Solutions Manual for Introduction to Formal Languages and Automata 7th Edition by Linz, Rodger

ISBN: 9781284231601

1 1.1 Solutions

1. With $S_1 = \{2, 3, 5, 7\}$, $S_2 = \{2, 4, 5, 8, 9\}$ and $U = \{1 : 10\}$ compute $\overline{S_1} \bigcup S_2$.

   Solution: $\overline{S_1} \bigcup S_2 = \{1, 2, 4, 5, 6, 8, 9, 10\}$.

2. With $S_1 = \{2, 3, 5, 7\}$ and $S_2 = \{2, 4, 5, 8, 9\}$, compute $S_1 \times S_2$ and $S_2 \times S_1$.

   Solution: $S_1 \times S_2 = \{(2, 2), (2, 4), (2, 5), (2, 8), (2, 9), (3, 2), (3, 4), (3, 5), (3, 8), (3, 9), (5, 2), (5, 4), (5, 5), (5, 8), (5, 9), (7, 2), (7, 4), (7, 5), (7, 8), (7, 9)\}$.

   $S_2 \times S_1 = \{(2, 2), (4, 2), (5, 2), (8, 2), (9, 2), (2, 3), (4, 3), (5, 3), (8, 3), (9, 3), (2, 5), (4, 5), (5, 5), (8, 5), (9, 5), (2, 7), (4, 7), (5, 7), (8, 7), (9, 7)\}$.

3. For $S = \{2, 5, 6, 8\}$ and $T = \{2, 4, 6, 8\}$, compute $|S \cap T| + |S \cup T|$.

   Solution: $|S \cap T| + |S \cup T| = 3 + 5 = 8$.

4. What relation between two sets S and T must hold so that $|S \cup T| = |S| + |T|$.

   Solution: S and T must be two disjoint sets.

5. ** Show that for all sets S and T, $S - T = S \cap \overline{T}$.

   Solution: Suppose $x \in S - T$. Then $x \in S$ and $x \notin T$, which means $x \in S$ and $x \in \overline{T}$, that is $x \in S \cap \overline{T}$. So $S - T \subseteq S \cap \overline{T}$. Conversely, if $x \in S \cap \overline{T}$, then $x \in S$ and $x \notin T$, which means $x \in S - T$. That is $S \cap \overline{T} \subseteq S - T$. Therefore $S - T = S \cap \overline{T}$.

6. ** Prove De Morgan's laws, Equations (1.2) and (1.3) by showing that an element x is in the set on one side of the equality, then it must also be in the set on the other side.

   Solution: For Equation (1.2), suppose $x \in \overline{S_1 \cup S_2}$. Then $x \notin S_1 \cup S_2$, which means that x cannot be in S_1 or in S_2, that is $x \in \overline{S}_1 \cap \overline{S}_2$. So $\overline{S_1 \cup S_2} \subseteq \overline{S_1} \cap \overline{S_2}$. Conversely, if $x \in \overline{S_1} \cap \overline{S_2}$, then x is not in S_1 and x is not in S_2, that is, $x \in \overline{S_1 \cup S_2}$. So $\overline{S_1} \cap \overline{S_2} \subseteq \overline{S_1 \cup S_2}$. Therefore $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$.

   Similar arguements apply for Equation (1.3).

7. Show that if $S_1 \subseteq S_2$ then $\overline{S_2} \subseteq \overline{S_1}$.

   Solution: Suppose $x \in \overline{S_2}$ which means x is not in S_2. Since S_1 is a subset of S_2, so x cannot be in S_1, that is $x \in \overline{S_1}$. So $\overline{S_2} \subseteq \overline{S_1}$.

8. Show that $S_1 = S_2$ if and only if $S_1 \cup S_2 = S_1 \cap S_2$.

   Solution: If $S_1 = S_2$, then obviously $S_1 \cup S_2 = S_1 \cap S_2 (= S_1 = S_2)$. Conversely, since $S_1 \cap S_2 \subseteq S_1 \subseteq S_1 \cup S_2$, so $S_1 \cup S_2 = S_1 \cap S_2$ means $S_1 = S_1 \cup S_2$. Ihe reasoning gives $S_2 = S_1 \cup S_2$. Therefore $S_1 = S_2$.

