbol{C} = \boldsymbol{0}$.

Exercise 2.11.4 If $\boldsymbol{A} = [A_{ij}]$ is an $m \times n$ rectangular matrix and its *transpose* $\boldsymbol{A}^T$ is the $n \times m$ matrix, $\boldsymbol{A}^T_{n\times m} = [A_{ji}]$. Prove that

(i) $(\boldsymbol{A}^T)^T = \boldsymbol{A}$.

$$\left((A_{ij})^T\right)^T = (A_{ji})^T = A_{ij}$$

(ii) $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$.

Particular case of Proposition 2.11.3(i).

(iii) $(\boldsymbol{ABC} \cdots \boldsymbol{XYZ})^T = \boldsymbol{Z}^T\boldsymbol{Y}^T\boldsymbol{X}^T \cdots \boldsymbol{C}^T\boldsymbol{B}^T\boldsymbol{A}^T$.

Use Proposition 2.11.3(ii) and recursion,

$$\begin{aligned}
(\boldsymbol{ABC} \ldots \boldsymbol{XYZ})^T &= (\boldsymbol{BC} \ldots \boldsymbol{XYZ})^T \boldsymbol{A}^T \\
&= (\boldsymbol{C} \ldots \boldsymbol{XYZ})^T \boldsymbol{B}^T \boldsymbol{A}^T \\
&\;\vdots \\
&= \boldsymbol{Z}^T\boldsymbol{Y}^T\boldsymbol{X}^T \ldots \boldsymbol{C}^T\boldsymbol{B}^T\boldsymbol{A}^T
\end{aligned}$$

(iv) $(q\boldsymbol{A})^T = q\boldsymbol{A}^T$.

Particular case of Proposition 2.11.3(i).

Exercise 2.11.5 In this exercise, we develop a classical formula for the inverse of a square matrix. Let $\boldsymbol{A} = [a_{ij}]$ be a matrix of order n. We define the *cofactor* A_{ij} of the element a_{ij} of the i-th column of $\boldsymbol{A}$ as the determinant of the matrix obtained by deleting the i-th row and j-th column of $\boldsymbol{A}$, multiplied by $(-1)^{i+j}$:

$$A_{ij} = \text{cofactor } a_{ij} \overset{\text{def}}{=} (-1)^{i+j} \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\
 & & & \cdots & \cdots & & \\
a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\
a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\
 & & & \cdots & \cdots & & \\
a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn}
\end{vmatrix}$$

(a) Show that

$$\delta_{ij} \det \boldsymbol{A} = \sum_{k=1}^{n} a_{ik} A_{jk}, \quad 1 \leq i, j \leq n$$

where δ_{ij} is the Kronecker delta.

Hint: Compare Exercise 2.13.4.

For $i = j$, the formula reduces to the Laplace Expansion Formula for determinants discussed in Exercise 2.13.4. For $i \neq j$, the right-hand side represents the Laplace expansion of the determinant of an array where two rows are identical. Antilinearity of determinant (comp. Section 2.13) implies then that the value must be zero.

(b) Using the result in (a), conclude that

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}[A_{ij}]^T$$

Divide both sides by $\det \boldsymbol{A}$.

(c) Use (b) to compute the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

and verify your answer by showing that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & \frac{4}{3} & -\frac{2}{3} \\ 1 & \frac{1}{3} & -\frac{2}{3} \\ -1 & -1 & 1 \end{bmatrix}$$

Exercise 2.11.6 Consider the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 4 & 1 \\ 2 & -1 & 3 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -1 & 4 \\ 12 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C} = [1, -1, 4, -3]$$

and

$$\boldsymbol{D} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 4 & 0 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

If possible, compute the following:

(a) $\boldsymbol{A}\boldsymbol{A}^T + 4\boldsymbol{D}^T\boldsymbol{D} + \boldsymbol{E}^T$

The expression is ill-defined, $\boldsymbol{A}\boldsymbol{A}^T \in \text{Matr}(2,2)$ and $\boldsymbol{E}^T \in \text{Matr}(4,4)$, so the two matrices cannot be added to each other.

