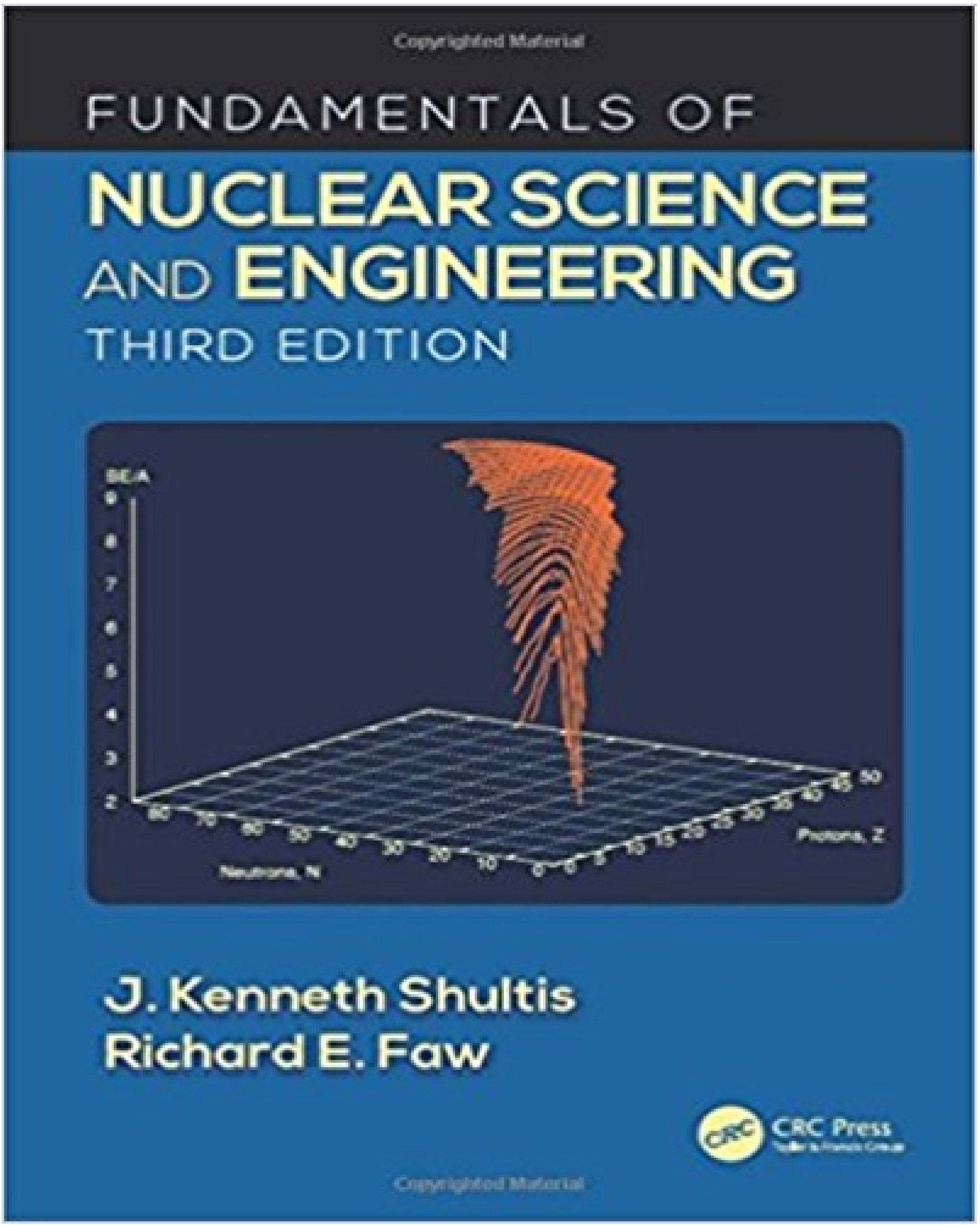
Solutions for Fundamentals of Nuclear Science and Engineering 3rd Edition by Shultis



Solutions

Chapter 2

Modern Physics Concepts

PROBLEMS

1. An accelerator increases the kinetic energy of electrons uniformly to 10 GeV over a 3000 m path. That means that at 30 m, 300 m, and 3000 m, the kinetic energy is 10^8, 10^9, and 10^{10} eV, respectively. At each of these distances, compute the velocity, relative to light (v/c), and the mass in atomic mass units.

