Solutions for Real Analysis and Foundations 4th Edition by Krantz

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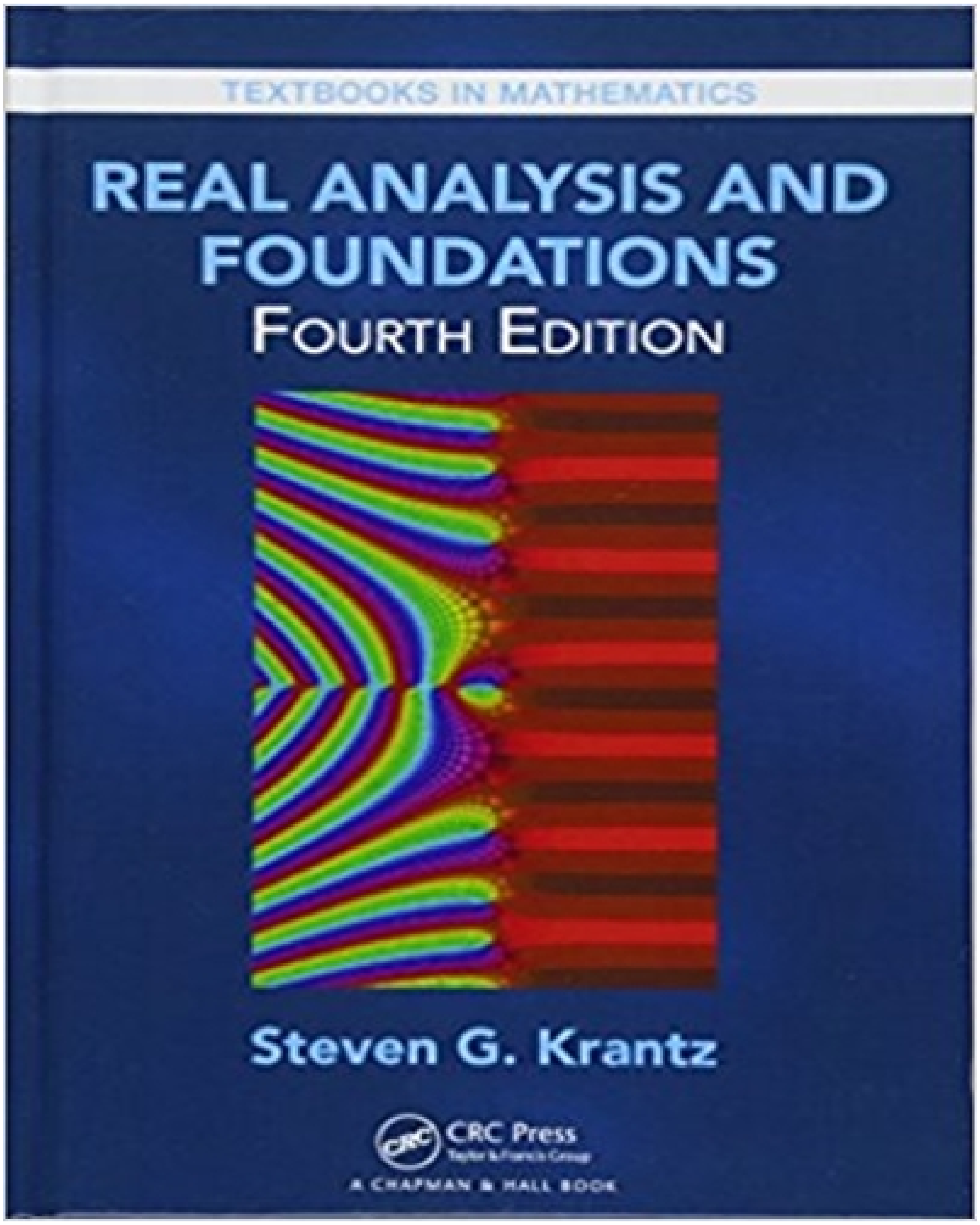

Solutions

Chapter 2

Sequences

2.1 Convergence of Sequences

1. The answer is no. We can even construct a sequence with arbitrarily long repetitive strings that has subsequences converging to *any* real number α. Indeed, order $\mathbb{Q}$ into a sequence $\{q_n\}$. Consider the following sequence

$$\{q_1, q_2, q_2, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3, q_1, q_1, q_1, q_1, q_1, q_1, \cdots\}.$$

In this way we have repeated each rational number infinitely many times, and with arbitrarily long strings. From the above sequence we can find subsequences that converge to any real number.