9. Use induction on the size of S to show that if S is a finite set, then $2^S \ = 2^{|S|}$.

   Solution: For $S =$, $2^S = \{\oslash\}$ which means $|S| = 0$ and $2^{|S|} = 1$. Suppose the conjecture, $2^S \ = 2^{|S|}$ is true for all sets S with $|S| = 0, 1, ..., n$.

   Let $S_{n+1} = S_n \cup \{x_{n+1}\}$ be a set of length $n+1$ where $|S_n| = n$ with $2^{S_n} = \{S_{n,1}, ..., S_{n,2^n}\}$. Since the power set $2^{S_{n+1}} = 2^{S_n} \cup (\{S_{n,1} \cup \{x_{n+1}\}, ..., \{S_{n,2^n} \cup \{x_{n+1}\}\})$, so $2^{S_{n+1}} \ = \ 2^{S_n} \ + \ 2^{S_n} \ = 2 \ 2^{S_n} \ = 2(2^n) = 2^{n+1}$. This completes the inductive proof.

10. Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then

$$|S_1 \cup S_2| \leq n + m.$$

   Solution: Since $S_1 \cup S_2 = S_1 \cup (S_2 - S_1)$ and $S_1 \cap (S_2 - S_1) =$, so $|S_1 \cup S_2| = |S_1| + |S_2 - S_1| \leq |S_1| + |S_2| = n + m$.

11. If S_1 and S_2 are finite sets, show that $|S_1 \times S_2| = |S_1|\,|S_2|$.

   Solution: If $S_1 = \{x_1, ..., x_n\}$ and $S_2 = \{y_1, ..., y_m\}$, then by definition $S_1 \times S_2 = \{(x_1, y_1), (x_1, y_2), ..., (x_n, y_m)\}$. That is $|S_1 \times S_2| = nm = |S_1|\,|S_2|$.

12. **Consider the relation between two sets defined by $S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$. Show that this is an equivalence relation.

Solution: Check the three rules of equivalence relation for any sets S_1, S_2 as follows :

a) reflexivity: Since $|S_1| = |S_1|$, so $S_1 \equiv S_1$.

b) symmetry: if $S_1 \equiv S_2$, then $|S_1| = |S_2|$. So $|S_2| = |S_1|$ which means $S_2 \equiv S_1$.

c) transitivity: if $S_1 \equiv S_3$ and $S_3 \equiv S_2$, then $|S_1| = |S_3| = |S_2|$. Hence $S_1 \equiv S_2$.

13. Occasionally, we need to use the union and intersection symbols in a manner analogous to the summation sign Σ. We define

$$\bigcup_{p \in \{i,j,k,\ldots\}} S_p = S_i \cup S_j \cup S_k \cdots$$

with an analogous notation for the intersection of several sets.

With this notation, the general DeMorgan's laws are written as

$$\overline{\bigcup_{p \in P} S_p} = \bigcap_{p \in P} \overline{S_p}$$

and

$$\overline{\bigcap_{p \in P} S_p} = \bigcup_{p \in P} \overline{S_p}.$$

Prove these identities when P is a finite set.

Solution: This can be proven by an induction on the number of sets. Let $Z = S_1 \cup S_2 ... \cup S_n$. Then $S_1 \cup S_2 ... \cup S_n \cup S_{n+1} = Z \cup S_{n+1}$. By the standard De Morgan's law,

$$\overline{Z \cup S_{n+1}} = \overline{Z} \cap \overline{S}_{n+1}.$$

With the inductive assumption, the relation is true for up to n sets, that is,

$$\overline{Z} = \overline{S}_1 \cap \overline{S}_2 \cap ...\overline{\cap S}_n.$$

Therefore,

$$\overline{Z \cup S_{n+1}} = \overline{S}_1 \cap \overline{S}_2 \cap ...\overline{\cap S}_n \cap \overline{S}_{n+1},$$

completing the inductive step.

