Solutions for Differential Equations for Engineers The Essentials 1st Edition by Kalbaugh

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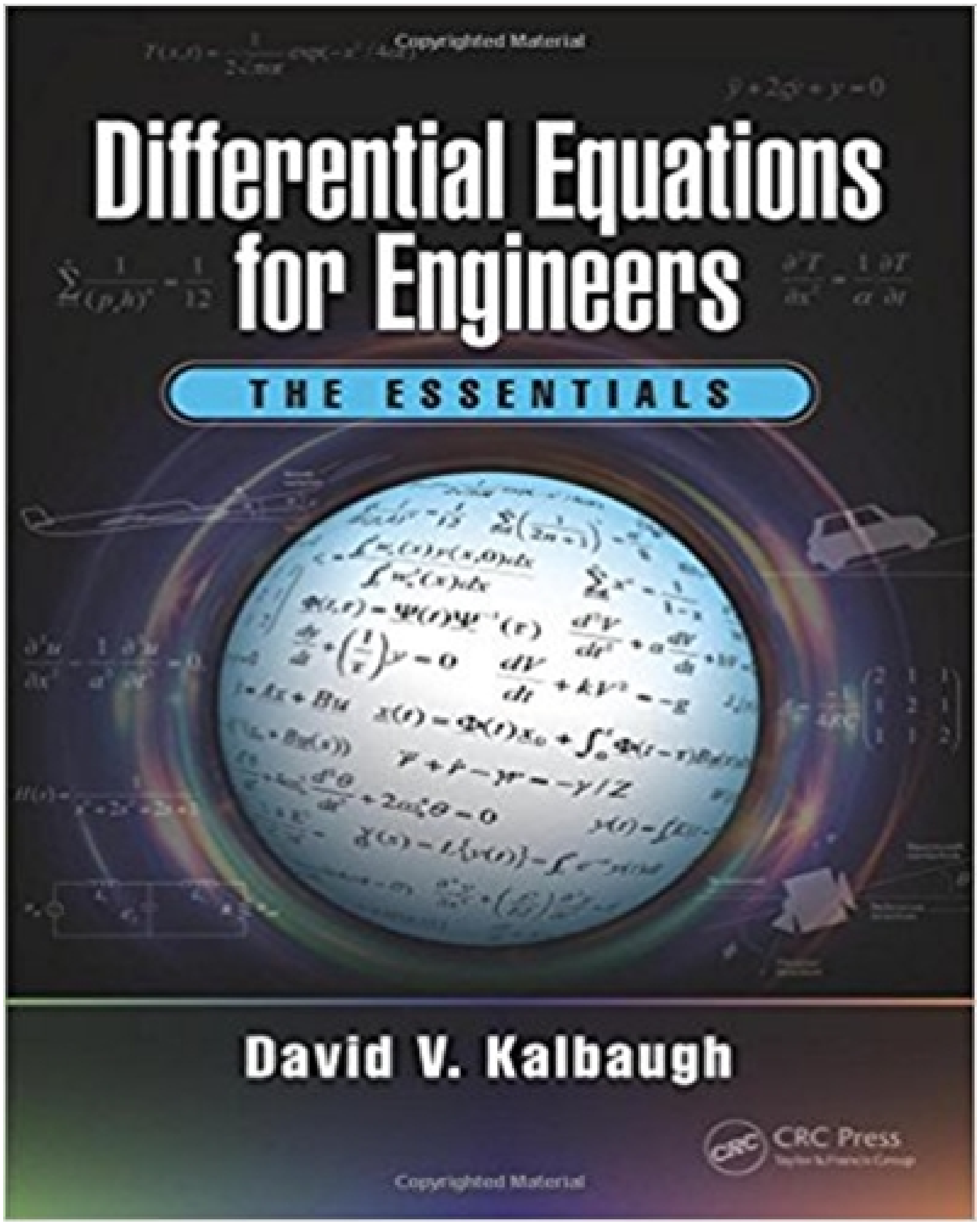


Solutions

Differential Equations for Engineers: the Essentials – Example Course Plan

Date	Class #	# of slides	Topics	Reading Assigned	Problems Assigned	Problems Reviewed
	1	36	Administrative matters, course objectives, importance of DEs to engineers, review of math foundations, classes of DEs	Chap 1 (14 pages)		_
	2	31	First order linear ODEs: RC circuit; general solution to homogeneous eqns; in-class homogeneous problems; water pipe temperature example; general solution nonhomogeneous; In-class nonhomogeneous problems	Chap 2 (11 pages)		_
	3	35	1st order linear ODEs: System viewpt First order nonlinear separable ODEs: General solution approach; In-class problems; Sounding rocket phases 1 & 2	Chap 3 thru Sec 3.1 (7 pages)		From Class #1
	4	21	First order nonlinear separable ODEs: Sounding rocket phase 3; Short quiz #1	Chap 3, Sec 3.2, 3.3 (5 pages)		From Class #2
	5	32	Review of short quiz #1 First order ODEs: successive approximations with example; in-class problems; existence and uniqueness	Chap 4, thru Sec 4.4 (9 pages)		From Class #3
	6A/B	30/36	Qualitative analysis Stability revisited Computing project phase 1	Chap 4, Sec 4.5, 4.6 (3 pages)	Computing project phase 1	From Class #4
	7	24	2nd order LTI homogeneous ODEs LRC circuit, characteristic equation, real and repeated roots; in-class problems	Chap 5 thru Sec 5.1.2 (6 pages)		From Class #5
	8		Test #1			_
	9	28	Review of Test #1 2nd order LTI homogeneous ODEs, LRC circuit, complex roots, fundamental solutions, In-class example problems	Chap 5, Sec 5.1.3, 5.1.4, 5.2 (5 pages)		From Class #7
	10	41	2nd order LTI Nonhomogeneous ODEs; Cruise control Undetermined coefficients method LRC circuit with sine source	Chap 5, Sec 5.3, 5.4 (15 pages)		From Class #6
	11	42	Process & example for kernel method Undetermined coefficients example Higher order ODEs	Chap 6 (10 pages)		From Class #9