(b) $\boldsymbol{C}^T\boldsymbol{C} + \boldsymbol{E} - \boldsymbol{E}^2$

$$= \begin{bmatrix} -1 & -4 & 3 & -17 \\ -7 & -12 & -1 & 1 \\ -2 & -8 & 16 & -27 \\ -1 & -2 & -9 & 10 \end{bmatrix}$$

(c) $\boldsymbol{B}^T\boldsymbol{D}$

Ill-defined, mismatched dimensions.

(d) $\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{D} - \boldsymbol{D}$

$$= \begin{bmatrix} 276 \\ 36 \end{bmatrix}$$

(e) $\boldsymbol{E}\boldsymbol{C} - \boldsymbol{A}^T\boldsymbol{A}$

$\boldsymbol{E}\boldsymbol{C}$ is not computable.

(f) $\boldsymbol{A}^T\boldsymbol{D}\boldsymbol{C}(\boldsymbol{E} - 2\boldsymbol{I})$

$$= \begin{bmatrix} 32 & -40 & 40 & 144 \\ -12 & 15 & -15 & -54 \\ 68 & -85 & 85 & 306 \\ 8 & -10 & 10 & 36 \end{bmatrix}$$

Exercise 2.11.7 Do the following vectors provide a basis for $\mathbb{R}^4$?

$$\boldsymbol{a} = (1, 0, -1, 1), \quad \boldsymbol{b} = (0, 1, 0, 22)$$
$$\boldsymbol{c} = (3, 3, -3, 9), \quad \boldsymbol{d} = (0, 0, 0, 1)$$

It is sufficient to check linear independence,

$$\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} + \delta\boldsymbol{d} = \boldsymbol{0} \quad \stackrel{?}{\Rightarrow} \quad \alpha = \beta = \gamma = \delta = 0$$

Computing

$$\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} + \delta\boldsymbol{d} = (\alpha + 3\gamma, \beta + 3\gamma, -\alpha - 3\gamma, \alpha + 22\beta + 9\gamma + \delta)$$

we arrive at the homogeneous system of equations

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 0 & -3 & 0 \\ 1 & 22 & 9 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a nontrivial solution iff the matrix is singular, i.e., $\det \boldsymbol{A} = 0$. By inspection, the third row equals minus the first one, so the determinant is zero. Vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are linearly dependent and, therefore, do not provide a basis for $\mathbb{R}^4$.

Exercise 2.11.8 Evaluate the determinant of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 4 \\ 1 & 0 & 2 & 1 \\ 4 & 7 & 1 & -1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

Use e.g. the Laplace expansion with respect to the last row and Sarrus' formulas,

$$\begin{vmatrix} 1 & -1 & 0 & 4 \\ 1 & 0 & 2 & 1 \\ 4 & 7 & 1 & -1 \\ 1 & 0 & 1 & 2 \end{vmatrix} = -(-1)\begin{vmatrix} -1 & 0 & 4 \\ 0 & 2 & 1 \\ 7 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 & 0 \\ 1 & 0 & 2 \\ 4 & 7 & 1 \end{vmatrix}$$
$$= 2 - 56 + 1 - (-2 + 4 - 32 - 1) + 2(-8 - 14 + 1) = -53 - (-31) - 42 = -64$$

Exercise 2.11.9 Invert the following matrices (see Exercise 2.11.5).

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \boldsymbol{B}^{-1} = \begin{bmatrix} \frac{4}{12} & -\frac{2}{12} & 0 \\ -\frac{2}{12} & \frac{7}{12} & -\frac{6}{12} \\ 0 & -\frac{6}{12} & 1 \end{bmatrix}$$

Exercise 2.11.10 Prove that if $\boldsymbol{A}$ is symmetric and nonsingular, so is $\boldsymbol{A}^{-1}$.