14. Show that
$$S_1 \cup S_2 = \overline{\overline{S}_1 \cap \overline{S}_2}.$$

Solution: By the DeMorgan's laws and the fact $\overline{\overline{S}} = S$, we have
$$\overline{\overline{S}_1 \cap \overline{S}_2} = \overline{\overline{S_1}} \cup \overline{\overline{S_2}} = S_1 \cup S_2.$$

15. Show that $S_1 = S_2$ if and only if
$$\left(S_1 \cap \overline{S}_2\right) \cup \left(\overline{S}_1 \cap S_2\right) = \quad .$$

Solution: Suppose $S_1 = S_2$. Then $S_1 \cap \overline{S}_2 = \overline{S_1} \cap S_2 = S_1 \cap \overline{S}_1 = \quad$, so $\left(S_1 \cap \overline{S}_2\right) \cup \left(\overline{S}_1 \cap S_2\right) = \quad \cup \quad = \quad$.

Suppose that $\left(S_1 \cap \overline{S}_2\right) \cup \left(\overline{S}_1 \cap S_2\right) = \quad$. Now if $S_1 \neq S_2$, then there must be an element x in S_1 but not in S_2 which means $S_1 \cap \overline{S}_2 \neq \quad$. So $\left(S_1 \cap \overline{S}_2\right) \cup \left(\overline{S}_1 \cap S_2\right) \neq \quad$. This contradics the assumption. So $S_1 = S_2$ and the proof is completed.

16. Show that
$$S_1 \cup S_2 - \left(S_1 \cap \overline{S}_2\right) = S_2.$$

Solution: If we use the distributive law in the next exercise problem, then $S_1 \cup S_2 - (S_1 \cap \overline{S_2}) = (S_1 \cup S_2) \cap \overline{(S_1 \cap \overline{S_2})} = (S_1 \cup S_2) \cap (\overline{S_1} \cup S_2)$ $= ((S_1 \cup S_2) \cap \overline{S_1}) \cup ((S_1 \cup S_2) \cap S_2)) = (S_2 \cap \overline{S_1}) \cup S_2 = S_2$.

17. Show that the distributive law
$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$$

holds for sets.

Solution: $x \in S_1 \cap (S_2 \cup S_3)$

if and only if $x \in S_1$ and $x \in (S_2 \cup S_3)$

if and only if $x \in S_1$ and $(x \in S_2$ or $x \in S_3)$

if and only if $(x \in S_1$ and $x \in S_2)$ or $(x \in S_1$ and $x \in S_3)$

if and only if $x \in (S_1 \cap S_2)$ or $(S_1 \cap S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$.

18. Show that
$$S_1 \times (S_2 \cup S_3) = (S_1 \times S_2) \cup (S_1 \times S_3)\,.$$

Solution: $(x, y) \in S_1 \times (S_2 \cup S_3)$ if and only if $x \in S_1$ and $y \in (S_2 \cup S_3)$

if and only if $x \in S_1$ and $(y \in S_2$ or $y \in S_3)$

if and only if $(x \in S_1$ and $y \in S_2)$ or $(x \in S_1$ and $y \in S_3)$

if and only if $(x, y) \in (S_1 \times S_2)$ or $(x, y) \in (S_1 \times S_3)$

if and only if $(x, y) \in (S_1 \times S_2) \cup (S_1 \times S_3)\,.$

19. Give conditions on S_1 and S_2 necessary and sufficient to ensure that
$$S_1 = (S_1 \cup S_2) - S_2.$$

Solution: $S_1 = (S_1 \cup S_2) - S_2 = (S_1 \cup S_2) \cap \overline{S_2} = (S_1 \cap \overline{S}_2) \cup (S_2 \cap \overline{S}_2) = S_1 \cap \overline{S}_2$. Since $S_1 = S_1 \cap \overline{S}_2$

if and only if $S_1 \subseteq \overline{S}_2$ if and only if $S_1 \cap S_2 =$.