Differential Equations for Engineers: the Essentials – Example Course Plan

			Satellite orbit decay			
	12		Review for Midterm			From Classes #10, 11
	13		Mid-term exam			_
	14A/B	42/44	Review of mid-term Laplace transforms: intro, homogeneous equations Computing project phase 2	Chap 7 thru Sec 7.5 (11 pages)	Computing project phase 2	
	15	46	Laplace transforms: Nonhomogeneous equations et al	Chap 7, Sec 7.6 (20 pages)		
	16	43	State space format: Numerical methods Review of matrix algebra Linear systems in state space format	Chap 8 thru Sec 8.4 (11 pages)		
	17	20	State space format: Heat transfer Short quiz #2	Chap 8, Sec 8.5.1.1 (8 pages)		From class #15
	18	35	Review of short quiz #2 State space format: Two-state electrical circuit; Aircraft dynamics	Chap 8, Sec 8.5.1.2, 8.5.1.3 (15 pages)		From classes #14,#16
	19	42	Three state electrical circuit Repeated eigenvalues; Coordinate systems: Vehicle suspension system	Chap 8, Sec 8.5.1.4 thru 8.5.1.7 (12 pages)		From classes #17,#18
	20		Test #2			_
	21	37	Review of Test #2 Coordinate systems; state transition matrix; nonhomogeneous equations, kernel method, 2-state electrical circuit example	Chap 8, Sec 8.5.1.5 thru 8.5.2.1 (15 pages)		From class #19
	22	35	Nonhomogeneous equations: Laplace transform method, trial and error method, PDEs: IV heat equation	Chap 8, Sec 8.5.2.2, 8.5.2.3 (6 pages)		
	23	24	PDEs: BV heat equation – Fourier series Short quiz #3	Chap 9 thru Sec 9.1 (11 pages)		From class #21
	24	35	Review of Short quiz #3 PDEs: wave equation; IV wave equation, BV problem: membrane – power series I	Chap 9, Sec 9.2 thru 9.2.3 (16 pages)		From class #22
	25	35	PDEs: higher order Bessell functions;	Chap 9,		From class

			power series II	Sec 9.2.4 thru 9.2.6 (8 pages)		#23
	26	34	PDEs: BV potential equation – Legendre's eq'n; cantilever beam	Chap 9, Sec 9.3, 9.4 (13 pages)		From class #24, #25
	27		Test #3			
	28		Review of Test #3 Review for final exam			
	29		Review for final exam			
			Final Exam			

Differential Equations for Engineers: the Essentials

Supplement to Class 4 Notes

Contents of Supplement

Differences Between Linear and Nonlinear ODEs in Their Input / Output Response

Input / Output Characteristics for a Linear System

In a linear system of any order (time-varying or time-invariant):

If the output is $y_1(t)$ **when the input is** $u_1(t)$ **and the output is** $y_2(t)$ **when the input is** $u_2(t)$
then the output is $ay_1(t)+by_2(t)$ **when the input is** $au_1(t)+bu_2(t)$ **for any constants** a **and** b

In a linear <u>time-invariant</u> system of any order, after transients have died away, when the input is a sine-wave of a given frequency the output is a steady-state oscillation of only that frequency.

These characteristics are <u>not</u> generally true of nonlinear systems

Existence and Uniqueness of Solutions to Linear ODEs

Theorem: Given a linear nth order ODE:

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^n y}{dt^n} + \ldots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

with initial conditions

$$\frac{d^{n-1}y}{dt^{n-1}}(0) = y_0^{(n-1)} \qquad \frac{d^{n-2}y}{dt^{n-2}}(0) = y_0^{(n-2)} \qquad \ldots \qquad \frac{dy}{dt}(0) = y_0' \qquad y(0) = y_0$$

If the coefficients $a_i(t)$ and the input $g(t)$ are continuous for all t then there exists a unique solution to the ODE satisfying the initial conditions for all t .

(Stated without proof.)

This is not generally true of nonlinear ODEs.

Key Points from the Following Nonlinear Example

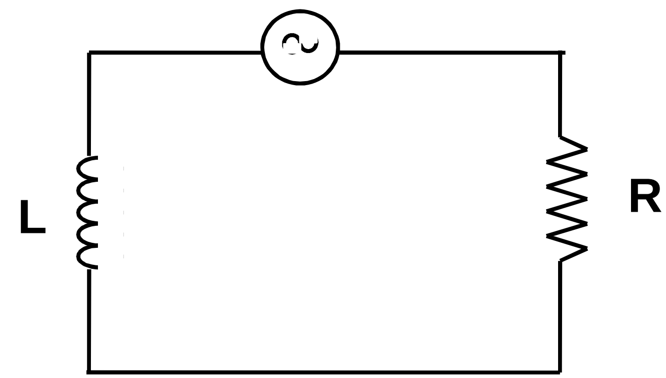
The method of successive approximations (solving a sequence of linear equations to approximate the solution of a nonlinear equation) is a powerful tool.