Use Proposition 2.11.3(iv).

$$(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1} = \boldsymbol{A}^{-1}$$

Exercise 2.11.11 Prove that if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, and $\boldsymbol{D}$ are nonsingular matrices of the same order then

$$(\boldsymbol{ABCD})^{-1} = \boldsymbol{D}^{-1}\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Use the fact that matrix product is associative,

$$\begin{aligned}(\boldsymbol{ABCD})(\boldsymbol{D}^{-1}\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) &= \boldsymbol{ABC}(\boldsymbol{DD}^{-1})\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \\ &= \boldsymbol{ABC}\,\boldsymbol{I}\,\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \ldots = \boldsymbol{I}\end{aligned}$$

In the same way,

$$(\boldsymbol{D}^{-1}\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{ABCD}) = \boldsymbol{I}$$

So, $(\boldsymbol{ABCD})^{-1} = \boldsymbol{D}^{-1}\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Exercise 2.11.12 Consider the linear problem

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

(i) Determine the rank of $\boldsymbol{T}$.

Multiplication of columns (rows) with non-zero factor, addition of columns (rows), and interchange of columns (rows), do not change the rank of a matrix. We may use those operations and mimic Gaussian elimination to compute the rank of matrices.

$$
\begin{aligned}
&\text{rank}\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} 1 & 3 & -2 & 0 \\ 1 & -4 & 3 & 2 \\ 3 & 2 & -1 & 2 \end{bmatrix} && \text{switch columns 1 and 4} \\
&= \text{rank}\begin{bmatrix} 1 & 3 & -2 & 0 \\ 1 & -4 & 3 & 2 \\ 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} && \text{divide row 3 by 3} \\
&= \text{rank}\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 5 & 2 \\ 0 & -\frac{7}{3} & \frac{5}{3} & \frac{2}{3} \end{bmatrix} && \text{subtract row 1 from rows 2 and 3} \\
&= \text{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 5 & 2 \\ 0 & -\frac{7}{3} & \frac{5}{3} & \frac{2}{3} \end{bmatrix} && \text{manipulate the same way columns to zero out the first row} \\
&= \text{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{7} & -\frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= 2
\end{aligned}
$$

(ii) Determine the null space of $\boldsymbol{T}$.

Set $x_3 = \alpha$ and $x_4 = \beta$ and solve for x_1, x_2 to obtain

$$\mathcal{N}(T) = \{(\frac{7}{2}\alpha - \frac{5}{2}\beta, -3\alpha + 2\beta, \alpha, \beta)^T \ : \ \alpha, \beta \in \mathbb{R}\}$$

(iii) Obtain a particular solution and the general solution.

Check that the rank of the augmented matrix is also equal 2. Set $x_3 = x_4 = 0$ to obtain a particular solution

$$\boldsymbol{x} = (2, 1, 0, 0)^T$$

The general solution is then

$$\boldsymbol{x} = (2 + \frac{7}{2}\alpha - \frac{5}{2}\beta, 1 - 3\alpha + 2\beta, \alpha, \beta)^T, \ \alpha, \beta \in \mathbb{R}$$

(iv) Determine the range space of $\boldsymbol{T}$.

As rank $T = 2$, we know that the range of T is two-dimensional. It is sufficient thus to find two linearly independent vectors that are in the range, e.g. we can take $T\boldsymbol{e}_1, T\boldsymbol{e}_2$ represented by the first two columns of the matrix,

$$\mathcal{R}(T) = \text{span}\{(0, 2, 2)^T, (1, 1, 3)^T\}$$

Exercise 2.11.13 Construct examples of linear systems of equations having (1) no solutions, (2) infinitely many solutions, (3) if possible, unique solutions for the following cases:

(a) 3 equations, 4 unknowns

(1)

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(2)

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(3) Unique solution is not possible.

(b) 3 equations, 3 unknowns

(1)

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(2)

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(3)

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 2.11.14 Determine the rank of the following matrices:

$$\boldsymbol{T} = \begin{bmatrix} 2 & 1 & 4 & 7 \\ 0 & 1 & 2 & 1 \\ 2 & 2 & 6 & 8 \\ 4 & 4 & 14 & 10 \end{bmatrix}, \quad \boldsymbol{T}_2 = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 & 4 \\ 2 & 0 & 3 & 2 & 1 & 5 \\ 1 & 1 & 1 & 2 & 1 & 3 \end{bmatrix}, \quad \boldsymbol{T}_3 = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

In all three cases, the rank is equal 3.