2. If, to the contrary, the β_j remain bounded by some number $M > 0$, then any such sequence α_j/β_j will have a subsequence α_{j_k}/β_{j_k} with all denominators having the same value β^*. But then the only way that α_{j_k}/β_{j_k} can converge is if the α_{j_k} eventually assume the same constant value. Thus α_{j_k}/β_{j_k} converges to a rational number. That is a contradiction.

4. Let $a_j = \alpha_j/\beta_j$ be a sequence of rational numbers with the property that all the β_j are powers of 2, and none of those powers is greater than 2^{10}. Then some subsequence α_{j_k}/β_{j_k} will have all the β_{j_k} equal. But then the only way that the sequence can converge is if the α_{j_k} eventually assume some constant value. Thus the subsequence must converge to a rational number with denominator a power of 2 not exceeding 2^{10}. Therefore the entire sequence must have the same property.

5. We know that

$$\int_0^1 \frac{dt}{1+t^2} = \text{Tan}^{-1}(t)\Big|_0^1 = \frac{\pi}{4}\,.$$

As we know from calculus (and shall learn in greater detail in Chapter 7 of the present text), the integral on the left can be approximated by its Riemann sums. So we obtain

$$\sum_{j=0}^{k} f(s_j)\Delta x_j \approx \frac{\pi}{4}\,.$$

Here $f(t) = 1/(1+t^2)$. Since the sum on the left can be written out explicitly, this gives a means of calculating π to any desired degree of accuracy.

7. Let $\epsilon > 0$. Choose an integer J so large that $j > J$ implies that $|a_j - \alpha| < \epsilon$. Also choose an integer K so large that $j > K$ implies that $|c_j - \alpha| < \epsilon$. Let $M = \max\{J, K\}$. Then, for $j > M$, we see that

$$\alpha - \epsilon < a_j \le b_j \le c_j < \alpha + \epsilon\,.$$

In other words,

$$|b_j - \alpha| < \epsilon\,.$$

But this says that $\lim_{j\to\infty} b_j = \alpha$.

9. The sequence

$$a_j = \pi + \frac{1}{j}\,, \quad j = 1, 2, \ldots$$

is decreasing and certainly converges to π.

10. Let $S\mathbb{R}$ be bounded above and let $t = \sup S$. Let $\epsilon > 0$. Then $t - \epsilon$ is *not* an upper bound for S. So there is an element $s \in S$ so that

$$t - \epsilon < s \le t\,.$$

11. If the assertion were not true then the sequence $\{a_j\}$ does not converge. So, for any $\epsilon > 0$ there exist arbitarily large j so that $|a_j - \alpha| > \epsilon$. Thus we may choose $j_1 < j_2 < \cdots$ so that $|a_{j_k} - \alpha| > \epsilon$. This says that the subsequence $\{a_{j_k}\}$ does not converge to α. Nor does it have a subsequence that converges to α. That is a contradiction.

* **12.** We know that, for any $N > M > 0$, it holds that

$$|a_M - a_{M+1}| + |a_{M+1} - a_{M+2}| + \cdots |a_{N-1} - a_N| \le 1 \, .$$

This tells us that the series

$$\sum_{j=1}^{\infty} |a_j - a_{j+1}|$$

converges. [Refer to Chapter 3 for more on series.] But then, given any $\epsilon > 0$, there are $N > M > 0$ so large that

$$\sum_{j=M}^{N} |a_j - a_{j+1}| < \epsilon \, .$$

From this we may conclude from the triangle inequality that

$$|a_M - a_{N+1}| \le \sum_{j=M}^{N} |a_j - a_{j+1}| < \epsilon \, .$$

It follows that the sequence is Cauchy, so it converges.