Nonlinearities in systems designed to be linear cause distortions in the frequency response, introducing "harmonics" (oscillations that are multiples of the input frequency).

Nonlinear Example: LR Circuit

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0



Voltage drop over an inductor:

$$V = L\frac{dI}{dt}$$

Voltage drop over a resistor:

$$V = IR$$

Voltage drop over source:

$$V = -V_0 \sin(\omega t)$$

Resulting equation :

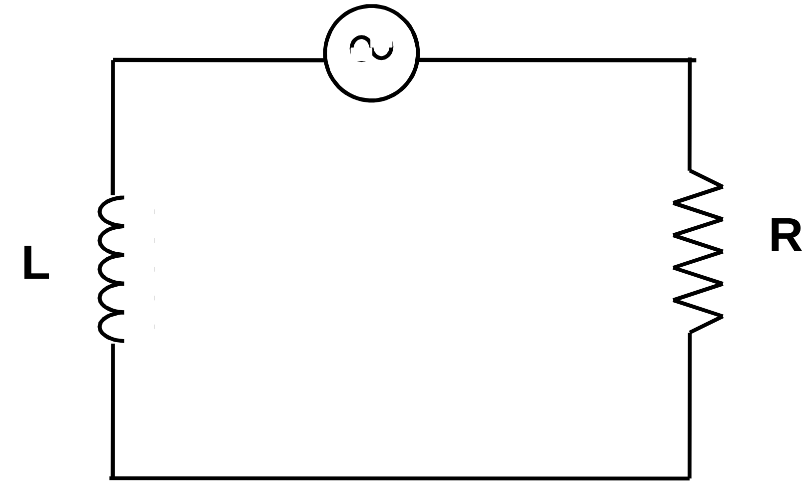
$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

Nonlinear Example: LR Circuit (2)

The solution to

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

$$I(0) = 0$$



is

$$I(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}}(V_0 / R)\sin(\omega t - \theta_1) + \frac{\lambda\omega}{\lambda^2 + \omega^2}(V_0 / R)e^{-\lambda t}$$

where

$$\theta_1 = \arctan(\omega / \lambda) \qquad \lambda = R / L$$

Nonlinear Example: Recalling the LR Circuit (3)

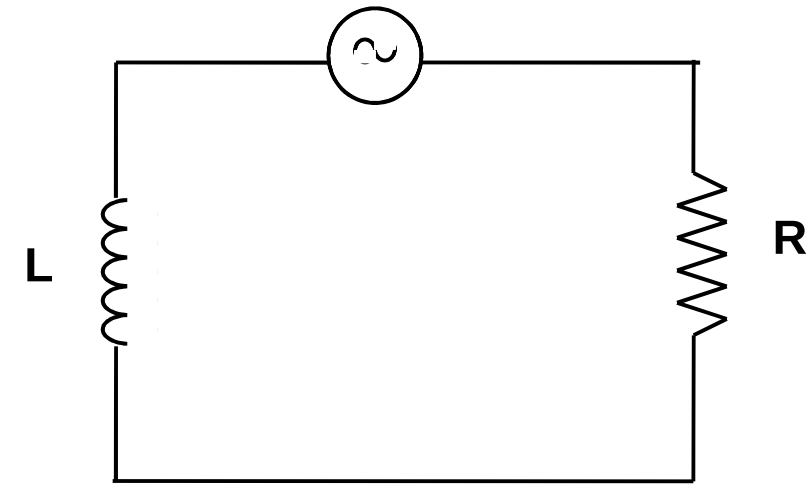
Input / output response:

The "input" to the "system" is the voltage source. The "output" is the current (or voltage) over the resistor.

Ignoring the transient, the system passes the sine wave frequency perfectly - it introduces no other frequencies.

We only discuss frequency response in the context of time-invariant systems.

What if the Resistor is Slightly Nonlinear?



Instead of

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

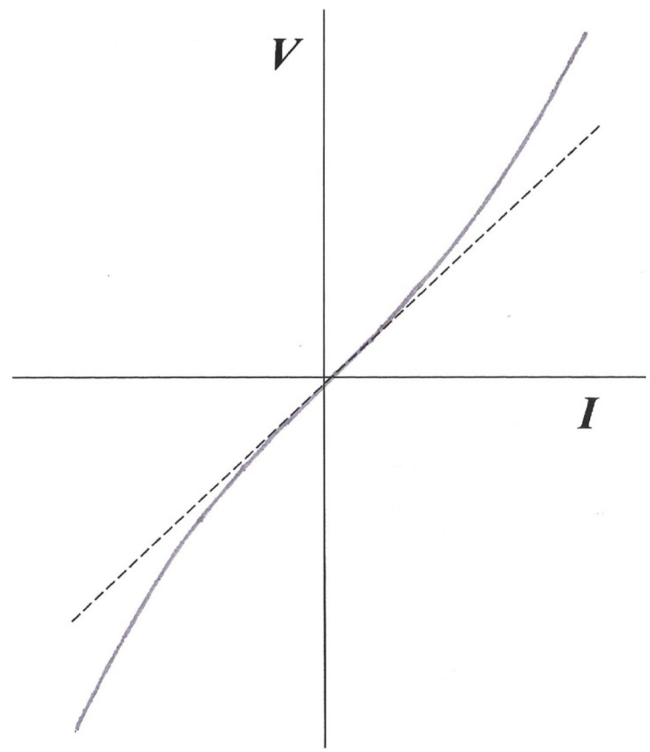
suppose we have

$$L\frac{dI}{dt} + R(I + \varepsilon I^3) = V_0 \sin \omega t$$

$$\frac{dI}{dt} + \lambda(I + \varepsilon I^3) = \lambda\left(\frac{V_0}{R}\right)\sin(\omega t) = \lambda I_0 \sin(\omega t)$$