Exercise 2.11.15 Solve, if possible, the following systems:

(a)

$$\begin{aligned} 4x_1 \qquad\quad + 3x_3 - x_4 + 2x_5 &= 2 \\ x_1 - x_2 + x_3 - x_4 + x_5 &= 1 \\ x_1 + x_2 + x_3 - x_4 + x_5 &= 1 \\ x_1 + 2x_2 + x_3 \qquad + x_5 &= 0 \end{aligned}$$

$$\boldsymbol{x} = \begin{bmatrix} t+3 \\ 0 \\ -2t-3 \\ -1 \\ t \end{bmatrix}, \quad t \in \mathbb{R}$$

(b)

$$\begin{aligned} -\ 4x_1 \ - \ 8x_2 \ + \ 5x_3 &= 1 \\ 2x_1 \ - 2x_2 \ + 3x_3 &= 2 \\ 5x_1 \ + \ x_2 \ + 2x_3 &= 4 \end{aligned}$$

$$\boldsymbol{x} = \left[\frac{19}{120}, -\frac{47}{120}, \frac{3}{10}\right]^T$$

(c)

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 3x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + \ x_2 + \ x_3 + \ x_4 &= 0 \end{aligned}$$

$$\boldsymbol{x} = \begin{bmatrix} \alpha+\beta \\ -2\alpha-\beta \\ \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

2.12 Tensor Products, Covariant and Contravariant Tensors

2.13 Elements of Multilinear Algebra

Exercises

Exercise 2.13.1 Let X be a finite-dimensional space of dimension n. Prove that the dimension of the space $M_m^s(X)$ of all m-linear *symmetric* functionals defined on X is given by the formula

$$\dim M_m^s(X) = \frac{n(n+1)\dots(n+m-1)}{1\cdot 2\cdot \dots \cdot m} = \frac{(n+m-1)!}{m!\,(n-1)!} = \binom{n+m-1}{m}$$

Proceed along the following steps.

(a) Let $P_{i,m}$ denote the number of increasing sequences of m natural numbers ending with i,

$$1 \le a_1 \le a_2 \le \dots \le a_m = i$$

Argue that

$$\dim M_m^s(X) = \sum_{i=1}^{n} P_{i,m}$$

Let a be a general m-linear functional defined on X. Let $e_1, \ldots, e_n$ be a basis for X, and let $v^j, j = 1, \ldots, n$, denote components of a vector v with respect to the basis. The multilinearity of a implies the representation formula,

$$a(v_1, \ldots, v_m) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \ldots \sum_{j_m=1}^{n} a(e_{j_1}, e_{j_2}, \ldots, e_{j_m})\, v_1^{j_1} v_2^{j_2} \ldots v_m^{j_m}$$

On the other side, if the form a is symmetric, we can interchange any two arguments in the coefficient $a(e_{j_1}, e_{j_2}, \ldots, e_{j_m})$ without changing the value. The form is thus determined by coefficients $a(e_{j_1}, e_{j_2}, \ldots, e_{j_m})$ where

$$1 \le j_1 \le \ldots \le j_m \le n$$

The number of such increasing sequences equals the dimension of space $M_m^s(X)$. Obviously, we can partition the set of such sequences into subsets that contain sequences ending at particular indices $1, 2, \ldots, n$, from which the identity above follows.

(b) Argue that

$$P_{i,m+1} = \sum_{j=1}^{i} P_{j,m}$$

The first m elements of an increasing sequence of $m+1$ integers ending at i, form an increasing sequence of m integers ending at $j \le i$.