Therefore the necessary and sufficient condition to ensure the given equation is $S_1 \cap S_2 =$.

20. ** Use the equivalence defined in Example 1.4 to partition the set $\{2, 4, 5, 6, 9, 23, 24, 25, 31, 37\}$ into equivalence classes.

Solution: The three equivalent classes are: $E_0 = \{6, 9, 24\}$; $E_1 = \{4, 25, 31, 37\}$ and $E_2 = \{2, 5, 23\}$.

21. Show that if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.

Solution: If $f(n) = O(g(n))$, then $f(n) \le c_1 |g(n)|$ for some constant c_1. If $g(n) = O(f(n))$, then $g(n) \le c_2 |f(n)|$ for some constant c_2 which means $\frac{1}{c_2} g(n) \le |f(n)|$. Therefore $\frac{1}{c_2} |g(n)| \le |f(n)| \le c_1 |g(n)|$, that is $f(n) = \Theta(g(n))$.

22. Show that $2^n = O(3^n)$ but $2^n \ne \Theta(3^n)$.

Solution: Since $2^n \le 3^n$ so $2^n = O(3^n)$. To show that $2^n \ne \Theta(3^n)$. Suppose $2^n = \Theta(3^n)$, then there must exists a constant $c > 0$ such that
$$2^n \le 3^n \le c2^n$$

for sufficiently large n. Dividing through the inequality by 2^n, then

$$\left(\frac{3}{2}\right)^n = \frac{3^n}{2^n} \leq c.$$

It is not possible, because $\left(\frac{3}{2}\right)^n$ is unbounded . This contradiction leads to the conclusion $2^n \neq \Theta\left(3^n\right)$.

23. Show that the following order-of-magnitude results hold.

    (a) $n^2 + 5\log n = O\left(n^2\right)$.

    (b) $3^n = O\left(n!\right)$.

    (c) $n! = O\left(n^n\right)$.

    Solution: (a) $n^2 + 5\log n \leq n^2 + 5n^2 = 6n^2$ for $n \geq 1$. Therefore, $n^2 + 5\log n = O\left(n^2\right)$.

    (b)

    $$\frac{3^n}{9n!} = \frac{1}{9}\left(\frac{3}{1}\frac{3}{2}\frac{3}{3}\frac{3}{4}....\frac{3}{n}\right) = \frac{1}{2}\left(\frac{3}{4}...\frac{3}{n}\right) \leq 1$$

    for all $n \geq 0$. Therefore, $3^n = O\left(n!\right)$.

    (c) Since

    $$\frac{n!}{n^n} = \frac{n}{n}\frac{n-1}{n}...\frac{2}{n}\frac{1}{n}$$

    is the product of factors less than or equal to one. Therefore, $n! = O\left(n^n\right)$.

24. Show that $(n^3 - 2n)/(n+1) = \Theta(n^2)$.

    Solution: Since $n^2 - n - 4 = (n-1/2)^2 - 17/4 > 0$, so $n^3 - n^2 - 4n > 0$ for all $n \geq 3$. Therefore, $n^3 - 4n > n^2$. Adding n^3 on both sides, we get $2(n^3 - 2n) > n^3 + n^2$ which is equivalent to $(n^3 - 2n) > (n^3 + n^2)/2$. Hence, $n^2/2 = \{(n^3 + n^2)/2\}/(n+1) < (n^3 - 2n)/(n+1)$.

    It is obvious that $(n^3 - 2n)/(n+1) < (n^3 + n^2)/(n+1) = n^2$ for all $n \geq 3$. Therefore, $n^2/2 < (n^3 - 2n)/(n+1) < n^2$ for $n \geq 3$ shows $(n^3 - 2n)/(n+1) = \Theta(n^2)$.

25. Show that $\frac{n^3}{\log(n+1)} = O(n^3)$ but not $O(n^2)$.

Solution: Since $\frac{n^3}{\log(n+1)} \leq \frac{n^3}{\log(2)}$ for $n \geq 1$, therefore $\frac{n^3}{\log(n+1)} = O(n^3)$.