Solution:

From Eq. (2.10) in the text $T = mc^2 - m_oc^2$ we obtain

$$m = T/c^2 + m_o. \qquad \text{(P2.1)}$$

From Eq. (2.5) in the text $m = m_o/\sqrt{1 - v^2/c^2}$, which can be solved for v/c to give

$$\frac{v}{c} = \sqrt{1 - \frac{m_o^2}{m^2}} \simeq 1 - \frac{1}{2}\frac{m_o^2}{m^2}, \quad \text{if} \quad \frac{m_o}{m} << 1. \qquad \text{(P2.2)}$$

(a) For an electron ($m_o = m_e$) with $T = 10^8$ eV = 100 MeV, Eq. (P2.1) gives

$$m = \frac{100 \text{ MeV}}{931.5 \text{ MeV/u}} + m_e = 0.1074 \text{ u} + 0.0005486 \text{ u} = \mathbf{0.1079 \text{ u}}.$$

Then $m_e^2/m^2 = (0.0005486/0.1079)^2 = 2.59 \times 10^{-5}$. Finally, from Eq. (P2.2) above, we obtain

$$\frac{v}{c} \simeq 1 - \frac{1}{2}\frac{m_o^2}{m^2} = 1 - 1.29 \times 10^{-5} = \mathbf{0.999987}.$$

(b) For an electron with $T = 10^9$ eV = 1000 MeV, we similarly obtain $m =$ **1.0741** u and $v/c =$ **0.99999987**.

(c) For an electron with $T = 10^{10}$ eV = 10^4 MeV, we similarly obtain $m =$ **10.736** u and $v/c =$ **0.9999999987**.

Alternative solution: Use Eq. (P2.4) developed in Problem 2-3, namely

$$\frac{v}{c} = \left\{1 - \left[\frac{m_ec^2}{T + m_ec^2}\right]^2\right\}^{1/2}.$$

2. Consider a fast moving particle whose relativistic mass m is 100ϵ percent greater than its rest mass m_o, i.e., $m = m_o(1+\epsilon)$. (a) Show that the particle's speed v, relative to that of light, is

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}}.$$

(b) For $v/c << 1$, show that this exact result reduces to $v/c \simeq \sqrt{2\epsilon}$.

Solution:

(a) We are given

$$\frac{m - m_o}{m_o} = \frac{m_o((1+\epsilon)-1)}{m_o} = \epsilon.$$

But we also have

$$\frac{m - m_o}{m_o} = \frac{1}{m_o}\left[\frac{m_o}{\sqrt{1 - v^2/c^2}} - m_o\right].$$

Equating these two results yields

$$\epsilon = \frac{1}{\sqrt{1 - v^2/c^2}} - 1.$$

Solving this result for v/c gives

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}}. \tag{P2.3}$$

(b) For $\epsilon << 1$ we have $(1+\epsilon)^{-2} \simeq 1 - 2\epsilon + \cdots$. Substitution of the approximation into Eq. (P2.3) above gives

$$\frac{v}{c} \simeq \sqrt{1 - (1 - 2\epsilon)} = \sqrt{2\epsilon}.$$

3. In fission reactors one deals with neutrons having kinetic energies as high as 10 MeV. How much error is incurred in computing the speed of 10-MeV neutrons by using the classical expression rather than the relativistic expression for kinetic energy?

Solution:

A neutron with rest mass $m_n = 1.6749288 \times 10^{-27}$ kg has a kinetic energy $T = (10^7 \text{ eV})(1.602177 \times 10^{-19} \text{ J/eV}) = 1.602177 \times 10^{-12}$ J. For the neutron $m_n c^2 = 939.56536$ MeV.

Classically:

$$v_c = \sqrt{2T/m_n} = \left[\frac{2 \times 1.602177 \times 10^{-12}}{1.6749288 \times 10^{-27}}\right]^{1/2} = 4.373993 \times 10^7 \text{ m/s}.$$

Relativistically: From the text we have

$$T = mc^2 - m_oc^2 = \frac{m_oc^2}{\sqrt{1-v^2/c^2}} - m_oc^2.$$

Solving this equation for v yields the relativistic speed v_r

$$v_r = c\left\{1 - \left[\frac{m_oc^2}{T+m_oc^2}\right]^2\right\}^{1/2}. \tag{P2.4}$$

Substitution then gives

$$v_r = c\left\{1 - \left[\frac{939.56536}{10+939.56536}\right]^2\right\}^{1/2} = 0.1447459c = 4.339373 \times 10^7 \text{ m/s}.$$

Thus the percent error in the classical speed is $= 100(v_c - v_r)/v_r = \mathbf{0.798\%}$.

