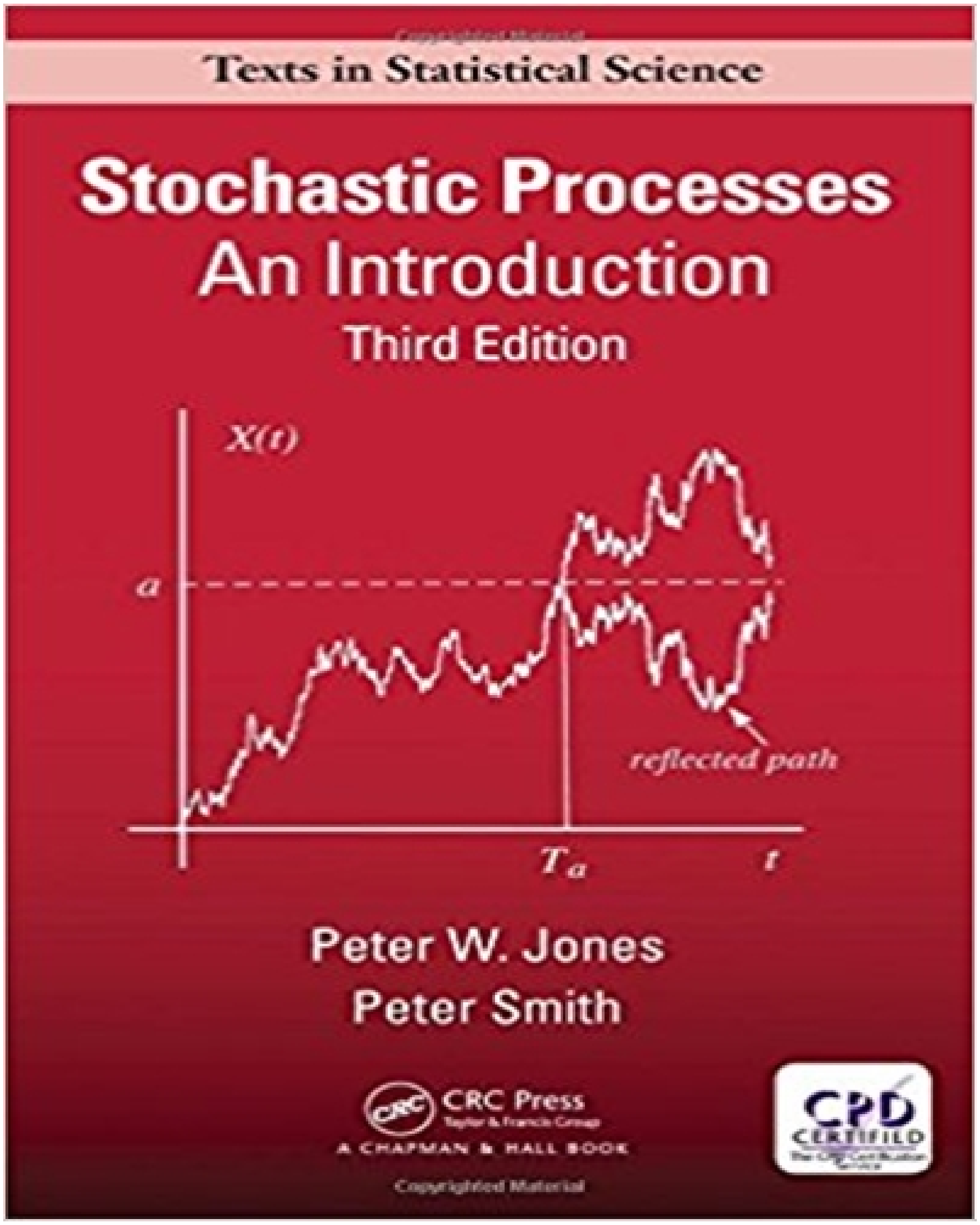
Solutions for Stochastic Processes An Introduction 3rd Edition by Jones



Solutions

Chapter 2

Some gambling problems

2.1. *In the standard gambler's ruin problem with total stake a and gambler's stake k and the gambler's probability of winning at each play is p, calculate the probability of ruin in the following cases;*
(a) $a = 100$, $k = 5$, $p = 0.6$;
(b) $a = 80$, $k = 70$, $p = 0.45$;
(c) $a = 50$, $k = 40$, $p = 0.5$.
Also find the expected duration in each case.

For $p \neq \frac{1}{2}$, the probability of ruin u_k and the expected duration of the game d_k are given by

$$u_k = \frac{s^k - s^a}{1 - s^a}, \qquad d_k = \frac{1}{1-2p}\left[k - \frac{a(1-s^k)}{(1-s^a)}\right].$$

(a) $u_k \approx 0.132$, $d_k \approx 409$.
(b) $u_k \approx 0.866$, $d_k \approx 592$.
(c) For $p = \frac{1}{2}$,

$$u_k = \frac{a-k}{a}, \qquad d_k = k(a-k).$$

so that $u_k = 0.2$, $d_k = 400$.