(c) Use the identity above and mathematical induction to prove that

$$P_{i,m} = \frac{i(i+1)\ldots(i+m-2)}{(m-1)!}$$

For $m = 1,\ P_{i,1} = 1$. For $m = 2$,

$$P_{i,2} = \sum_{j=1}^{i} 1 = i$$

For $m = 3$,

$$P_{i,3} = \sum_{j=1}^{i} j = \frac{i(i+1)}{2}$$

Assume the formula is true for a particular m. Then

$$P_{i,m+1} = \sum_{j=1}^{i} \frac{j(j+1)\ldots(j+m-2)}{(m-1)!}$$

We shall use induction in i to prove that

$$\sum_{j=1}^{i} \frac{j(j+1)\ldots(j+m-2)}{(m-1)!} = \frac{i(i+1)\ldots(i+m-1)}{m!}$$

The case $i = 1$ is obvious. Suppose the formula is true for a particular value of i. Then,

$$\begin{aligned}
\sum_{j=1}^{i+1} \frac{j(j+1)\dots(j+m-2)}{(m-1)!} &= \sum_{j=1}^{i} \frac{j(j+1)\dots(j+m-2)}{(m-1)!} + \frac{(i+1)(i+2)\dots(i+m-1)}{(m-1)!} \\
&= \frac{i(i+1)\dots(i+m-1)}{m!} + \frac{m(i+1)(i+2)\dots(i+m-1)}{m!} \\
&= \frac{(i+1)(i+2)\dots(i+m-1)(i+m)}{m!} \\
&= \frac{(i+1)(i+2)\dots(i+m-1)((i+1)+m-1)}{m!}
\end{aligned}$$

(d) Conclude the final formula.

Just use the formula above.

Exercise 2.13.2 Prove that any bilinear functional can be decomposed into a unique way into the sum of a symmetric functional and an antisymmetric functional. In other words,

$$M_2(V) = M_2^s(V) \oplus M_2^a(V)$$

Does this result hold for a general m-linear functional with $m > 2$?

The result follows from the simple decomposition,

$$a(u,v) = \frac{1}{2}(a(u,v) + a(v,u)) \; + \; \frac{1}{2}(a(u,v) - a(v,u))$$

Unfortunately, it does not generalize to $m > 2$. This can for instance be seen from the simple comparison of dimensions of the involved spaces in the finite-dimensional case,

$$n^m > \binom{n+m-1}{m} + \binom{n}{m}$$

for $2 < m \leq n$.

Exercise 2.13.3 Antisymmetric linear functionals are a great tool to check for linear independence of vectors. Let a be an m-linear antisymmetric functional defined on a vector space V. Let $v_1, \dots, v_m$ be m vectors in space V such that $a(v_1, \dots, v_m) \neq 0$. Prove that vectors $v_1, \dots, v_n$ are linearly independent. Is the converse true? In other words, if vectors $v_1, \dots, v_n$ are linearly independent, and a is a nontrivial m-linear antisymmetric form, is $a(v_1, \dots, v_m) \neq 0$?

Assume in contrary that there exists an index i such that

$$v_i = \sum_{j \neq i} \beta_j v_j$$

for some constants $\beta_j, j \neq i$. Substituting into the functional a, we get,

$$a(v_1, \dots, v_i, \dots, v_m) = a(v_1, \dots, \sum_{j \neq i} \beta_j v_j, \dots, v_m) = \sum_{j \neq i} \beta_j a(v_1, \dots, v_j, \dots, v_m) = 0$$

since in each of the terms $a(v_1, \ldots, v_j, \ldots, v_m)$, two arguments are the same.

The converse is not true. Consider for instance a bilinear, antisymmetric form defined on a three-dimensional space. Let e_1, e_2, e_3 be a basis for the space. As discussed in the text, the form is uniquely determined by its values on pairs of basis vectors: $a(e_1, e_2), a(e_1, e_3), a(e_2, e_3)$. It is sufficient for one of these numbers to be non-zero in order to have a nontrivial form. Thus we may have $a(e_1, e_2) = 0$ for the linearly independent vectors e_1, e_2, and a nontrivial form a. The discussed criterion is only a sufficient condition for the linear independence but not necessary.

Exercise 2.13.4 Use the fact that the determinant of matrix $\boldsymbol{A}$ is a multilinear antisymmetric functional of matrix columns and rows to prove the *Laplace Expansion Formula*. Select a particular column of matrix A_{ij}, say the j-th column. Let $\boldsymbol{A}^{ij}$ denote the submatrix of $\boldsymbol{A}$ obtained by removing i-th row and j-th column (do not confuse it with a matrix representation). Prove that

$$\det \boldsymbol{A} = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det \boldsymbol{A}^{ij}$$

Formulate and prove an analogous expansion formula with respect to an i-th row.