4. What speed (m s^{-1}) and kinetic energy (MeV) would a neutron have if its relativistic mass were 10% greater than its rest mass?

Solution:

We are given $(m - m_o)/m_o \equiv \epsilon = 0.1$. From Problem 2-2

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}} = \sqrt{1 - \frac{1}{1.1^2}} = 0.4167.$$

Thus the neutron's speed is $v = 0.4167c = \mathbf{1.25 \times 10^8}$ **m/s**.

The kinetic energy can be calculated from

$$T = mc^2 - m_oc^2 = m_oc^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right].$$

For $m_oc^2 = 939.6$ MeV and $v/c = 0.4167$ we obtain

$$T = 939.6\left[\frac{1}{\sqrt{1-0.4167^2}} - 1\right] = \mathbf{94.0 \text{ MeV}}.$$

5. Show that for a relativistic particle the kinetic energy is given in terms of the particl's momentum by

$$T = \sqrt{p^2c^2 + m_o^2c^4} - m_cc^2.$$

Solution:

Squaring Eq. (2.17) and rearranging the terms one obtains

$$T^2 + 2Tm_oc^2 - p^2c^2 = 0$$

The solution of this quadratic equation gives

$$T = \frac{1}{2}\left\{-2m_oc^2 \pm \sqrt{4m_o^2c^4 + 4p^2c^2}\right\}$$

Only the + sign gives a physically meaningful result. Rearrangement gives the desired realtion.

6. For a relativistic particle show that Eq. (2.21) is valid.

Solution:

From the definition of η one has

$$\eta^2+1 = \frac{P^2}{(m_oc)^2}+1 = \frac{p^2c^2}{(m_oc^2)^2}+1 = \frac{(mc^2)^2 - (m_oc^2)^2}{(m_oc^2)^2}+1 = (W^2-1)+1 = W^2.$$

7. Prove the relationships given in (a) Eq. (2.19), (b) Eq. (2.20), and (c) Eq. (2.21).

Solution:

(a) From the definition of η and W one immediately has

$$\beta = \frac{v}{c} = \frac{p}{mc} = \frac{\eta}{W}.$$

(b) Because $W^2 = 1 + \eta^2$, then

$$\beta^2 = \left(\frac{v}{c}\right)^2 = \frac{\eta^2}{W^2} = \frac{\eta^2}{1+\eta^2}.$$

(c) Because $\beta = \eta/W$ and $W^2 = 1 + \eta^2$, one has

$$\frac{\beta^2}{1-\beta^2} = \frac{\eta^2/W^2}{1-\eta^2/W^2} = \frac{\eta^2/(1+\eta^2)}{1-\eta^2/(1+\eta^2)} = \frac{\eta^2}{(1+\eta^2)-\eta^2} = \eta^2.$$

From this result we see

$$\frac{\beta^2}{1-\beta^2} = \frac{p^2}{m_o^2c^2} = \frac{c^2p^2}{(m_oc^2)^2},$$

but we know $p^2c^2 = T^2 + 2Tm_oc^2$, so

$$\frac{\beta^2}{1-\beta^2} = \frac{T^2 + 2Tm_oc^2}{(m_oc^2)^2} = \left(\frac{T}{m_oc^2}\right)^2 + \frac{2T}{m_oc^2} = \left(\frac{T}{m_oc^2}\right)^2\left(1 + \frac{2m_oc^2}{T}\right).$$

8. In the Relativistic Heavy Ion Collider, nuclei of gold are accelerated to speeds of 99.95% the speed of light. These nuclei are almost spherical when at rest; however, as they move past the experimenters they appear considerably flattened in the direction of motion because of relativistic effects. Calculate the apparent diameter of such a gold nucleus in its direction of motion relative to that perpendicular to the motion.

Solution: The relativistically contracted diameter D to the uncontracted diameter D_o when $v/c = 0.9995$ is

$$D/D_o = \sqrt{1 - v^2/c^2} = \sqrt{1 - 0.9995^2} = \sqrt{1 - (1 - 0.0005)^2}$$

$$\simeq \sqrt{1 - (1 - 2 \times 0.0005)} = \sqrt{0.001} = 0.031.$$

Hence the gold nucleus appears to flatten to **3.1%** of its at-rest width.