2.2. *In a casino game based on the standard gambler's ruin, the gambler and the dealer each start with 20 tokens and one token is bet on at each play. The game continues until one player has no further tokens. It is decreed that the probability that any gambler is ruined is 0.52 to protect the casino's profit. What should the probability that the gambler wins at each play be?*

The probability of ruin is

$$u = \frac{s^k - s^a}{1 - s^a},$$

where $k = 20$, $a = 40$, p is the probability that the gambler wins at each play, and $s = (1-p)/p$. Let $r = s^{20}$. Then $u = r/(1+r)$, so that $r = u/(1-u)$ and

$$s = \left(\frac{u}{1-u}\right)^{1/20}.$$

Finally, with $u = 0.52$,

$$p = \frac{1}{1+s} = \frac{(1-u)^{1/20}}{(1-u)^{1/20} + u^{1/20}} \approx 0.498999.$$

2.3. *Find general solutions of the following difference equations:*
(a) $u_{k+1} - 4u_k + 3u_{k-1} = 0$;
(b) $7u_{k+2} - 8u_{k+1} + u_k = 0$;
(c) $u_{k+1} - 3u_k + u_{k-1} + u_{k-2} = 0$.

(d) $pu_{k+2} - u_k + (1-p)u_{k-1} = 0, \quad (0 < p < 1).$

(a) The characteristic equation is

$$m^2 - 4m + 3 = 0$$

which has the solutions $m_1 = 1$ and $m_2 = 3$. The general solution is

$$u_k = Am_1^k + Bm_2^k = A + 3^k B,$$

where A and B are any constants.
(b) The characteristic equation is

$$7m^2 - 8m + 1 = 0,$$

which has the solutions $m_1 = 1$ and $m_2 = \frac{1}{7}$. The general solution is

$$u_k = A + \frac{1}{7^k} B.$$

(c) The characteristic equation is the cubic equation

$$m^3 - 3m^2 + m + 1 = (m-1)(m^2 - 2m - 1) = 0,$$

which has the solutions $m_1 = 1$, $m_2 = 1 + \sqrt{2}$, and $m_3 = 1 - \sqrt{2}$. The general solution is

$$u_k = A + B(1+\sqrt{2})^k + C(1-\sqrt{2})^k.$$

(d) The characteristic equation is the cubic equation

$$pm^3 - m + (1-p) = (m-1)(pm^2 + pm - (1-p)) = 0,$$

which has the solutions $m_1 = 1$, $m_2 = -\frac{1}{2} + \frac{1}{2}\sqrt{[(4-3p)/p]}$ and $m_3 = -\frac{1}{2} - \frac{1}{2}\sqrt{[(4-3p)/p]}$. The general solution is

$$u_k = A + Bm_2^k + Cm_3^k.$$

2.4 *Solve the following difference equations subject to the given boundary conditions:*
(a) $u_{k+1} - 6u_k + 5u_{k-1} = 0, \quad u_0 = 1, u_4 = 0;$
(b) $u_{k+1} - 2u_k + u_{k-1} = 0, \quad u_0 = 1, u_{20} = 0;$
(c) $d_{k+1} - 2d_k + d_{k-1} = -2, \quad d_0 = 0, d_{10} = 0;$
(d) $u_{k+2} - 3u_k + 2u_{k-1} = 0, \quad u_0 = 1, u_{10} = 0, 3u_9 = 2u_8.$

(a) The charactedristic equation is

$$m^2 - 6m + 5 = 0,$$

which has the solutions $m_1 = 1$ and $m_2 = 5$. Therefore the general solution is given by

$$u_k = A + 5^k B.$$

The boundary conditions $u_0 = 1$, $u_4 = 0$ imply

$$A + B = 1, \qquad A + 5^4 B = 0,$$

which have the solutions $A = 625/624$ and $B = -1/624$. The required solution is

$$u_k = \frac{625}{624} - \frac{5^k}{624}.$$

(b) The characteristic equation is

$$m^2 - 2m + 1 = (m-1)^2 = 0,$$

which has the repeated solution $m = 1$. Using the rule for repeated roots,

$$u_k = A + Bk.$$

The boundary conditions $u_0 = 1$ and $u_{20} = 0$ imply $A = 1$ and $B = -1/20$. The required solution is $u_k = (20-k)/20$.
(c) This is an inhomogeneous equation. The characteristic equation is

$$m^2 - 2m + 1 = (m-1)^2 = 0,$$

which has the repeated solution $m = 1$. Hence the complementary function is $A + Bk$. For a particular solution, we must try $d_k = Ck^2$. Then

$$d_{k+1} - 2d_k + d_{k-1} = C(k+1)^2 - 2Ck^2 + C(k-1)^2 = 2C = -2$$

if $C = -1$. Hence the general solution is

$$d_k = A + Bk - k^2.$$

The boundary conditions $d_0 = d_{10} = 0$ imply $A = 0$ and $B = 10$. Therefore the required solution is $d_k = k(10-k)$.
(d) The characteristic equation is

$$m^3 - 3m + 2 = (m-1)^2(m+2) = 0,$$

which has two solutions $m_1 = 1$ (repeated) and $m_2 = -2$. The general solution is given by

$$u_k = A + Bk + C(-2)^k.$$

The boundary conditions imply

$$A + C = 1, \quad A + 10B + C(-2)^{10} = 0, \quad 3A + 27B + 3C(-2)^9 = 2[A + 8B + C(-2)^8].$$

The solutions of these linear equations are

$$A = \frac{31744}{31743}, \quad B = \frac{3072}{31743}, \quad C = -\frac{1}{31743}$$

so that the required solution is

$$u_k = \frac{1024(31-2k) - (-2)^k}{31743}.$$

2.5. *Show that a difference equation of the form*

$$au_{k+2} + bu_{k+1} - u_k + cu_{k-1} = 0,$$

where $a, b, c \geq 0$ are probabilities with $a + b + c = 1$, can never have a characteristic equation with complex roots.