Equation 1

What if the Resistor is Slightly Nonlinear? (2)



Nonlinear resistor characteristic

Approach to Solution of Slightly Nonlinear ODE

Since ε is small, consider the <u>successive approximations</u>

$$\frac{dI_1}{dt} + \lambda I_1 = \lambda I_0 \sin(\omega t)$$
$$I_1(0) = 0$$

Equation 2

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_1^3(t)$$
$$I_2(0) = 0$$

Equation 3

$$\frac{dI_3}{dt} + \lambda I_3 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_2^3(t)$$
$$I_3(0) = 0$$

Equation 4

Approach to Solution of Slightly Nonlinear ODE (2)

We are solving the <u>nonlinear</u> Equation 1 approximately by the sequential solution of a series of <u>linear</u> Equations 2, 3 and 4.

The general solution of

$$\frac{dI}{dt} + \lambda I = g(t) \qquad \text{is} \qquad I(t) = \int_0^t e^{-\lambda(t-\tau)} g(\tau)\, d\tau \qquad \text{Equation 5}$$

$$I(0) = 0$$

Using Equation 5, we have already found that the solution to Equation (2) is

$$I_1(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} I_0 \sin(\omega t - \theta_1) + \frac{\lambda \omega}{\lambda^2 + \omega^2} I_0 e^{-\lambda t}$$

In what follows we will ignore the transient. We are examining the steady state frequency response.

Approach to Solution of Slightly Nonlinear ODE (3)

Repeating Equation 3:

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin \omega t - \lambda \varepsilon I_1^3$$

This has solution

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)} \left(\lambda I_0 \sin(\omega\tau) - \lambda\varepsilon I_1^3\right) d\tau$$

Now

$$\int_0^t e^{-\lambda(t-\tau)} \lambda I_0 \sin(\omega\tau) d\tau = I_1(t)$$

so

$$I_2(t) = I_1(t) - \lambda\varepsilon \int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau) d\tau$$

Approach to Solution of Slightly Nonlinear ODE (4)

Continuing in this way, we find

$$I_{n+1}(t) = I_1(t) - \lambda\varepsilon \int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau)\, d\tau$$

which one can show can be written as

$$I_n(t) = I_1(t) - G_1 I_0 \eta_n(t)$$

where

$$G_1 = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} \qquad I_0 = V_0 / R$$

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn} \sin(m\omega t - \phi_{mn})$$

Note the higher frequencies – the harmonics

Approach to Solution of Slightly Nonlinear ODE (4)

***Challenge problem*:**

Given the steady state solution

$$I_1(t) = G_1 I_0 \sin(\omega t - \theta_1)$$

find the steady state component of $I_2(t)$

Differential Equations for Engineers: the Essentials

Class 2 notes

Agenda: Class 2

First order linear differential equations:

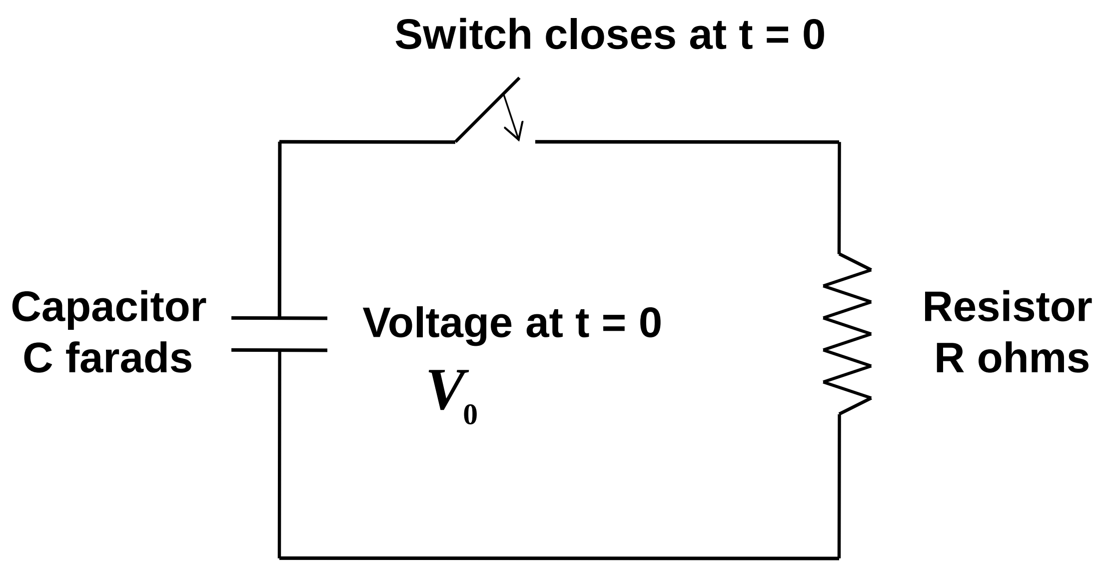
(1) Engineering example: RC circuit
(2) General solution of homogeneous equation
(3) In-class homogeneous problems
(4) Example: Exposed water pipe in cyclical air temperature
(5) General solution of nonhomogeneous equation
(6) In-class nonhomogeneous problems