It follows from the linearity of the determinant with respect to the j-th column that,

$$\det \begin{bmatrix} \ldots A_{1j} \ldots \\ \vdots \quad \vdots \quad \vdots \\ \ldots A_{nj} \ldots \end{bmatrix} = A_{1j} \det \begin{bmatrix} \ldots 1 \ldots \\ \vdots \; \vdots \; \vdots \\ \ldots 0 \ldots \end{bmatrix} + \ldots + A_{nj} \det \begin{bmatrix} \ldots 0 \ldots \\ \vdots \; \vdots \; \vdots \\ \ldots 1 \ldots \end{bmatrix}$$

On the other side, the determinant of matrix,

$$\begin{bmatrix} & & (j) & \\ & \ldots & 0 & \ldots \\ (i) & \vdots & 1 & \vdots \\ & \ldots & 0 & \ldots \end{bmatrix}$$

is a multilinear functional of the remaining columns (and rows) and, for $\boldsymbol{A}^{ij} = \boldsymbol{I}$ (The $\boldsymbol{I}$ denote here the identity matrix in $\mathbb{R}^{n-1}$), its value reduces to $(-1)^{i+j}$. Hence,

$$\det \begin{bmatrix} & & (j) & \\ & \ldots & 0 & \ldots \\ (i) & \vdots & 1 & \vdots \\ & \ldots & 0 & \ldots \end{bmatrix} = (-1)^{i+j} \det \boldsymbol{A}^{i+j}$$

The reasoning follows identical lines for the expansion with respect to the i-th column.

Exercise 2.13.5 Prove the Kramer's formulas for the solution of a nonsingular system of n equations with n unknowns,

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Hint: In order to develop the formula for the j-th unknown, rewrite the system in the form:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & \dots & x_1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \underset{(j)}{x_n} & \dots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & & & & \vdots \\ a_{n1} & \dots & \underset{(j)}{b_n} & \dots & a_{nn} \end{bmatrix}$$

Compute the determinant of both sides of the identity, and use Cauchy's Theorem for Determinants for the left-hand side.

Exercise 2.13.6 Explain why the rank of a (not necessarily square) matrix is equal to the maximum size of a square submatrix with a non-zero determinant.

Consider an $m \times n$ matrix A_{ij}. The matrix can be considered to be a representation of a linear map A from an n-dimensional space X with a basis $e_i, i = 1, \dots, n$, into an m-dimensional space Y with a basis $g_1, \dots, g_m$. The transpose of the matrix represents the transpose operator A^T mapping dual space Y^* into the dual space X^*, with respect to the dual bases $g_1^*, \dots, g_m^*$ and $e_1^*, \dots, e_n^*$. The rank of the matrix is equal to the dimension of the range space of operator A and operator A^T. Let $e_{j_1}, \dots, e_{j_k}$ be such vectors that $Ae_{j_1}, \dots, Ae_{j_k}$ is the basis for the range of operator A. The corresponding submatrix represents a restriction B of operator A to a subspace $X_0 = \text{span}(e_{j_1}, \dots, e_{j_k})$ and has the same rank as the original whole matrix. Its transpose has the same rank equal to k. By the same argument, there exist k vectors $g_{i_1}, \dots, g_{i_k}$ such that $A^T g_{i_1}^*, \dots, A^T g_{i_k}^*$ are linearly independent. The corresponding $k \times k$ submatrix represents the restriction of the transpose operator to the k-dimensional subspace $Y_0^* = \text{span}(g_{i_1}^*, \dots, g_{i_k}^*)$, with values in the dual space X_0^*, and has the same rank equal k. Thus, the final submatrix represents an isomorphism from a k-dimensional space into a k-dimensional space and, consequently, must have a non-zero determinant.