The characteristic equation can be expressed in the form

$$am^3 + bm^2 - m + c = (m-1)[am^2 + (a+b)m - (1-a-b)] = 0,$$

since $a + b + c = 1$. One solution is $m_1 = 1$, and the others satisfy the quadratic equation

$$am^2 + (a+b)m - (1-a-b) = 0.$$

The discriminant is given by

$$(a+b)^2 + 4(1-a-b) = (a-b)^2 + 4a(1-a) \geq 0,$$

since $0 \leq a \leq 1$.

2.6. *In the standard gambler's ruin problem with equal probabilities $p = q = \frac{1}{2}$, find the expected duration of the game given the usual initial stakes of k units for the gambler and $a - k$ units for the opponent.*

The expected duration d_k satisfies

$$d_{k+1} - 2d_k + d_{k-1} = -2.$$

The complementary function is $A + Bk$, and for a particular solution try $d_k = Ck^2$. Then

$$d_{k+1} - 2d_k + d_{k-1} + 2 = C(k+1)^2 - 2Ck^2 + C(k-1)^2 + 2 = 2C + 2 = 0$$

if $C = -1$. Hence

$$d_k = A + Bk - k^2.$$

The boundary conditions $d_0 = d_a = 0$ imply $A = 0$ and $B = a$. The required solution is therefore

$$d_k = k(a - k).$$

2.7. *In a gambler's ruin problem the possibility of a draw is included. Let the probability that the gambler wins, loses or draws against an opponent be respectively,* $p, p, 1-2p$, $(0 < p < \frac{1}{2})$. *Find the probability that the gambler loses the game given the usual initial stakes of* k *units for the gambler and* $a - k$ *units for the opponent. Show that* d_k*, the expected duration of the game, satisfies*

$$pd_{k+1} - 2pd_k + pd_{k-1} = -1.$$

Solve the difference equation and find the expected duration of the game.

The difference equation for the probability of ruin u_k is

$$u_k = pu_{k+1} + (1-2p)u_k + pu_{k-1} \quad \text{or} \quad u_{k+1} - 2u_k + u_{k-1} = 0.$$

The general solution is $u_k = A + Bk$. The boundary conditions $u_0 = 1$ and $u_a = 0$ imply $A = 1$ and $B = -1/a$, so that the required probability is given by $u_k = (a-k)/a$.

The expected duration d_k satisfies

$$d_{k+1} - 2d_k + d_{k+1} = -1/p.$$

The complementary function is $A + Bk$. For the particular solution try $d_k = Ck^2$. Then

$$C(k+1)^2 - 2Ck^2 + C(k-1)^2 = 2C = -1/p.$$

Hence $C = -1/(2p)$. The boundary conditions $d_0 = d_a = 0$ imply $A = 0$ and $B = a/(2p)$, so that the required solution is

$$d_k = k(a - 2p)/(2p).$$

2.8. *In the changing stakes game in which a game is replayed with each player having twice as many units,* $2k$ *and* $2(a-k)$ *respectively, suppose that the probability of a win for the gambler at each play is* $\frac{1}{2}$. *Whilst the probability of ruin is unaffected by how much is the expected duration of the game extended compared with the original game?*

With initial stakes of k and $a - k$, the expected duration is $d_k = k(a-k)$. If the initial stakes are doubled to $2k$ and $2a - 2k$, then the expected duration becomes, using the same formula,

$$d_{2k} = 2k(2a - 2k) = 4k(a-k) = 4d_k.$$

2.9. *A roulette wheel has 37 radial slots of which 18 are red, 18 are black and 1 is green. The gambler bets one unit on either red or black. If the ball falls into a slot of the same colour, then the gambler wins one unit, and if the ball falls into the other colour (red or black), then the casino wins. If the ball lands in the green slot, then the bet remains for the next spin of the wheel or more if necessary until the ball lands on a red or black. The original bet is either returned or lost depending on whether the outcome matches the* *original bet or not (this is the Monte Carlo system). Show that the probability u_k of ruin for a gambler who starts with k chips with the casino holdiing $a-k$ chips satisfies the difference equation*

$$36u_{k+1} - 73u_k + 37u_{k-1} = 0.$$

Solve the difference equation for u_k. If the house starts with €1,000,000 at the roulette wheel and the gambler starts with €10,000, what is the probability that the gambler breaks the bank if €5,000 are bet at each play.

In the US system the rules are less generous to the players. If the ball lands on green then the player simply loses. What is the probability now that the player wins given the same initial stakes? (see Luenberger (1979))

There is the possibility of a draw (see Example 2.1). At each play the probability that the gambler wins is $p = \frac{18}{37}$. The stake is returned with probability

$$\frac{1}{37}\left(\frac{18}{37}\right) + \frac{1}{37^2}\left(\frac{18}{37}\right) + \cdots = \frac{1}{36}\frac{18}{37} = \frac{1}{74},$$

or the gambler loses after one or more greens also with probability 1/74 by the same argument. Hence u_k, the probability that the gambler loses satisfies

$$u_k = \frac{18}{37}u_{k+1} + \frac{1}{74}(u_k + u_{k-1}) + \frac{18}{37}u_{k+1},$$

or

$$36u_{k+1} - 73u_k + 37u_{k-1} = 0.$$

The charactersitic equation is

$$36m^2 - 73m + 37 = (m-1)(36m-37) = 0,$$

which has the solutions $m_1 = 1$ and $m_2 = 37/36$. With $u_0 = 1$ and $u_a = 0$, the required solution is

$$u_k = \frac{s^k - s^a}{1 - s^a}, \qquad s = \frac{37}{36}.$$

The bets are equivalent to $k = 10000/5000 = 2$, $a = 1010000/5000 = 202$. The probability that the gambler *wins* is

$$1 - u_k = \frac{1-s^k}{1-s^a} = \frac{1-s^2}{1-s^{202}} = 2.23 \times 10^{-4}.$$

In the US system, u_k satisfies

$$u_k = \frac{18}{37}u_{k+1} + \frac{19}{37}u_{k-1}, \qquad \text{or} \qquad 18u_{k+1} - 37u_k + 19u_{k-1} = 0.$$

in this case the ratio is $s' = 19/18$. Hence the probability the the gambler wins is

$$1 - u_k = \frac{1-s'^2}{1-s'^{202}} = 2.06 \times 10^{-6},$$

which is less than the previous value.