Homework Assignment 2

First Order Linear Differential Equations

Example: The RC Electrical Circuit

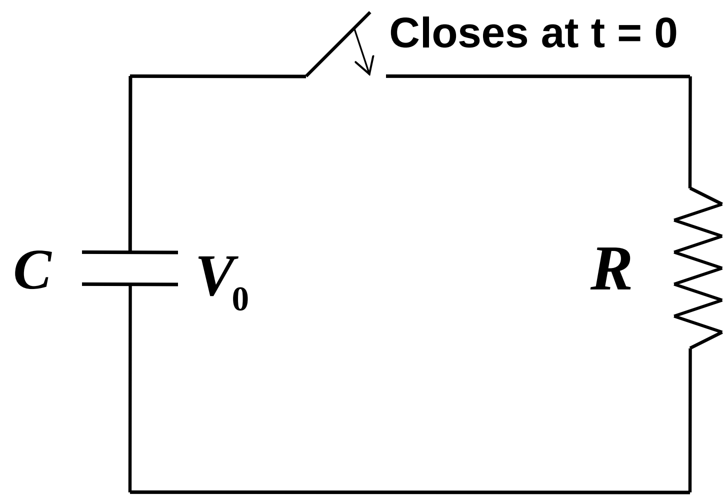
Example: RC Electrical Circuit



Example: RC Circuit (2)

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0



Voltage drop over a capacitor:

$$V = q/C$$ $q =$ **charge on capacitor**

Voltage drop over a resistor:

$$V = IR$$ $I =$ **current through resistor**

Conservation of electrical charge:

$$\frac{dq}{dt} = I$$

Example: RC Circuit (3)

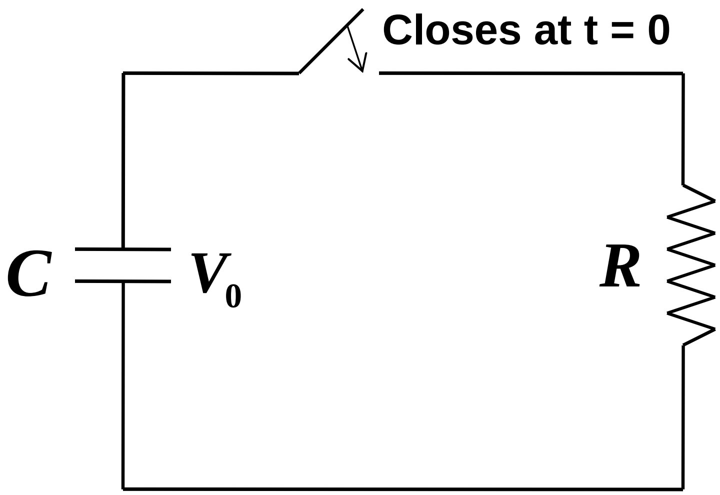
Resulting differential equation

$$IR + V = 0$$

$$\frac{dq}{dt}R + V = 0$$

$$RC\frac{dV}{dt} + V = 0$$

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$



Initial condition:

$$V(0) = V_0$$

Example: RC Circuit (4)

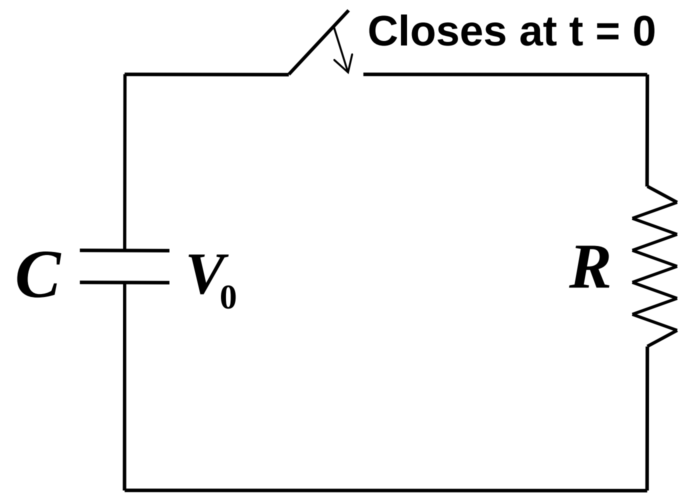
This is a linear first order ODE. To solve it, we separate variables: i.e., we put all terms involving *V* on the left side and all terms involving *t* on the right: specifically, we divide by *V*, move *1/RC* to the right side and multiply by *dt* :

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$\frac{1}{V}\frac{dV}{dt} + \frac{1}{RC} = 0$$

$$\frac{1}{V}\frac{dV}{dt} = -\frac{1}{RC}$$

$$\frac{dV}{V} = -\frac{1}{RC}dt$$



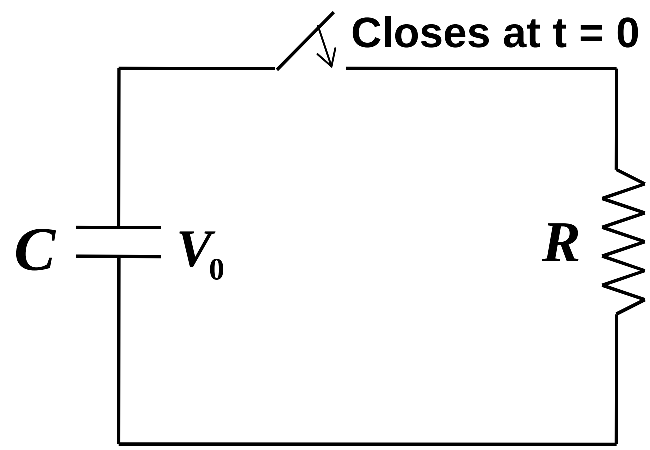
The text justifies this short-cut procedure