Conversely, let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ be k column vectors in $\mathbb{R}^m$. Consider a matrix composed of the columns. If there exists a square submatrix of the matrix with a non-zero determinant, the vectors must be linearly independent. Indeed, the determinant of any square submatrix of the matrix represents a k-linear, antisymmetric functional of the column vectors, so, by Exercise 2.13.3, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are linearly independent vectors. The same argument applies to the rows of the matrix.

Euclidean Spaces

2.14 Scalar (Inner) Product. Representation Theorem in Finite-Dimensional Spaces

2.15 Basis and Cobasis. Adjoint of a Transformation. Contra- and Covariant Components of Tensors

Exercises

Exercise 2.15.1 Go back to Exercise 2.11.1 and consider the following product in $\mathbb{R}^2$,

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \to (x, y)_V = x_1 y_1 + 2x_2 y_2$$

Prove that $(x, y)_V$ satisfies the axioms for an inner product. Determine the adjoint of map A from Exercise 2.11.1 with respect to *this inner product*.

The product is bilinear, symmetric and positive definite, since $(x, x)_V = x_1^2 + 2x_2^2 \geq 0$, and $x_1^2 + 2x_2^2 = 0$ implies $x_1 = x_2 = 0$. The easiest way to determine a matrix representation of A^* is to determine the cobasis of the (canonical) basis used to define the map A. Assume that $a^1 = (\alpha, \beta)$. Then

$$\begin{aligned} &(a_1, a^1) = \alpha = 1 \\ &(a_2, a^1) = \alpha + 2\beta = 0 \quad \implies \quad \beta = -\tfrac{1}{2} \end{aligned}$$

so $a^1 = (1, -\frac{1}{2})$. Similarly, if $a^2 = (\alpha, \beta)$ then,

$$\begin{aligned} &(a_1, a^2) = \alpha = 0 \\ &(a_2, a^2) = \alpha + 2\beta = 1 \quad \implies \quad \beta = \tfrac{1}{2} \end{aligned}$$

so $a^2 = (0, \frac{1}{2})$. The matrix representation of A^* in the cobasis is simply the transpose of the original matrix,

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

In order to represent A^* in the original, canonical basis, we need to switch in between the bases.

$$\begin{array}{l} a^1 = e_1 - \frac{1}{2} e_2 \\ a^2 = \frac{1}{2} e_2 \end{array} \quad \implies \quad \begin{array}{l} e_1 = a^1 + a^2 \\ e_2 = 2a^2 \end{array}$$

Then,

$$\begin{aligned} A^*y = A^*(y_1e_1 + y_2e_2) &= A^*(y_1(a^1 + a^2) + y_2 2a^2) = A^*(y_1a^1 + (y_1 + 2y_2)a^2) \\ &= y_1A^*a^1 + (y_1 + 2y_2)A^*a^2 = y_1a^1 + (y_1 + 2y_2)(a^1 + 2a^2) \\ &= y_1(e_1 - \tfrac{1}{2}e_2) + (y_1 + 2y_2)(e_1 + \tfrac{1}{2}e_2) = 2(y_1 + y_2)e_1 - \tfrac{1}{2}y_1e_2 \\ &= (2(y_1 + y_2), y_2) \end{aligned}$$

Now, let us check our calculations. First, let us compute the original map (that has been given to us in basis a_1, a_2), in the canonical basis,

$$\begin{aligned} A(x_1, x_2) = A(x_1e_1 + x_2e_2) &= A(x_1a_1 + x_2(a_2 - a_1)) \\ &= A((x_1 - x_2)a_1 + x_2a_2) = (x_1 - x_2)(a_1 + a_2) + x_2 2a_2 \\ &= (x_1 - x_2)(2e_1 + e_2) + 2x_2(e_1 + e_2) \\ &= (2x_1, x_1 + x_2) \end{aligned}$$

If our calculations are correct then,

$$(Ax, y)_V = 2x_1y_1 + 2(x_1 + x_2)y_2$$

must match

$$(x, A^*y)_V = x_1 2(y_1 + y_2) + 2x_2y_2$$

which it does! Needless to say, you can solve this problem in many other ways.