2.10. *In a single trial the possible scores 1 and 2 can occur each with probability $\frac{1}{2}$. If p_n is the probability of scoring exactly n points at some stage, that is score after several trials is the sum of individual scores in each trial. Show that*

$$p_n = \tfrac{1}{2}p_{n-1} + \tfrac{1}{2}p_{n-2},$$

Calculate p_1 and p_2, and find a formula for p_m. How does p_m behave as m becomes large? How do you interpret the result?

Let A_n be the event that the score is n at some stage. Let B_1 be the event score 1, and B_2 score 2. Then

$$\mathbf{P}(A_n) = \mathbf{P}(A_n|B_1)\mathbf{P}(B_1) + \mathbf{P}(A_n|B_2)\mathbf{P}(B_2) = \mathbf{P}(A_{n-1})\tfrac{1}{2} + \mathbf{P}(A_{n-2})\tfrac{1}{2},$$

or

$$p_n = \tfrac{1}{2}p_{n-1} + \tfrac{1}{2}p_{n-2}.$$

Hence

$$2p_n - p_{n-1} - p_{n-2} = 0.$$

The characteristic equation is

$$2m^2 - m - 1 = (m-1)(2m+1) = 0,$$

which has the solutions $m_1 = 1$ and $m_2 = -\frac{1}{2}$. Hence

$$p_n = A + B(-\tfrac{1}{2})^n.$$

The initial conditions are $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{3}{4}$. Hence

$$A - \tfrac{1}{2}B = \tfrac{1}{2}, \qquad A + \tfrac{1}{4}B = \tfrac{3}{4},$$

so that $A = \frac{2}{3}$, $B = \frac{1}{3}$. Hence

$$p_n = \tfrac{2}{3} + \tfrac{1}{3}(-\tfrac{1}{2})^n, \qquad (n = 1, 2, \ldots).$$

As $n \to \infty$, $p_n \to \frac{2}{3}$.

2.11. *In a single trial the possible scores 1 and 2 can occur with probabilities q and $1-q$, where $0 < q < 1$. Find the probability of scoring exactly n points at some stage in an indefinite succession of trials. Show that*

$$p_n \to \frac{1}{2-q},$$

as $n \to \infty$.

Let p_n be the probability. Then

$$p_n = qp_{n-1} + (1-q)p_{n-2}, \qquad \text{or} \qquad p_n - qp_{n-1} - (1-q)p_{n-2} = 0.$$

The characteristic equation is

$$m^2 - qm - (1-q) = (m-1)[m + (1-q)] = 0,$$

which has the solutions $m_1 = 1$ and $m_2 = -(1-q)$. Hence

$$p_n = A + B(q-1)^n.$$

The initial conditions are $p_1 = q$, $p_2 = 1 - q + q^2$, which imply

$$q = A + B(q-1), \qquad 1 - q + q^2 = A + B(q-1)^2.$$

The solution of these equations leads to $A = 1/(2-q)$ and $B = (q-1)/(q-2)$, so that

$$p_n = \frac{1}{2-q}[1 - (q-1)^{n+1}].$$

2.12. *The probability of success in a single trial is $\frac{1}{3}$. If u_n is the probability that there are no two consecutive successes in n trials, show that u_n satisfies*

$$u_{n+1} = \tfrac{2}{3}u_n + \tfrac{2}{9}u_{n-1}.$$

What are the values of u_1 and u_2? Hence show that

$$u_n = \frac{1}{6}\left[(3+2\sqrt{3})\left(\frac{1+\sqrt{3}}{3}\right)^n + (3-2\sqrt{3})\left(\frac{1-\sqrt{3}}{3}\right)^n\right].$$

Let A_n be the event that there have *not* been two consecutive successes in the first n trials. Let B_1 be the event of success and B_2 the event of failure in a single trial. Then

$$\mathbf{P}(A_n) = \mathbf{P}(A_n|B_1)\mathbf{P}(B_1) + \mathbf{P}(A_n|B_2)\mathbf{P}(B_2).$$

Now $\mathbf{P}(A_n|B_2) = \mathbf{P}(A_{n-1})$: failure will not change the probability. Also

$$\mathbf{P}(A_n|B_1) = \mathbf{P}(A_{n-1}|B_2)\mathbf{P}(B_2) = \mathbf{P}(A_{n-2})\mathbf{P}(B_2).$$

Since $\mathbf{P}(B_1) = \frac{1}{3}$, $\mathbf{P}(B_2) = \frac{2}{3}$,

$$u_n = \tfrac{2}{9}u_{n-2} + \tfrac{2}{3}u_{n-1} \quad \text{or} \quad 9u_n - 6u_{n-1} - 2u_n = 0,$$

where $u_n = \mathbf{P}(A_n)$.