Example: RC Circuit (5)

Next, we integrate both sides:

$$\int_{V(0)}^{V(t)} \frac{dV}{V} = -\int_0^t \frac{dt}{RC}$$

$$\ln\left(\frac{V(t)}{V(0)}\right) = -t/RC$$



Taking the exponential of both sides:

$$\exp\left(\ln\left(\frac{V(t)}{V(0)}\right)\right) = \exp(-t/RC)$$

$$\frac{V(t)}{V(0)} = e^{-t/RC}$$

Example: RC Circuit (6)

Hence:

$$V(t) = V(0)e^{-t/RC}$$

We require that the voltage over the capacitor at time 0 be given by

$$V(0) = V_0$$

and so

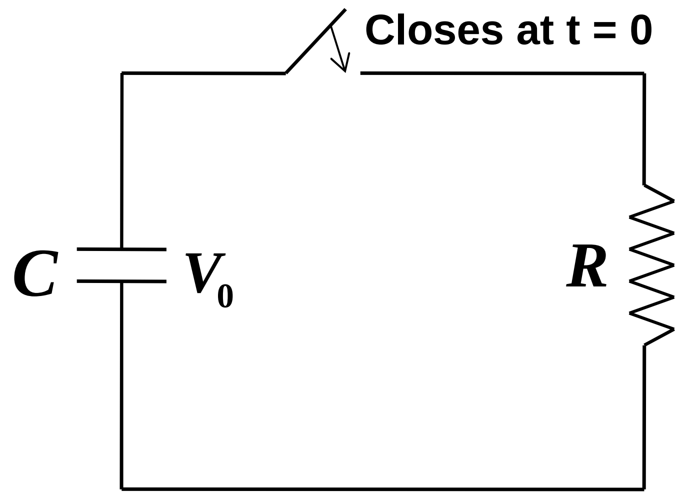
$$V(t) = V_0 e^{-t/RC}$$

Example: RC Circuit (7)

In summary, the solution to

$$\frac{dV}{dt} + \frac{1}{RC} V = 0$$

$$V(0) = V_0$$

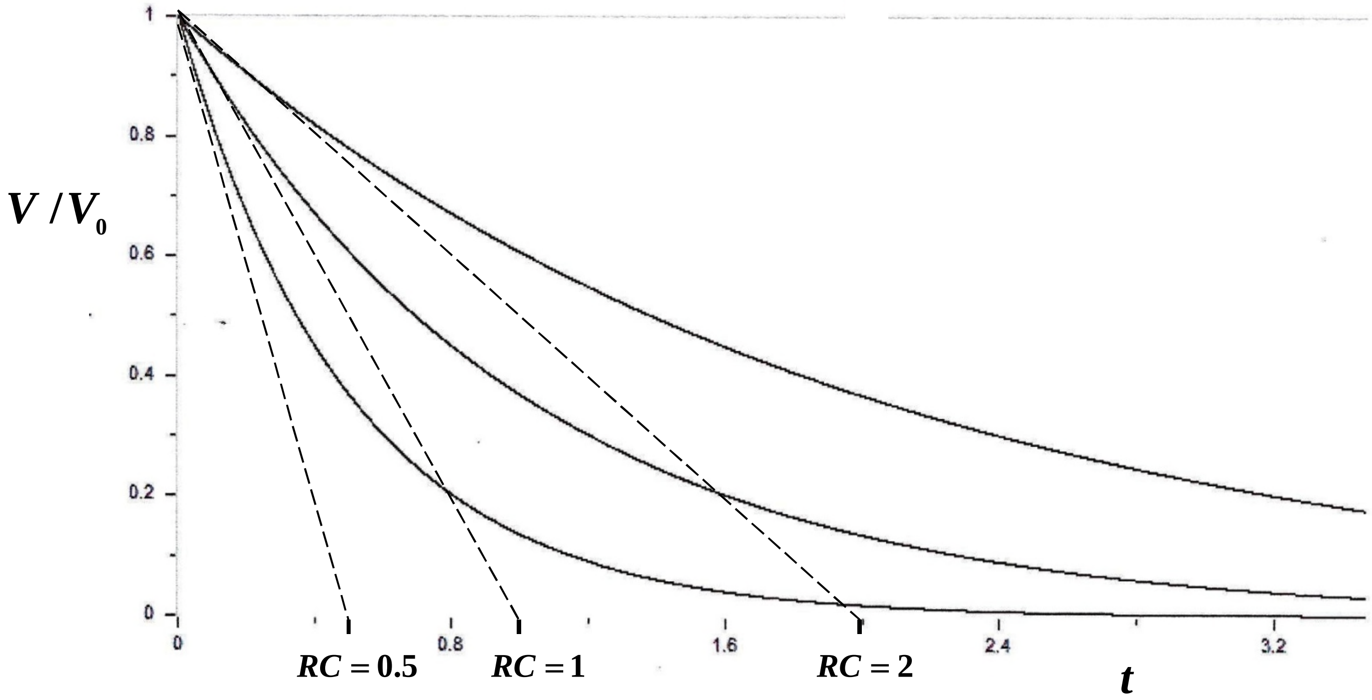


is

$$V(t) = V_0 e^{-t/RC}$$

The product RC has the dimension of time and is called the time constant for the circuit

Example: RC Circuit (8)



First Order Linear Differential Equations

General Solution of Homogeneous Equation

General Solution of Homogeneous Equation

The general form of a first order linear time-varying ordinary differential equation (ODE) is

$$\frac{dy}{dt} + p(t)y = 0 \qquad y(t_0) = y_0 \qquad \text{Equation 1}$$

How do we solve it?