Suppose $\frac{n^3}{\log{(n+1)}} = O(n^2)$, then there exists a constant c such that $n^3 \leq cn^2 \log(n+1)$ which means $n \leq c \log(n+1) \leq 2c \log(n)$ for sufficiently large n. This is not possible because $n \neq O(\log(n))$. Therefore, $\frac{n^3}{\log{(n+1)}}$ is not $O(n^2)$.

26. ** What is wrong with the following argument? $x = O(n^4), y = O(n^2)$, therefore $x/y = O(n^2)$.

    Solution: Since $O(x)$ is any upper bound for x, it is not unique. Same for $O(y)$. So ordinary arithmetic operations are not neceesary applicable. For example, take $x = n^3$ and $y = \frac{1}{n^2}$, then by the definition $x = O(n^4), y = O(n^2)$. However, $x/y = n^5$ which is not $O(n^2)$.

27. What is wrong with the following argument? $x = \Theta(n^4), y = \Theta(n^2)$, therefore $\frac{x}{y} = \Theta(n^2)$.

    Solution: Nothing is wrong. The argument is correct. Since $x = \Theta(n^4), y = \Theta(n^2)$, gives $c_1 n^4 \leq x \leq c_2 n^4$ and $c_3 n^2 \leq y \leq c_4 n^2$ for constants c_1, c_2, c_3, c_4 and sufficiently large n. Therefore, $\frac{c_1}{c_4} n^2 \leq \frac{x}{y} \leq \frac{c_2}{c_3} n^2$. That is $\frac{x}{y} = \Theta(n^2)$.

28. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

    Solution: If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) \leq c_1 |g(n)|$ and $g(n) \leq c_2 |h(n)|$ for constants c_1, c_2 and sufficiently large n. Thereofe, $f(n) \leq c_1 c_2 |h(n)|$ which means $f(n) = O(h(n))$.

29. Show that if $f(n) = O(n^2)$ and $g(n) = O(n^3)$, then

    $$f(n) + g(n) = O\left(n^3\right)$$

    and

    $$f(n)\, g(n) = O\left(n^6\right).$$

    In this case, is it true that $g(n)/f(n) = O(n)$?

Solution: If $f(n) = O(n^2)$ and $g(n) = O(n^3)$, then $f(n) \leq c_1 n^2$ and $g(n) \leq c_2 n^3$ for constants c_1, c_2 and sufficiently large n. So $f(n) + g(n) \leq c_1 n^2 + c_2 n^3 < (c_1 + c_2) n^3$ and $f(n) g(n) \leq (c_1 n^2)(c_2 n^3) < (c_1 c_2) n^5 < (c_1 c_2) n^6$. That is $f(n) + g(n) = O(n^3)$ and $f(n) g(n) = O(n^6)$.

By the same reason as Exercise 26, $g(n)/f(n) = O(n)$ is not always true.

30. ** Assume that $f(n) = 2n^2 + n$ and $g(n) = O(n^2)$. What is wrong with the following argument?

$$f(n) = O\left(n^2\right) + O(n),$$

so that

$$f(n) - g(n) = O\left(n^2\right) + O(n) - O\left(n^2\right).$$

Therefore,

$$f(n) - g(n) = O(n).$$

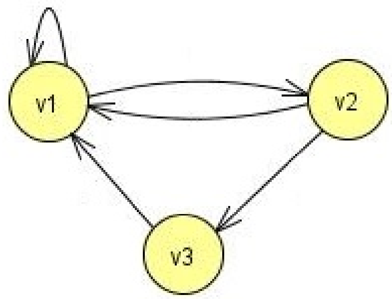
Solution: Ordinary arithmetic operations do not apply to order of magnitud eargumentsl. That is $O(n^2) - O(n^2)$ does not necessarily equal to 0. For example, in this problem if $g(n) = n^2$, then $f(n) - g(n) = n^2 + n = O(n^2)$.