9. Muons are subatomic particles that have the negative charge of an electron but are 206.77 times more massive. They are produced high in the atmosphere by cosmic rays colliding with nuclei of oxygen or nitrogen, and muons are the dominant cosmic-ray contribution to background radiation at the earth's surface. A muon, however, rapidly decays into an energetic electron, existing, from its point of view, for only 2.20 μs, on the average. Cosmic-ray generated muons typically have speeds of about $0.998c$ and thus should travel only a few hundred meters in air before decaying. Yet muons travel through several kilometers of air to reach the earth's surface. Using the results of special relativity explain how this is possible. HINT: consider the atmospheric travel distance as it appears to a muon, and the muon lifetime as it appears to an observer on the earth's surface.

Solution:

Muon's Point of View: A muon, with a lifetime $t_o = 2.20 \times 10^{-6}$ s and traveling with a speed $v = 0.998c$, travels on the average a distance $d = vt_o = 0.998(3.00 \times 10^8 \text{ m/s})(2.29 \times 10^{-6} \text{ s}) = 660$ m.

If the muon is created at an altitude L_o, from the muon's point of view the distance to the surface (approaching with speed $v = 0.998c$) is relativistically narrowed or contracted to a distance

$$L = L_o\sqrt{1 - v^2/c^2} = L_o\sqrt{1 - 0.998^2} = 0.063L_o.$$

For example, if $L_o = 10$ km, $L = 630$ m, so that, on the average, almost half of the muons will reach the surface.

Surface Observer's Point of View: An observer on the earth's surface observes the muon approaching at a speed $v = 0.998c$ and the muon's lifetime appears to expand (the muon's internal clock appears to slow) as

$$t = \frac{t_o}{\sqrt{1 - v^2/c^2}} = \frac{t_o}{\sqrt{1 - 0.998^2}} = 15.9t_o = 3.49 \times 10^{-5} \text{ s}.$$

In such a lifetime, the muon can travel $d = 0.998c \times t = 10,500$ m so that it can reach the surface from an altitude of 10 km before decaying.

10. A 1-MeV gamma ray loses 200 keV in a Compton scatter. Calculate the scattering angle.

Solution:

From Eq. (2.26) in the text we find

$$1 - \cos\theta_s = m_e c^2 \left[\frac{1}{E'} - \frac{1}{E}\right]$$

or

$$\cos\theta_s = 1 - m_e c^2 \left[\frac{1}{E'} - \frac{1}{E}\right].$$

Here $m_e c^2 = 0.511$ MeV, $E' = 0.8$ MeV, and $E = 1$ MeV so that

$$\cos\theta_s = 1 - 0.511 \left[\frac{1}{0.8} - \frac{1}{1}\right] = 0.87225.$$

Thus the scattering angle $\theta_s = \cos^{-1}(0.87225) = \mathbf{29.3^o}$

11. At what energy (in MeV) can a photon lose at most one-half of its energy in Compton scattering?

Solution:

Eq. (2.26) in the text gives the basic Compton scattering relation:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}(1 - \cos\theta_s).$$

By inspection, the maximum energy loss (the smallest E') occurs when $\theta_s = \pi$. Here we are told $E' = E/2$

$$\frac{2}{E} - \frac{1}{E} = \frac{1}{E} = \frac{2}{m_e c^2} = \frac{2}{0.511 \text{ MeV}}.$$

From this result, we find $E = \mathbf{0.255}$ **MeV**. *Above* this incident photon energy, the minimum scattered photon energy is less than one-half of the initial energy.

12. Derive for the Compton scattering process the recoil electron energy T as a function of the incident photon energy E and the electron angle of scattering ϕ_e. Show that ϕ_e is never greater than $\pi/2$ radians.

Solution:

Application of the law of cosines to the triangle in text Fig. 2.5 leads to

$$p_{\lambda'}{}^2 = p_{\lambda}{}^2 + p_e{}^2 - 2p_{\lambda} p_e \cos\phi_e.$$

Substitute E/c for p_{λ}, $(E - T)/c$ for $p_{\lambda'}$, and $(1/c)\sqrt{T^2 + 2Tm_e c^2}$ for p_e. Then solve for T, with the result

$$T = \frac{2m_e c^2 E^2 \cos^2\phi_e}{(E + m_e c^2)^2 - E^2 \cos^2\phi_e}.$$

Examination of the triangle in Fig. 2.5 reveals that, since $p_{\lambda'} \le p_{\lambda}$, $0 \le \phi_e \le \pi/2$, confirming the commonsense observation that the target electron, initially at rest, can recoil only in the forward hemisphere.

13. A 1 MeV photon is Compton scattered at an angle of 55 degrees. Calculate (a) the energy of the scattered photon, (b) the change in wavelength, and (c) the recoil energy of the electron.