The characteristic equation is

$$9m^2 - 6m - 2 = 0,$$

which has the solutions $m_1 = \frac{1}{3}(1+\sqrt{3})$ and $m_2 = \frac{1}{3}(1-\sqrt{3})$. Hence

$$u_n = A\frac{1}{3^n}(1+\sqrt{3})^n + B\frac{1}{3^n}(1-\sqrt{3})^n.$$

The initial conditions are $u_1 = 1$ and $u_2 = 1 - \frac{1}{3}\frac{1}{3} = \frac{8}{9}$. Therefore A and B are defined by

$$1 = \frac{A}{3}(1+\sqrt{3}) + \frac{B}{3}(1-\sqrt{3}),$$

$$\frac{8}{9} = \frac{A}{9}(1+\sqrt{3})^2 + \frac{B}{9}(1-\sqrt{3})^2 = \frac{A}{9}(4+2\sqrt{3}) + \frac{B}{9}(4-2\sqrt{3}).$$

The solutions are $A = \frac{1}{6}(2\sqrt{3}+3)$ and $B = \frac{1}{6}(-2\sqrt{3}+3)$. Finally

$$u_n = \frac{1}{6\cdot 3^n}[(2\sqrt{3}+3)(1+\sqrt{3})^n + (-2\sqrt{3}+3)(1-\sqrt{3})^n].$$

2.13. *A gambler with initial capital k units plays against an opponent with initial capital $a-k$ units. At each play of the game the gambler either wins one unit or loses one unit with probability $\frac{1}{2}$. Whenever the opponent loses the game, the gambler returns one unit so that the game may continue. Show that the expected duration of the game is $k(2a-1-k)$ plays.*

The expected duration d_k satisfies

$$d_{k+1} - 2d_k + d_{k-1} = -2, \quad (k = 1, 2, \ldots, a-1).$$

The boundary conditions are $d_0 = 0$ and $d_a = d_{a-1}$, indicating the return of one unit when the gambler loses. The general solution for the duration is

$$d_k = A + Bk - k^2.$$

The boundary conditions imply

$$A = 0, \qquad A + Ba - a^2 = A + B(a-1) - (a-1)^2,$$

so that $B = 2a-1$. Hence $d_k = k(2a-1-k)$.

2.14. *In the usual gambler's ruin problem, the probability that the gambler is eventually ruined is*

$$u_k = \frac{s^k - s^a}{1-s^a}, \qquad s = \frac{q}{p}, \qquad (p \neq \tfrac{1}{2}).$$

In a new game the stakes are halved, whilst the players start with the same initial sums. How does this affect the probability of losing by the gambler? Should the gambler agree to this change of rule if $p < \frac{1}{2}$? By how many plays is the expected duration of the game extended?

The new probability of ruin v_k (with the stakes halved) is, adapting the formula for u_k,

$$v_k = u_{2k} = \frac{s^{2k} - s^{2a}}{1 - s^{2a}} = \frac{(s^k + s^a)(s^k - s^a)}{(1 - s^a)(1 + s^a)} = u_k \left(\frac{s^k + s^a}{1 + s^a} \right).$$

Given $p < \frac{1}{2}$, then $s = (1 - p)/p > 1$ and $s^k > 1$. It follows that

$$v_k > u_k \left(\frac{1 + s^a}{1 + s^a} \right) = u_k.$$

With this change the gambler is more likely to lose.

From (2.9), the expected duration of the standard game is given by

$$d_k = \frac{1}{1 - 2p} \left[k - \frac{a(1 - s^k)}{(1 - s^a)} \right].$$

With the stakes halved the expected duration h_k is

$$h_k = d_{2k} = \frac{1}{1 - 2p} \left[2k - \frac{2a(1 - s^{2k})}{(1 - s^{2a})} \right].$$

The expected duration is extended by

$$\begin{aligned} h_k - d_k &= \frac{1}{1 - 2p} \left[k - \frac{2a(1 - s^{2k})}{(1 - s^{2a})} + \frac{a(1 - s^k)}{(1 - s^a)} \right] \\ &= \frac{1}{1 - 2p} \left[k + \frac{a(1 - s^k)(s^a - 1 - 2s^k)}{(1 - s^{2a})} \right]. \end{aligned}$$

2.15. *In a gambler's ruin game, suppose that the gambler can win £2 with probability $\frac{1}{3}$ or lose £1 with probability $\frac{2}{3}$. Show that*

$$u_k = \frac{(3k - 1 - 3a)(-2)^a + (-2)^k}{1 - (3a + 1)(-2)^a}.$$

Compute u_k if $a = 9$ for $k = 1, 2, \ldots, 8$.

The probability of ruin u_k satisfies

$$u_k = \tfrac{1}{3} u_{k+2} + \tfrac{2}{3} u_{k-1} \quad \text{or} \quad u_{k+2} - 3u_k + 2u_k = 0.$$

The characteristic equation is

$$m^3 - 3m + 2 = (m - 1)^2 (m + 2) = 0,$$

which has the solutions $m_1 = 1$ (repeated) and $m_2 = -2$. Hence

$$u_k = A + Bk + C(-2)^k.$$

The boundary conditions are $u_0 = 1$, $u_a = 0$, $u_{a-1} = \frac{2}{3} u_{a-2}$. The constants A, B and C satisfy

$$A + C = 1, \qquad A + Ba + C(-2)^a = 0,$$

$$3[A + B(a + 1) + C(-2)^{a-1}] = 2[A + B(a - 2) + C(-2)^{a-2}],$$

or

$$A + B(a + 1) - 8C(-2)^{a-2} = 0.$$

The solution of these equations is

$$A = \frac{-(-2)^a (3a + 1)}{1 - (-2)^a (3a + 1)}, \quad B = \frac{3(-2)^a}{1 - (-2)^a (3a + 1)}, \quad C = \frac{1}{1 - (-2)^a (3a + 1)}.$$