General Solution of Homogeneous Equation (2)

We separate variables

$$\frac{dy}{y} = -p(t)dt$$

Integrate both sides

$$\int_{y_0}^{y(t)} \frac{dy}{y} = -\int_{t_0}^{t} p(\tau)d\tau$$

$$\ln\left(\frac{y(t)}{y_0}\right) = -\int_{t_0}^{t} p(\tau)d\tau$$

Take the exponential of both sides

$$\exp\left(\ln\left(\frac{y(t)}{y_0}\right)\right) = \exp(-\int_{t_0}^{t} p(\tau)d\tau)$$

$$\frac{y(t)}{y_0} = \exp(-\int_{t_0}^{t} p(\tau)d\tau)$$

General Solution of Homogeneous Equation (3)

Summary:

The solution of

$$\frac{dy}{dt} + p(t)y = 0 \qquad y(t_0) = y_0$$

is

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} p(\tau)\,d\tau\right)$$

Success depends entirely on being able to do the integral

Homogeneous First Order Linear ODEs: In-class problems

$$\frac{dy}{dt} + ky = 0$$
$$y(0) = a$$

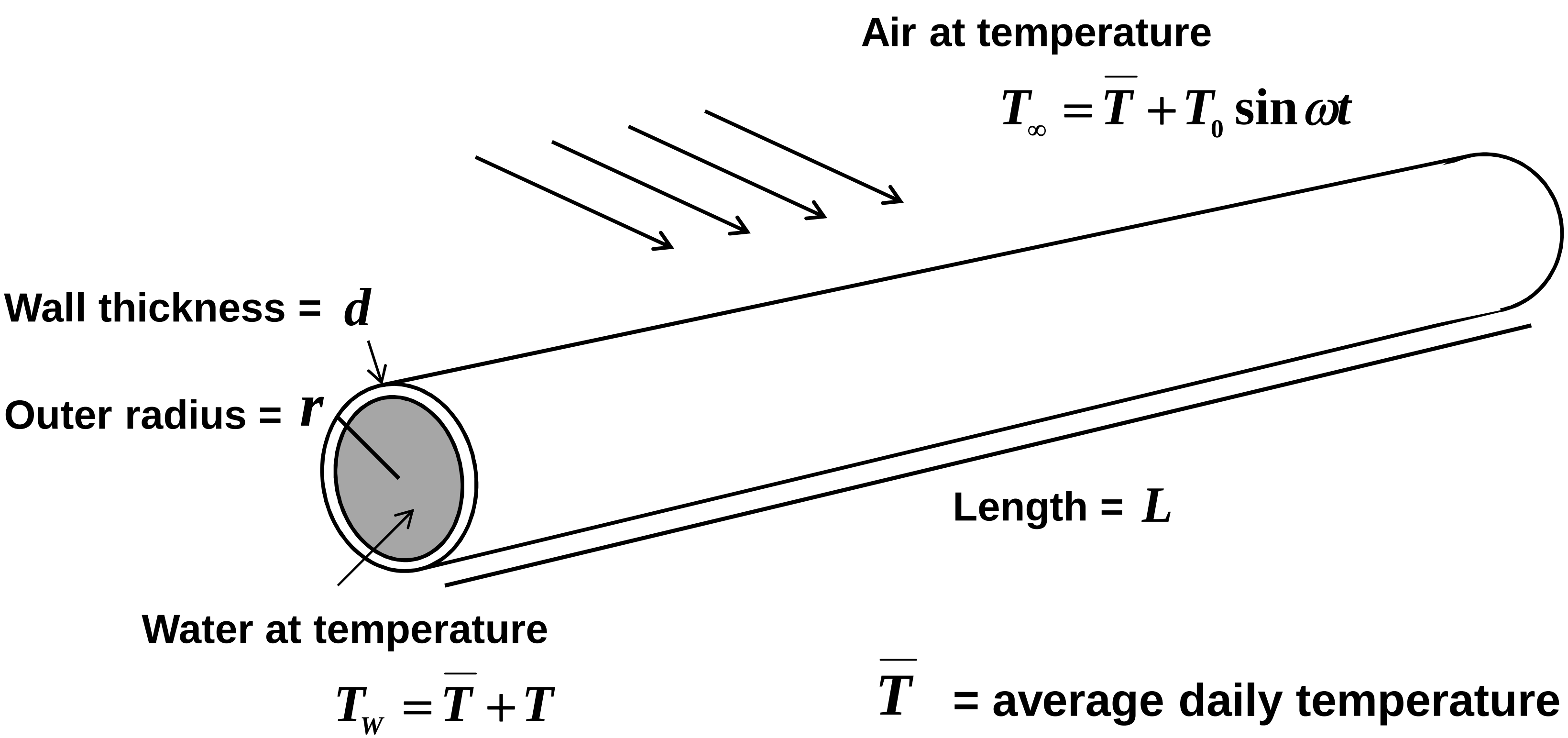
$$\frac{dy}{dt} + ty = 0$$
$$y(0) = b$$

$$\frac{dy}{dt} + \left(\frac{1}{t}\right)y = 0$$
$$y(1) = c$$

First Order Linear Differential Equations

Example: Exposed water pipe in cyclical ambient temperature

Exposed Water Pipe in Cyclical Ambient Temperature



Exposed Water Pipe in Cyclical Ambient Temperature (2)