2.2 Subsequences

1. Let $a_1 \ge a_2 \ge \cdots$ be a decreasing sequence that is bounded below by some number M. Of course the sequence is bounded above by a_1. So the sequence is bounded. By the Bolzano-Weierstrass theorem, there is a subsequence $\{a_{j_k}\}$ that converges to a limit α.

Let $\epsilon > 0$. Choose $K > 0$ so that, when $k \ge K$, $|a_{j_k} - \alpha| < \epsilon$. Then, when $j > j_K$,

$$\alpha - \epsilon < a_j \le a_{j_K} < \alpha + \epsilon \, .$$

Thus

$$|a_j - \alpha| < \alpha \, .$$

So the sequence converges to α.

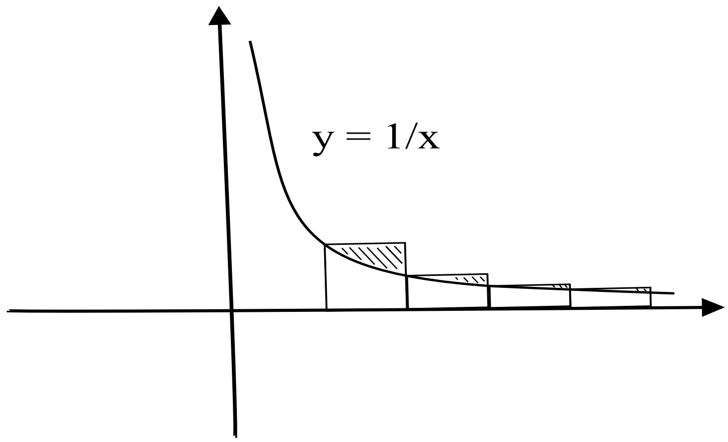

Figure 2.1: Sum of shaded regions is $1 + 1/2 + \cdots 1/j - \log j$.

2. Let $\{q_j\}$ be an enumeration of the rationals. Consider the sequence

$$q_1, q_1, q_2, q_1, q_2, q_3, q_1, q_2, q_3, q_4, \ldots .$$

Call these sequence elements $\{r_j\}$. We see that each q_j is repeated infinitely many times in the sequence.

If α is any real number, then select a sequence p_j of rational numbers so that $p_j \to \alpha$. Then p_1 occurs in the sequence $\{r_j\}$. Call the first occurrence r_{j_1}. And p_2 occurs in the sequence $\{r_j\}$ *after* r_{j_1}. Call that occurrence r_{j_2}. Continuing in this fashion, we can realize the numbers $p_1, p_2, p_3, \ldots$ as a subsequence of $\{r_j\}$. And $p_j \to \alpha$. So that does the job.

A similar argument applies when $\alpha = \pm\infty$.

3. Suppose that $\{a_\}$ has a subsequence diverging to $+\infty$. If in fact $\{a_j\}$ converges to some finite real number α, then every subsequence converges to α. But that is a contradiction.

5. Consider Figure 2.1.

The sum of the areas of the four shaded regions is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \log j \,,$$

where of course we use the natural logarithm. All four of these shaded regions may be slid to the left so that they lie in the first, large box. And they do not overlap. This assertion is true not just for the first four summands but for any number of summands. So we see that the value of

$$\lim_{j\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j}\right)-\log j$$

is not greater that $1\times 1=1$. In particular, the sequence is increasing and bounded above. So it converges.

6. Similar to the solution of Exercise 1 above.

7. Similar to the solution of Exercise 13 in Section 1.1 above.

8. Similar to the solution of Exercise 13 in Section 1.1 above.

9. Define the sequence a_j by

$$0,0,1,0,1,1/2,0,1,1/2,1/3,0,1,1/2,1/3,1/4,\ldots\,.$$

Then, given an element $1/j$ in S, we may simply choose the subsequence

$$1/j,1/j,1/j,\ldots$$

from the sequence a_j to converge to $1/j$. And it is clear that the subsequences of a_j have no other limits.