Solution:

(a) From Eq. (2.26)

$$\frac{1}{E'} = \frac{1}{E} + \frac{1-\cos\theta_s}{m_e c^2} = \frac{1}{1\text{ MeV}} + \frac{1-\cos 55}{0.511\text{ MeV}} = 1.835\text{ MeV}^{-1}.$$

Thus the scattered photon energy is $E' = 1/1.835 = \mathbf{0.545\ MeV}$.

(b) From Eq. (2.25) we have

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta_s) = \frac{hc}{m_e c^2}(1-\cos\theta_s)$$

$$= \frac{(4.135\times10^{-21}\text{ MeV s})(3.00\times10^{8}\text{ m/s})}{0.511\text{ MeV}}(1-\cos 55)$$

$$= \mathbf{1.04\times10^{-12}\ m}.$$

(c) The kinetic energy of the recoil electron is $E_r = E - E' = 1 - 0.545 = \mathbf{0.455\ MeV}$.

14. When light with wavelengths > 475 nm $= \lambda_{\max}$ impinges on of a certain metalic surface photoelectrons are observed to be emitted. What is the work functiion of this metal in eV?

Solution:

The frequency of light corresponding the the maximum wavelgth is $\nu_{\min} = c/\lambda_{\max} = (2.998\times10^{8}\text{m s}^{-1}/(475\times10^{-9}\text{ m}) = 6.31\times10^{14}\text{ s}^{-1}$. From Example 2.3, the work function is $A = h\nu_{\min} = (4.136\times10^{-15}\text{ eV s})(6.31\times10^{14}\text{ s}^{-1}) = \mathbf{2.61\ eV}$.

15. Consider the experimental arrangement shown in Fig. 2.3. The surface of a sodium sample was illuminated by monochromatic light of various wavelengths, and the retarding potentials required to stop the collection of the photoelectrons were observed. The results are shown below.

wavelemgth (nm)	253.6	283.0	303.9	330.2	366.3	435.8
retarding potential (V)	2.60	2.11	1.81	1.47	1.10	0.57

Present these data graphically to verify the photoelectric equation $eV_o = h\nu - A$. From the graph estimate the value of Planck's constant h and the work function A for sodium.

Solution:

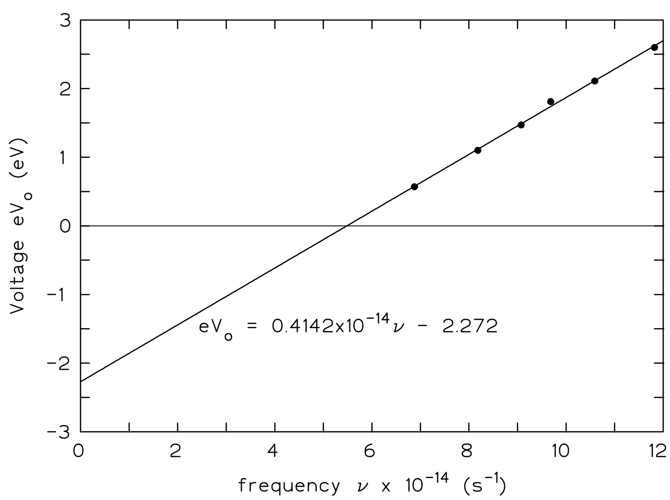
The frequency of the light is related to the wavelength by

$$[\nu = \frac{c}{\lambda} = \frac{2.997 \times 10^{17}}{\lambda \text{ (nm)}} \text{ s}^{-1}.$$

Then plot the following data:

eV_o (eV)	2.60	2.11	1.81	1.47	1.10	0.57
$\nu \times 10^{-14}$	11.82	10.59	9.682	9.076	8.182	6.877

Fit a straight line to the plotted data as shown below.



From the least-squares fit it is found that $h = \mathbf{4.142 \times 10^{-15}}$ **eV s** and that the work function for sodium is $A = $ **2.271 eV**.

16. Consider the electron scattering experiment of Davisson and Germer described in Section 2.2.4. For the nickel crystal they used the interatomic spacing was $d = 2.15$ Å $= 2.15 \times 10^{-10}$ m. (a) For an incident electrons with an arbitrary energy of T eV, show that the constructive interference peaks occur at angles

$$\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right) = \sin^{-1}\left(\frac{5.705n}{\sqrt{T\ \text{eV}}}\right), \qquad n = 1, 2, 3, \ldots.$$

(b) What are the angles of the peaks when $T = 54$ eV (as used by Davisson and Germer) and when $T = 300$ eV?