Finally

$$u_k = \frac{(3k - 1 - 3a)(-2)^a + (-2)^k}{1 - (-2)^a (3a + 1)}.$$

The values of the probabilities u_k for $a = 9$ are shown in the table below.

k	1	2	3	4	5	6	7	8
u_k	0.893	0.786	0.678	0.575	0.462	0.362	0.241	0.161

2.16. *Find the general solution of the difference equation*

$$u_{k+2} - 3u_k + 2u_{k-1} = 0.$$

A reservoir with total capacity of a volume units of water has, during each day, either a net inflow of two units with probability $\frac{1}{3}$ *or a net outflow of one unit with probability* $\frac{2}{3}$. *If the reservoir is full or nearly full any excess inflow is lost in an overflow. Derive a difference equation for this model for* u_k, *the probability that the reservoir will eventually become empty given that it initially contains k units. Explain why the upper boundary conditions can be written* $u_a = u_{a-1}$ *and* $u_a = u_{a-2}$. *Show that the reservoir is certain to be empty at some time in the future.*

The characteristic equation is

$$m^3 - 3m + 2 = (m-1)^2(m+2) = 0.$$

The general solution is (see Problem 2.15)

$$u_k = A + Bk + C(-1)^k.$$

The boundary conditions for the reservoir are

$$u_0 = 1, \quad u_a = \tfrac{1}{3}u_a + \tfrac{2}{3}u_{a-1}, \quad u_{a-1} = \tfrac{1}{3}u_a + \tfrac{2}{3}u_{a-2}.$$

The latter two conditions are equivalent to $u_a = u_{a-1} = u_{a-2}$. Hence

$$A + C = 1, \quad A + Ba + C(-2)^a = A + B(a-1) + C(-2)^{a-1} = A + B(a-2) + C(-2)^{a-2}.$$

which have the solutions $A = 1$, $B = C = 0$. The solution is $u_k = 1$, which means that that the reservoir is certain to empty at some future date.

2.17. *Consider the standard gambler's ruin problem in which the total stake is a and gambler's stake is k, and the gambler's probability of winning at each play is p and losing is* $q = 1 - p$. *Find* u_k, *the probability of the gambler losing the game, by the following alternative method. List the difference equation (2.2) as*

$$\begin{aligned} u_2 - u_1 &= s(u_1 - u_0) = s(u_1 - 1) \\ u_3 - u_2 &= s(u_2 - u_1) = s^2(u_1 - 1) \\ &\vdots \\ u_k - u_{k-1} &= s(u_{k-1} - u_{k-2}) = s^{k-1}(u_1 - 1), \end{aligned}$$

where $s = q/p \neq \frac{1}{2}$ *and* $k = 2, 3, \ldots a$. *The boundary condition* $u_0 = 1$ *has been used in the first equation. By adding the equations show that*

$$u_k = u_1 + (u_1 - 1)\frac{s - s^k}{1 - s}.$$

Determine u_1 *from the other boundary condition* $u_a = 0$, *and hence find* u_k. *Adapt the same method for the special case* $p = q = \frac{1}{2}$.

Addition of the equations gives

$$u_k - u_1 = (u_1 - 1)(s + s^2 + \cdots + s^{k-1}) = (u_1 - 1)\frac{s - s^k}{1 - s}$$

summing the geometric series. The condition $u_a = 0$ implies

$$-u_1 = (u_1 - 1)\frac{s - s^a}{1 - s}.$$

Hence

$$u_1 = \frac{s - s^a}{1 - s^a},$$

so that

$$u_k = \frac{s^k - s^a}{1 - s}.$$

2.18. *A car park has 100 parking spaces. Cars arrive and leave randomly. Arrivals or departures of cars are equally likely, and it is assumed that simultaneous events have negligible probability. The 'state' of the car park changes whenever a car arrives or departs. Given that at some instant there are k cars in the car park, let u_k be the probability that the car park first becomes full before it becomes empty. What are the boundary conditions for u_0 and u_{100}? How many car movements can be expected before this occurs?*

The probability u_k satisfies the difference equation

$$u_k = \frac{1}{2}u_{k+1} + \frac{1}{2}u_{k-1} \quad \text{or} \quad u_{k+1} - 2u_k + u_{k-1}.$$

The general solution is $u_k = A + Bk$. The boundary conditions are $u_0 = 0$ and $u_{100} = 1$. Hence $A = 0$ and $B = 1/100$, and $u_k = k/100$.

The expected duration of car movements until the car park becomes full is $d_k = k(100 - k)$.

2.19. *In a standard gambler's ruin problem with the usual parameters, the probability that the gambler loses is given by*

$$u_k = \frac{s^k - s^a}{1 - s^a}, \qquad s = \frac{1-p}{p}.$$

If p is close to $\frac{1}{2}$, given say by $p = \frac{1}{2} + \varepsilon$ where $|\varepsilon|$ is small, show, by using binomial expansions, that

$$u_k = \frac{a-k}{a}\left[1 - 2k\varepsilon - \frac{4}{3}(a - 2k)\varepsilon^2 + O(\varepsilon^3)\right]$$

as $\varepsilon \to 0$. (The order O terminology is defined as follows: we say that a function $g(\varepsilon) = O(\varepsilon^b)$ as $\varepsilon \to 0$ if $g(\varepsilon)/\varepsilon^b$ is bounded in a neighbourhood which contains $\varepsilon = 0$. See also the Appendix in the book.)