Assuming pipe wall is thin and made of material that is a good heat conductor, by Newton's law of cooling, the heat transferred from air to water is

$$q = hA(T_\infty - T_W)$$

where

$A =$ **Exposed surface area of the pipe**

$h =$ **Convection coefficient**

Exposed Water Pipe in Cyclical Ambient Temperature (3)

The thermal energy stored in the water is

$$E = mcT_w$$

where

$m =$ **mass of the water**

$c =$ **specific heat of water**

Exposed Water Pipe in Cyclical Ambient Temperature (4)

Key physical principle:

$$\frac{dE}{dt} = q$$

which leads to

$$mc\frac{d}{dt}\left(\overline{T} + T\right) = hA\left((\overline{T} + T_0 \sin \omega t) - (\overline{T} + T)\right)$$

$$mc\frac{dT}{dt} + (hA)T = (hA)T_0 \sin \omega t$$

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \qquad \text{where} \qquad \lambda = \frac{hA}{mc}$$

Exposed Water Pipe in Cyclical Ambient Temperature (5)

How do we solve

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \qquad ? \qquad \textbf{Equation 2}$$

Let's be more inclusive and ask how do we solve the general linear first order nonhomogeneous equation

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \textbf{Equation 3}$$

$$y(t_0) = y_0$$

General Solution to Nonhomogeneous Linear First Order ODEs

We begin by searching for an integrating factor $\mu(t)$ that, when multiplied into the equation, turns the left-hand side into

$$\frac{d}{dt}(\mu(t)y)$$

Multiplying Equation 3 by $\mu(t)$:

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\mu(t)g(t)$$

Search for $\mu(t)$ such that left hand side is

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\frac{d}{dt}(\mu(t)y)$$

General Solution to Nonhomogeneous Linear First Order ODEs (2)

We must have

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

which means that

$$\frac{d\mu}{dt} = p(t)\mu(t)$$

$$\frac{1}{\mu(t)}\frac{d\mu}{dt} = p(t)$$

General Solution to Nonhomogeneous Linear First Order ODEs (3)

$$\frac{d}{dt}(\ln \mu(t)) = p(t)$$

$$\ln(\mu(t)/\mu(t_0)) = \int_{t_0}^{t} p(u)du$$

$$\mu(t) = \mu(t_0)\exp\int_{t_0}^{t} p(u)du \qquad \textbf{Equation 4}$$

This is the desired integrating factor.

But we can simplify the form.

General Solution to Nonhomogeneous Linear First Order ODEs (4)

We do not know what value to assign to $\mu(t_0)$ but it turns out not to matter. (The value cancels out.) So we set

$$\mu(t_0) = 1$$

It also suffices to use the indefinite integral form:

$$\mu(t) = \exp \int^{t} p(u)du \qquad \text{Equation 5}$$

You should remember, or be able to derive, Equation 5

Exposed Water Pipe in Cyclical Ambient Temperature (6)

For the water pipe temperature problem (Equation 2):

$$\frac{dy}{dt} + p(t)y = g(t)$$

becomes

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t$$

so

$$p(t) = \lambda$$

$$\mu(t) = \exp\left(\int^t p(u)du\right) = \exp\left(\int^t \lambda du\right) = e^{\lambda t}$$

Exposed Water Pipe in Cyclical Ambient Temperature (7)

Applying the integration factor to Equation 2:

$$e^{\lambda t}\left(\frac{dT}{dt}+\lambda T\right)=e^{\lambda t}(\lambda T_0 \sin \omega t)$$

$$\frac{d}{dt}\left(e^{\lambda t}T\right)=\lambda T_0 e^{\lambda t}\sin \omega t$$

Now the value of the integration factor becomes clear:
We can solve the problem with an integration:

$$e^{\lambda t}T(t)-T(0)=\lambda T_0\int_0^t e^{\lambda \tau}\sin \omega\tau d\tau$$

$$T(t)=T(0)e^{-\lambda t}+e^{-\lambda t}\int_0^t e^{\lambda \tau}\sin \omega\tau d\tau$$

Exposed Water Pipe in Cyclical Ambient Temperature (9)

After performing the integral we have

$$T(t) = T_I e^{-\lambda t} + \left(\frac{\lambda}{\lambda^2 + \omega^2}\right) T_0 (\lambda \sin(\omega t) - \omega \cos(\omega t) + \omega e^{-\lambda t})$$

where

$$T_I = T(0) = T_W(0) - \overline{T}$$

Inhomogeneous First Order Linear ODEs: In-class Problems

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = 4$$

$$y(1) = 2$$

$$\frac{dy}{dt} + 4\left(\frac{e^{4t} - e^{-4t}}{e^{4t} + e^{-4t}}\right)y = e^{3t}$$

$$y(0) = 6$$

$$\frac{dy}{dt} - (\tan t)y = \sec t$$

$$y(0) = 0$$

Homework Assignment 2

<u>In text</u>:

Read: Chapter 2

Work: On course website: Homework Assignment #2 Problems

Solutions for Homework Assignment #2 Problems will be provided on course website on (date)

Always read over the day's lecture notes and be sure you understand them.