10. Let $\{a_j\}$ be a bounded sequence. Define $b_j=\inf\{a_j,a_{j+1},\ldots\}$. Certainly each b_j is finite and $b_1\le b_2\le\cdots$. Also $\{b_j\}$ is bounded above because $\{a_j\}$ is. So, by Proposition 2.16, the sequence $\{b_j\}$ converges to some number α. But the sequence $\{b_j\}$ also converges to the liminf of the a_j. So some subsequence of the a_j will converge to α.

* **12.** We see that

$$\frac{\sin^2 m}{m}=\frac{1-\cos 2m}{2}\,.$$

Now

$$\sum_m \frac{1}{2}$$

diverges because this is the harmonic series. And

$$\sum_m \frac{\cos 2m}{m}$$

converges by an argument similar to that for Exercise 11. It follows then that

$$\sum_m \frac{\sin^2 m}{m}$$

diverges.

2.3 Lim sup and Lim inf

1. Consider the sequence

$$1, -1, 1, -1, 5, -5, 1, -1, 1, -1, \ldots .$$

Then, considered as a sequence, the limsup is plainly 1. But the supremum of the set of numbers listed above is 5. Also the liminf is -1. But the infimum of the set of numbers listed above is -5.

What is true is that

$$\limsup a_j \leq \sup\{a_j\}$$

and

$$\liminf a_j \geq \inf\{a_j\} .$$

We shall not prove these two inequalities here.

2. Let $\alpha = \limsup a_j$. Then there is a subsequence $\{a_{j_k}\}$ that converges to α. Assume that $\alpha \neq 0$. Then the subsequence $\{1/a_{j_k}\}$ converges to $1/\alpha$. If there were a subsequence $\{1/a_{j_\ell}\}$ that converges to some number $\beta < 1/\alpha$ then $\{a_{j_\ell}\}$ would converge to $1/\beta > \alpha$, and that is impossible. Hence $1/\alpha$ is the liminf of the $1/a_j$.

A similar argument applies to the situation where $\gamma = \liminf a_j$ and then we consider $1/a_j$.

3. Let $\alpha = \limsup a_j$. Then there is a subsequence $\{a_{j_k}\}$ that converges to α. But then $\{-a_{j_k}\}$ converges to $-\alpha$. If there is some other subsequence $\{-a_{j_\ell}\}$ that converges to some number $\beta < -\alpha$ then $\{a_{j_\ell}\}$

would converge to $-\beta > \alpha$. And that is impossible. Hence $-\alpha$ is the liminf of $\{-a_j\}$.

A similar argument applies to $\gamma = \liminf a_j$ and the consideration of $\{-a_j\}$.

4. Let $\alpha = \limsup a_j = \liminf a_j$. But the limsup is the greatest subsequential limit of the a_j and the liminf is the least subsequential limit of the a_j. Since they are equal, we must conclude that all subsequential limits are the same, and they all equal α. So the full sequence converges to α.

Conversely, if the full sequence converges to some number α then every subsequence converges to α. But then the subsequence that converges to the limsup converges to α and the subsequence that converges to the liminf converges to α. We must conclude that both the limsup and the liminf are equal to α.

5. Consider the sequence

$$a, b, a, b, a, b, a, b, \ldots .$$

Then clearly the limsup of this sequence is equal to b and the liminf of this sequence is equal to a.

6. The complex numbers do not form an ordered field, so we cannot speak of the least or greatest subsequential limit.

8. Consider the sequence $\{a_j\}$ given by

$$\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots .$$

Then $\liminf a_j = -1/2$ and $\limsup a_j = 1/2$. So

$$\limsup a_j - \liminf a_j = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 .$$

9. The limsup is defined to be the limit of the sequence $b_j = \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$. Clearly $b_j \geq a_j$. Therefore

$$\lim_{j\to\infty} b_j = \lim_{k\to\infty} b_{j_k} \geq \lim_{k\to\infty} a_{j_k} .$$

So

$$\lim_{k\to\infty} a_{j_k} \le \limsup a_j \,.$$

A similar argument shows that

$$\lim_{k\to\infty} a_{j_k} \ge \liminf a_j \,.$$

11. Let $\{a_{j_\ell}\}$ be any subsequence of the given sequence. Define $b_{j_\ell} = \sup\{a_{j_\ell}, a_{j_{\ell+1}}, \ldots\}$. Then

$$b_{j_\ell} \ge a_{j_\ell}$$

so

$$\limsup_{\ell\to\infty} b_{j_\ell} \ge \limsup \ell \to \infty a_{j_\ell}$$

so that

$$\limsup a_{j_\ell} \ge \limsup a_{j_\ell} \,.$$

A similar argument applies to the liminf.