Solution:

(a) From Eq. (2.30) for non-relativistic electrons $\lambda = h/\sqrt{2m_eT}$. Recall the rest mass of the electron is $m_e/c^2 = 5.11 \times 10^6$ eV. Substitution of of these values gives

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{nhc}{d\sqrt{2m_eT}}\right)\\ &= \sin^{-1}\left(\frac{n(4.136\times10^{-15}\ \text{eV s})(2.998\times10^{8}\ \text{m s}^{-1})}{(2.15\times10^{-10}\ \text{m})\sqrt{(2\times0.555\times10^{6}\ \text{eV})(T\ \text{eV})}}\right)\\ &= \sin^{-1}\left(\frac{5.705n}{\sqrt{T\ \text{eV}}}\right). \qquad \text{(P2.5)}\end{aligned}$$

(b) For $T = 54$ eV the only angle is $\theta = 50.9°$ $(n = 1)$. For $T = 300$ eV the angles are $\theta = 19.2°$ $(n = 1)$, $41.2°$ $(n = 2)$, and$81.2°$ $(n = 3)$.

17. Show that the de Broglie wavelength of a particle with kinetic energy T can be written as

$$\lambda = \frac{h}{\sqrt{m_o}}\frac{1}{\sqrt{T}}\left[1 + \frac{m}{m_o}\right]^{-1/2}$$

where m_o is the particles's rest mass and m is its relativistic mass.

Solution: From Eq. (2.17)

$$p = \frac{1}{c}\sqrt{T^2 + 2Tm_oc^2} = \frac{\sqrt{T}}{c}\sqrt{T + 2m_oc^2}.$$

But $T = mc^2 - m_oc^2$ so the above result can be written as

$$p = \frac{\sqrt{T}}{c}\sqrt{mc^2 + m_oc^2} = \sqrt{T}\sqrt{m_o}\sqrt{1 + (m/m_o)}.$$

Finally, use of the de Broglie relation $\lambda = h/p$ in the above result gives

$$\lambda = \frac{h}{\sqrt{m_o}}\frac{1}{\sqrt{T}}\left[1 + \frac{m}{m_o}\right]^{-1/2}.$$

18. Apply the result of the previous problem to an electron. (a) Show that when the electron's kinetic energy is expressed in units of eV, its de Broglie wavelength can be written as

$$\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \left[1 + \frac{m}{m_o}\right]^{-1/2} \text{ cm.}$$

(b) For non-relativistic electrons, i.e., $m \simeq m_o$, show that this result reduces to

$$\lambda = \frac{12.27 \times 10^{-8}}{\sqrt{T}} \text{ cm.}$$

(c) For very relativistic electrons, i.e., $m >> m_o$, show that the de Broglie wavelength is given by

$$\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \sqrt{\frac{m_o}{m}} \text{ cm.}$$

Solution:

(a) Rewrite the result of Problem 2-10 as

$$\lambda = \frac{hc}{\sqrt{m_o c^2}} \frac{1}{\sqrt{T}} \left[1 + \frac{m}{m_o}\right]^{-1/2}.$$

Substitute for the constants and use $m_o = m_e = 0.511 \text{ MeV}/c^2$ to obtain

$$\lambda = \frac{(4.1357 \times 10^{-15} \text{ eV s})(2.998 \times 10^{10} \text{ cm/s})}{\sqrt{0.5110 \times 10^6 \text{ eV}}} \frac{(1 + m/m_o)^{-1/2}}{\sqrt{T \text{ (eV)}}}$$

$$= \frac{17.35 \times 10^{-8}}{\sqrt{T \text{ (eV)}}} \left[1 + \frac{m}{m_o}\right]^{-1/2} \text{ cm.} \quad \text{(P2.6)}$$

(b) For non-relativistic electrons $m \simeq m_o$, so that $1/\sqrt{1 + (m/m_o)} \simeq 1/\sqrt{2}$, and the above result becomes

$$\lambda = \frac{12.27 \times 10^{-8}}{\sqrt{T \text{ (eV)}}} \text{ cm.}$$

(c) For very relativistic particles, $m >> m_o$ so that $1/\sqrt{1 + (m/m_o)} \simeq \sqrt{m_o/m}$. Eq. (2.4) above then becomes

$$\lambda = \frac{17.35 \times \sqrt{m_o/m}}{\sqrt{T \text{ (eV)}}} \times 10^{-8} \text{ cm.}$$

19. What are the wavelengths of electrons with kinetic energies of (a) 10 eV, (b) 1000 eV, and (c) 10^7 eV?

Solution: From Eq. (2.17) $p = (1/c)\sqrt{T^2 + 2Tm_oc^2}$ and using the de Broglie relation $\lambda = h/p$ we obtain the de Broglie wavelength as

$$\lambda = \frac{hc}{\sqrt{T^2 + 2Tm_oc^2}}. \tag{P2.7}$$

Now apply this equation to the three electron energies.