Let $p = \frac{1}{2} + \varepsilon$. Then $s = (1 - 2\varepsilon)/(1 + 2\varepsilon)$, and

$$u_k = \frac{(1-2\varepsilon)^k(1+2\varepsilon)^{-k} - (1-2\varepsilon)^a(1+2\varepsilon)^{-a}}{1 - (1-2\varepsilon)^a(1+2\varepsilon)^{-a}}.$$

Apply the binomial theorem to each term. The result is

$$u_k = \frac{a-k}{a}\left[1 - 2k\varepsilon - \frac{4}{3}(a - 2k)\varepsilon^2 + O(\varepsilon^3)\right].$$

[Symbolic computation of the series is a useful check.]

2.20. *A gambler plays a game against a casino according to the following rules. The gambler and casino each start with 10 chips. From a deck of 53 playing cards which includes a joker, cards are randomly and successively drawn with replacement. If the card is red or the joker the casino wins 1 chip from the gambler, and if the card is black the gambler wins 1 chip from the casino. The game continues until either player has no chips. What is the probability that the gambler wins? What will be the expected duration of the game?*

From (2.4) the probability u_k that the gambler loses is

$$u_k = \frac{s^k - s^a}{1 - s^a},$$

with $k = 10$, $a = 20$, $p = 26/53$, and $s = 27/26$. Hence

$$u_{10} = \frac{(27/26)^{10} - (27/26)^{20}}{1 - (27/26)^{20}} \approx 0.593.$$

Therefore the probability that the gambler wins is approximately 0.407.

From (2.10)

$$d_k = \frac{1}{1-2p}\left[k - \frac{a(1-s^k}{1-s^a}\right] = 98.84,$$

for the given data.

2.21. *In the standard gambler's ruin problem with total stake a and gambler's stake k, the probability that the gambler loses is*

$$u_k = \frac{s^k - s^a}{1-s^a},$$

where $s = (1-p)/p$. Suppose that $u_k = \frac{1}{2}$, that is fair odds. Express k as a function of a. Show that,

$$k = \frac{\ln[\frac{1}{2}(1+s^a)]}{\ln s}.$$

Given

$$u_k = \frac{s^k - s^a}{1-s^a} \quad \text{and} \quad u_k = \tfrac{1}{2},$$

then $1 - s^a = 2(s^k - s^a)$ or $s^k = \frac{1}{2}(1+s^a)$. Hence

$$k = \frac{\ln[\frac{1}{2}(1+s^a)]}{\ln s},$$

but generally k will not be an integer.

2.22. *In a gambler's ruin game the probability that the gambler wins at each play is α_k and loses is $1-\alpha_k$, $(0 < \alpha_k < 1,\ 0 \le k \le a-1)$, that is, the probability varies with the current stake. The probability u_k that the gambler eventually loses satisfies*

$$u_k = \alpha_k u_{k+1} + (1-\alpha_k)u_{k-1}, \quad u_o = 1, \quad u_a = 0.$$

Suppose that u_k is a specified function such that $0 < u_k < 1$, $(1 \le k \le a-1)$, $u_0 = 1$ and $u_a = 0$. Express α_k in terms of u_{k-1}, u_k and u_{k+1}.

Find α_k in the following cases:
(a) $u_k = (a-k)/a$;
(b) $u_k = (a^2-k^2)/a^2$;
(c) $u_k = \frac{1}{2}[1+\cos(k\pi/a)]$.

From the difference equation

$$\alpha_k = \frac{u_k - u_{k-1}}{u_{k+1} - u_{k-1}}.$$

(a) $u_k = (a-k)/a$. Then

$$\alpha_k = \frac{(a-k)-(a-k+1)}{(a-k-1)-(a-k+1)} = \frac{1}{2},$$

which is to be anticipated from eqn (2.5).

(b) $u_k = (a^2-k^2)/a^2$. Then

$$\alpha_k = \frac{(a^2-k^2)-[a^2-(k-1)^2]}{[a^2-(k+1)^2]-[a^2-(k-1)^2]} = \frac{2k-1}{4k}.$$

(c) $u_k = 1/(a+k)$. Then

$$\alpha_k = \frac{[1/(a+k)]-[1/(a+k-1)]}{[1/(a+k+1)]-[1/(a+k-1)]} = \frac{a+k+1}{2(a+k)}.$$

2.23. *In a gambler's ruin game the probability that the gambler wins at each play is α_k and loses is $1-\alpha_k$, $(0 < \alpha_k < 1,\ 1 \le k \le a-1)$, that is, the probability varies with the current stake. The probability u_k that the gambler eventually loses satisfies*

$$u_k = \alpha_k u_{k+1} + (1-\alpha_k)u_{k-1}, \quad u_o = 1, \quad u_a = 0.$$

Reformulate the difference equation as

$$u_{k+1} - u_k = \beta_k(u_k - u_{k-1}),$$

where $\beta_k = (1-\alpha_k)/\alpha_k$. Hence show that

$$u_k = u_1 + \gamma_{k-1}(u_1 - 1), \quad (k = 2, 3, \ldots, a)$$

where

$$\gamma_k = \beta_1 + \beta_1\beta_2 + \cdots + \beta_1\beta_2\ldots\beta_k.$$

Using the boundary condition at $k = a$, confirm that

$$u_k = \frac{\gamma_{a-1} - \gamma_{k-1}}{1 + \gamma_{a-1}}.$$

Check that this formula gives the usual answer if $\alpha_k = p \neq \frac{1}{2}$, a constant.