* **13.** The numbers $\{\sin j\}$ are dense in the interval $[-1,1]$ (see Exercise 7 of Section 2.2). Thus, given $\epsilon > 0$, there is an integer j so that $|\sin j - 1| < \epsilon$. But then

$$|\sin j|^{\sin j} > (1-\epsilon)^{1-\epsilon} \,.$$

It follows that

$$\limsup |\sin j|^{\sin j} = 1 \,.$$

A similar argument shows that

$$\liminf |\sin j|^{\sin j} = (1/e)^{1/e} \,.$$

2.4 Some Special Sequences

1. Let $r = p/q = m/n$ be two representations of the rational number r. Recall that for any real α, the number α^r is defined as the real number β for which

$$\alpha^m = \beta^n.$$

Let β' satisfy

$$\alpha^p = \beta'^q.$$

We want to show that $\beta = \beta'$. we have

$$\begin{aligned} \beta^{n\cdot q} &= \alpha^{m\cdot q} \\ &= \alpha^{p\cdot n} \\ &= \beta'^{q\cdot n}. \end{aligned}$$

By the uniqueness of the $(n \cdot q)^{th}$ root of a real number it follows that

$$\beta = \beta',$$

proving the desired equality. The second equality follows in the same way. Let

$$\alpha = \gamma^n.$$

Then

$$\alpha^m = \gamma^{n\cdot m}.$$

Therefore, if we take the n^{th} root on both sides of the above inequality, we obtain

$$\gamma^m = (\alpha^m)^{1/n}.$$

Recall that γ is the n^{th} root of α. Then we find that

$$(\alpha^{1/n})^m = (\alpha^m)^{1/n}.$$

Using similar arguments, one can show that for all real numbers α and β and $q \in \mathbb{Q}$

$$(\alpha \cdot \beta)^q = \alpha^q \cdot \beta^q.$$

Finally, let α, β, and γ be positive real numbers. Then

$$\begin{aligned} (\alpha \cdot \beta)^\gamma &= \sup\{(\alpha \cdot \beta)^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q \beta^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q : q \in \mathbb{Q}, q \leq \gamma\} \cdot \sup\{\beta^q : q \in \mathbb{Q},\ q \leq \gamma\} \\ &= \alpha^\gamma \cdot \beta^\gamma. \end{aligned}$$

2. By Example 2.44, $j^{1/j}$ converges to 1. It follows that $(1/j)^{1/j}$ converges to 1.

3. We write

$$\frac{j^j}{(2j)!} = \frac{1}{1 \cdot 2 \cdots (j-1) \cdot (j)} \cdot \frac{j \cdot j \cdot \cdots \cdot j}{(j+1) \cdot (j+2) \cdot \cdots \cdot 2j} \,.$$

Now the second fraction is clearly bounded by 1, while the first fraction is bounded by $1/((j-1)j)$. Altogether then,

$$0 \le \frac{j^j}{(2j)!} \le \frac{1}{j^2 - j} \,.$$

The righthand side clearly tends to 0. So

$$\lim_{j \to \infty} \frac{j^j}{(2j)!} = 0 \,.$$

* **6.** The first two terms a_0 and a_1 are each defined to be 1. Then, for $j \ge 0$, we define

$$a_{j+2} = a_j + a_{j+1} \,.$$

We shall show that the following formula for the Fibonacci sequence is valid:

$$a_j = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j}{\sqrt{5}} .$$

We shall use the method of *generating functions*.