(a) Substitute $m_oc^2 = m_ec^2 = 0.5110$ MeV and $T = 10$ eV into Eq. (P2.6) to obtain

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^2 + 2(10)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{3.88 \times 10^{-10} \text{ m}}.$$

(b) similarly, for $T = 10^3$ eV we find

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^6 + 2(10^3)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{3.87 \times 10^{-11} \text{ m}}.$$

(c) similarly, for $T = 10^7$ eV we find

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^{14} + 2(10^7)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{1.18 \times 10^{-13} \text{ m}}.$$

20. Low energy neutrons are often referred to by their de Broglie wavelength as measured in angstoms (Å) with 1 Å$= 1 \times 10^{-10}$ m. (a) Derive a formula that gives the kinetic energy of such a neutron in terms of its de Broglie wavelength. (b) What is the energy of a neutron (in eV) of a 6-Å neutron.

Solution:

(a) Equation (2.30) for a non-relativistic particle reduces to

$$\lambda = h/\sqrt{2m_oT},$$

which, upon solving to T gives

$$T = \frac{h^2}{2\lambda^2 m_o}.$$

(b) Here $\lambda = 6 \times 10^{-10}$ m and $m_o/c^2 = 931.49 \times 10^6$ eV, so

$$T = \frac{(4.135 \times 10^{-15} \text{ eV s})^2}{(2)(6 \times 10^{-10}\text{m}})^2(931.49 \times 10^6 \text{ eV})/(2.998 \times 10^8 \text{ m s}^{-1})^2$$
$$= \mathbf{0.00229 \text{ eV}}.$$

21. What is the de Broglie wavelength of a water molecule moving at a speed of 2400 m/s? What is the wavelength of a 3-g bullet moving at 400 m/s?

Solution:

(a) A water molecule (H_2O) has a rest mass of about $m = (18 \text{ u})(1.661 \times 10^{-27} \text{ kg/u}) = 2.989 \times 10^{-26}$ kg.

Its momentum when traveling at 2400 m/s is $p = mv = (2.989 \times 10^{-26} \text{ kg}) \times (2400 \text{ m/s}) = 7.18 \times 10^{-23} \text{ kg m s}^{-1} = 7.18 \times 10^{-23} \text{ J s m}^{-1}$.

Thus the de Broglie wavelength of the water molecule is

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J s}}{7.18 \times 10^{-23} \text{ J s m}^{-1}} = \mathbf{9.23 \times 10^{-12} \text{ m}}.$$

(b) A 3-g bullet moving at 400 m/s has a momentum $p = mv = (0.003 \text{ kg}) \times (400 \text{ m/s}) = 1.2 \text{ kg m s}^{-1} = 1.2 \text{ J s m}^{-1}$. Its de Broglie wavelength is thus

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J s}}{1.2 \text{ J s m}^{-1}} = \mathbf{5.53 \times 10^{-34} \text{ m}}.$$

22. If a neutron is confined somewhere inside a nucleus of characteristic dimension $\Delta x \simeq 10^{-14}$ m, what is the uncertainty in its momentum Δp? For a neutron with momentum equal to Δp, what is its total energy and its kinetic energy in MeV? Verify that classical expressions for momentum and kinetic energy may be used.

Solution:

From the uncertainty principle, $\Delta p \Delta x \gtrsim h/(2\pi)$ so that for $\Delta x \simeq 10^{-14}$ m

$$\Delta p = \frac{h}{2\pi\Delta x} = \frac{6.626 \times 10^{-34} \text{ J s}}{2\pi \times 10^{-14} \text{ m}} = 1.05 \times 10^{-20} \text{ J s m}^{-1}.$$

A non-relativistic (classical) particle has kinetic energy $T = (1/2)mv^2 = p^2/(2m)$. For a neutron with $p \simeq \Delta p = 1.05 \times 10^{-20}$ J s m^{-1}

$$T = \frac{(\Delta p)^2}{2m_n} = \frac{(1.05 \times 10^{-20} \text{ J s m}^{-1})^2}{2(1.6749 \times 10^{-27} \text{ kg})} = 3.32 \times 10^{-14} \text{ J}$$

$$= \frac{3.32 \times 10^{-14} \text{ J}}{1.602 \times 10^{-13} \text{ J/MeV}} = \mathbf{0.208 \text{ MeV}}.$$

This energy is well below the energy at which a neutron becomes relativistic, and hence justifies the use of classical mechanics.