The difference equation can be expressed in the equivalent form

$$u_{k+1} - u_k = \beta_k(u_k - u_{k-1}),$$

where $\beta_k = (1-\alpha_k)/\alpha_k$. Now list the equations as follows, noting that $u_0 = 0$,:

$$\begin{aligned} u_2 - u_1 &= \beta_1(u_1 - 1) \\ u_3 - u_2 &= \beta_1\beta_2(u_1 - 1) \\ \cdots &= \cdots \\ u_k - u_{k-1} &= \beta_1\beta_2\cdots\beta_{k-1}(u_1 - 1) \end{aligned}$$

Adding these equations, we obtain

$$u_k - u_1 = \gamma_{k-1}(u_1 - 1),$$

where

$$\gamma_{k-1} = \beta_1 + \beta_1\beta_2 + \cdots + \beta_1\beta_2\cdots\beta_{k-1}.$$

The condition $u_a = 0$ implies

$$-u_1 = \gamma_{a-1}(u_1 - 1),$$

so that

$$u_1 + \frac{\gamma_{a-1}}{1 + \gamma_{a-1}}.$$

Finally

$$u_k = \frac{\gamma_{a-1} - \gamma_{k-1}}{1 + \gamma_{a-1}}.$$

If $\alpha_k = p \neq \frac{1}{2}$, then $\beta_k = (1-p)/p = s$, say, and

$$\gamma_k = s + s^2 + \cdots + s^k = \frac{s - s^{k+1}}{1 - s}.$$

Hence

$$u_k = \frac{(s - s^a)/(1-s) - (s - s^k)/(1-s)}{1 + (s - s^a)/(1-s)} = \frac{s^k - s^a}{1 - s^a}$$

as required.

2.24. *Suppose that a fair n-sided die is rolled n independent times. A match is said to occur if side i is observed on the ith trial, where $i = 1, 2, \ldots, n$.*

(a) Show that the probability of at least one match is

$$1 - \left(1 - \frac{1}{n}\right)^n.$$

(b) What is the limit of this probability as $n \to \infty$?
(c) What is the probability that just one match occurs in n trials?
(d) What value does this probability approach as $n \to \infty$?
(e) What is the probability that two or more matches occur in n trials?

(a) The probability of *no* matches is

$$\left(\frac{n-1}{n}\right)^n.$$

The probability of at least one match is

$$1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n.$$

(b) As $n \to \infty$,

$$\left(1 - \frac{1}{n}\right)^n \to e^{-1}.$$

Hence for large n, the probability of at least one match approaches $1 - e^{-1} = (e-1)/e$.
(c) There is only one match with probability

$$\left(\frac{n-1}{n}\right)^{n-1}.$$

(d) As $n \to \infty$

$$\left(\frac{n-1}{n}\right)^{n-1} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^n = e^{-1}.$$

(e) Probability of two or more matches is

$$\left(\frac{n-1}{n}\right)^{n-1} - \left(\frac{n-1}{n}\right)^n = \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}.$$

2.25. *(Kelly's strategy) A gambler plays a repeated favourable game in which the gambler wins with probability $p > \frac{1}{2}$ and loses with probability $q = 1 - p$. The gambler starts with an initial outlay K_0 (in some currency). For the first game the player bets a proportion rK_0, $(0 < r < 1)$. Hence after this play the stake is $K_0(1+r)$ after a win or $K_0(1-r)$ after losing. Subsequently, the gambler bets the same proportion of the current stake at each play. Hence, after n plays of which w_n are wins the stake S_r will be*

$$K_n(r) = K_0(1+r)^{w_n}(1-r)^{n-w_n}.$$

Construct the function

$$G_n(r) = \frac{1}{n}\ln\left[\frac{K_n(r)}{K_0}\right].$$

What is the expected value of $G_n(r)$ for large n? For what values of r is this expected value a maximum? This value of r indicates a safe betting level to maximise the gain, although at a slow rate. You might consider why the logarithm is chosen: this is known as a **utility function** *in gambling and economics. It is a matter of choice and is a balance between having a reasonable gain against having a high risk gain. Calculate r if $p = 0.55$. [At the extremes $r = 0$ corresponds to no bet whilst $r = 1$ to betting K_0 the whole stake in one go which could be catastrophic.]*

From

$$K_n(r) = K_0(1+r)^{w_n}(1-r)^{n-w_n},$$

it follows that

$$G_n(r) = \frac{1}{n}\ln[(1+r)^{w_n}(1-r)^{n-w_n} = \frac{w_n}{n}\ln(1+r) + \frac{(n-w_n)}{n}\ln(1-r).$$

$G+n(r)$ is our chosen utility function. For large n, the ratio w_n/n is approximately the probability p, and $(n-w_n)/n$ the probability q. Hence $G_n(r)$ approaches

$$H(r) = p\ln(1+r) + q\ln(1-r).$$

This function is zero where $r = 0$ and approaches $-\infty$ as $r \to 1-$. Also

$$H'(r) = \frac{p}{1+r} - \frac{q}{1-r}.$$

Hence $H'(0) = p + q = 1 > 0$, so that the slope is positive which implies that $H(r)$ has a maximum for in $0 < r < 1$. This occurs at $H'(r) = 0$ where where $r_m = 2p - 1$ so that

$$H(r_m) = p\ln(2p) + (1-p)\ln(2-2p).$$

The function $G_n(r)$ is an example of a utility function which attempts to avoid possible catastrophic losses, and keeps gains modest.

As an example let $p = 0.55$. Then $r_m = 2p - 1 = 0.1$. Suppose that $n = 100$ and $w_n = 55$ (reflecting the odds). Then for this value of r_m it can be calculated that $K_{100}(r_m) \approx 1.65K_0$.