We write $F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Here the a_j's are the terms of the Fibonacci sequence and the letter x denotes an unspecified variable. What is curious here is that we do not care about what x is. We intend to manipulate the function F in such a fashion that we will be able to solve for the coefficients a_j. Just think of $F(x)$ as a polynomial with a *lot* of coefficients.

Notice that

$$xF(x) = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots$$

and

$$x^2 F(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots \,.$$

Thus, grouping like powers of x, we see that

$$\begin{aligned} &F(x) - xF(x) - x^2F(x) \\ &\quad = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 \\ &\qquad +(a_3 - a_2 - a_1)x^3 + (a_4 - a_3 - a_2)x^4 + \cdots . \end{aligned}$$

But the basic property that defines the Fibonacci sequence is that $a_2 - a_1 - a_0 = 0$, $a_3 - a_2 - a_1 = 0$, etc. Thus our equation simplifies drastically to

$$F(x) - xF(x) - x^2F(x) = a_0 + (a_1 - a_0)x.$$

We also know that $a_0 = a_1 = 1$. Thus the equation becomes

$$(1 - x - x^2)F(x) = 1$$

or

$$F(x) = \frac{1}{1 - x - x^2}. \qquad (***)$$

It is convenient to factor the denominator as follows:

$$F(x) = \frac{1}{\left[1 - \frac{-2}{1-\sqrt{5}}x\right] \cdot \left[1 - \frac{-2}{1+\sqrt{5}}x\right]}$$

(just simplify the right hand side to see that it equals $(***)$).

A little more algebraic manipulation yields that

$$F(x) = \frac{5+\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1-\sqrt{5}}x}\right] + \frac{5-\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1+\sqrt{5}}x}\right].$$

Now we want to apply the formula $(**)$ to each of the fractions in brackets ([ ]). For the first fraction, we think of $-\frac{2}{1-\sqrt{5}}x$ as λ. Thus the first expression in brackets equals

$$\sum_{j=0}^{\infty}\left(-\frac{2}{1-\sqrt{5}}x\right)^j.$$

Likewise the second sum equals

$$\sum_{j=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

All told, we find that

$$F(x) = \frac{5+\sqrt{5}}{10} \sum_{j=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^j + \frac{5-\sqrt{5}}{10} \sum_{j=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

Grouping terms with like powers of x, we finally conclude that

$$F(x) = \sum_{j=0}^{\infty} \left[\frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}}x\right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}}x\right)^j \right] x^j .$$

But we began our solution of this problem with the formula

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The two different formulas for $F(x)$ must agree. In particular, the coefficients of the different powers of x must match up. We conclude that

$$a_j = \frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}}\right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}}\right)^j .$$

We rewrite

$$\frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} \qquad \frac{5-\sqrt{5}}{10} = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$$

and

$$-\frac{2}{1-\sqrt{5}} = \frac{1+\sqrt{5}}{2} \qquad -\frac{2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2} .$$

Making these four substitutions into our formula for a_j, and doing a few algebraic simplifications, yields

$$a_j = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j}{\sqrt{5}}$$

as desired.

7. Use a generating function as in the solution of Exercise 6 above.

8. Use a generating function as in the solution of Exercise 6 above.

* **9.** Notice that

$$\left(1+\frac{1}{j^2}\right) \le \exp(1/j^2)$$

(just examine the power series expansion for the exponential function). Thus

$$\begin{aligned} a_j &= \left(1+\frac{1}{1^2}\right)\cdot\left(1+\frac{1}{2^2}\right)\cdot\left(1+\frac{1}{1^3}\right)\cdot\cdots\cdot\left(1+\frac{1}{j^2}\right) \\ &\le \exp(1/1^2)\cdot\exp(1/2^2)\cdot\exp(1/3^2)\cdot\cdots\cdot\exp(1/j^2) \\ &= \exp(1/1^2+1/2^2+1/3^2+\cdots+1/j^2)\,. \end{aligned}$$

Of course the series in the exponent on the right converges. So we may conclude that the infinite product converges.

10. Imitate Example 2.47. Use the binomial expansion.