The neutron's total energy is thus $E = T + m_n c^2 = 0.207 \text{ MeV} + 939 \text{ MeV} \simeq m_n c^2$.

23. Repeat the previous problem for an electron trapped in the nucleus. HINT: relativistic expressions for momentum and kinetic energy must be used.

Solution:

From the uncertainty principle, $\Delta p \Delta x \gtrsim h/(2\pi)$ so that for $\Delta x \simeq 10^{-14}$ m

$$\Delta p = \frac{h}{2\pi\Delta x} = \frac{6.626 \times 10^{-34} \text{ J s}}{2\pi \times 10^{-14} \text{ m}} = 1.05 \times 10^{-20} \text{ J s m}^{-1}.$$

For an electron with $p \simeq \Delta p = 1.05 \times 10^{-20}$ J s m^{-1}

$$\begin{aligned} p^2c^2 &= (1.05 \times 10^{-20} \text{ J s m}^{-1})^2 (3.00 \times 10^8 \text{ m/s})^2 \\ &= (3.15 \times 10^{-12} \text{ J})^2 = (19.7 \text{ MeV})^2. \end{aligned}$$

From the equation above Eq. (2.16) in the text, we see that $p^2c^2 = (mc^2)^2 - (m_oc^2)^2 = E^2 - (m_oc^2)^2$. We use this relation to find the electron's total energy E as

$$E = \sqrt{p^2c^2 + (m_ec^2)^2} = \sqrt{19.7^2 + 0.511^2} \text{ MeV} \simeq 20 \text{ MeV}.$$

Since the electron's total energy E is related to the kinetic energy T by $E = T + m_ec^2 = T + 0.511$ MeV, in this problem the total energy is essentially the electron's kinetic energy, i.e., $E \simeq T$.

24. The wavefunction for the electron in a hydrogen atom in its ground state (the 1s state for which $n = 0,\ \ell = 0,$ and $m = 0$ is spherically symmetric as shown in Fig. 2.14. For this state the wavefuntion is real and is given by

$$\psi_0(r) = \frac{1}{\sqrt{\pi a_0^3}} \exp[-r/a_0],$$

where $a_o = h^2\epsilon_o/(4\pi^2 m_e e^2) \simeq 5.29 \times 10^{-11}$ m. This quantity is the radius of the first Bohr orbit for hydrogen (see next chapter). Because of the spherical symmetry of ψ_o, dV in Eq. (2.40) is $dV = 4\pi r^2\, dr$ and the integral in Eq. (2.40) can be written as

$$\int_0^\infty \psi_0(r)\psi_0^*(r) 4\pi dr = \frac{4}{a_0^3}\int_0^\infty r^2 e^{-\alpha r} dr,$$

where $\alpha \equiv 2/a_0$. (a) Verify that the required normalization required by Eq. (2.40) is satisfied, i.e., the electron is somewhere in the space around the proton. (b) What is the probability the electron is found a radial distance $r < a_0$ from the proton?

Solution:

(a) Integration by parts twice gives

$$\frac{4}{a_0^3}\int_0^\infty r^2 e^{-\alpha r} dr = \frac{4}{a_0^3}\frac{2}{\alpha^3} = \frac{4}{a_0^3}\frac{a_0^3}{4} = 1.$$

(b) Replace upper limit in the above itegral by a_0. Then integration by parts twice gives

$$
\begin{aligned}
\text{Prob}\{\text{electron is inside } r \le a_0\} &= \frac{4}{a_0^3}\int_0^{a_0} r^2 e^{-\alpha r}\,dr \\
&= 1 - \frac{4}{a_0^3} e^{-\alpha a_0}\left\{\frac{a_0^2}{\alpha} + \frac{2a_0}{\alpha^2} + \frac{2}{\alpha^3}\right\} \\
&= 1 - \frac{4}{a_0^3} e^{-2}\left\{\frac{a_0^3}{2} + \frac{2a_0^3}{4} + \frac{2a_0^3}{8}\right\} \\
&= 1 - 5e^{-2} = \mathbf{0.323}.
\end{aligned}
$$

Thus the electron has a 32.3% of being at a radial distance less that a_0.