31. Show that if $f(n) = \Theta(\log_2 n)$, then $f(n) = \Theta(\log_{10} n)$.

Solution: If $f(n) = \Theta(\log_2 n)$, then $c_1 \log_2 n \leq |f(n)| \leq c_2 \log_2 n$ for constant c_1, c_2 and sufficiently large n. Since $\log_2 n = \log_2 10 \log_{10} n$, so $d_1 \log_{10} n \leq |f(n)| \leq d_2 \log_{10} n$ where $d_1 = c_1 \log_2 10$ and $d_2 = c_2 \log_2 10$. That is $f(n) = \Theta(\log_{10} n)$.

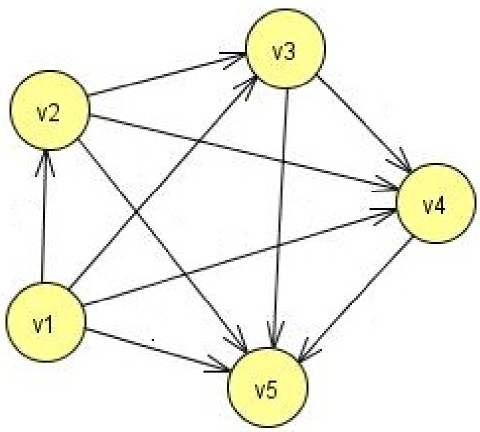
32. Draw a picture of the graph with vertices $\{v_1, v_2, v_3\}$ and edges $\{(v_1, v_1), (v_1, v_2), (v_2, v_3), (v_2, v_1), (v_3, v_1)\}$. Enumerate all cycles with base v_1.

Solution: The graph is given as follows.



33. Construct a graph with five vertices, ten edges, and no cycles.

    Solution: The graph with five vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and ten edges $\{(v_1, v_2),$
    $(v_1, v_3), (v_1, v_4), (v_1, v_5), (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_5), (v_4, v_5)\}$ has no cycles.



34. Let $G = (V, E)$ be any graph. Prove the following claim: If there is any walk between $v_i \in V$ and $v_j \in V$, then there must be a path of length no larger than $|V| - 1$ between these two vertices.

    Solution: To simplify the problem, we assume that there is no edge of form (v, v) in a walk. Otherwise we remove it from the walk and will not affect the result of this problem.

    If the walk from v_i to v_j has no repeated vertices, then it is a simple path. In this case, every vertex in the path has been passed only once. Hence the length of the path cannot exceed $|V| - 1$.

Suppose $(v_i, v_k), (v_k, v_l), (v_l, v_m), ..., (v_p, v_j)$ is a walk from v_i to v_j which has repeated vertices. Then we can find a subsequence as follows so that it travels from v_i to v_j will be a simple path, that is, without any repeating vertices.

Step 1 : Determine if the second element (or edge) (v_k, v_l) is in the final simple path. We first find all the edges in the original sequence whose first component is v_k. If (v_k, v_l) is the only such edge, then it is the second edge in the final simple and Step1 is completed. If it is not the case, then there are, say, r such edges, $(v_k, v_l), (v_k, v_{l2}), ..., (v_k, v_{lr})$. Assuming the edges are listed in the original order in the sequence, then we remove all the edges from (v_k, v_l) till (v_k, v_{lr}) and make (v_k, v_{lr}) the second edge in the sequence. The step is completed and the new sequence,$(v_i, v_k), (v_k, v_{l_r}), ..., (v_p, v_j)$, is generated. Note in the new sequence, (v_k, v_{lr}) will be the unique edge of the form (v_k, v_{li}) in the final path.

Step 2: Repeat Step 1 to select the unique edge on the most currently updated sequence (as resulted in Step 1) for the third, fourth, etc. elements until the final edge (v_p, v_j) is reached.

Since in the above two steps the vertices in the second component of edges are alll distinct, the resulting sequence formed from the original walk must be a simple path. So its length is no larger than $|V| - 1$.

35. Consider graphs in which there is at most one edge between any two vertices. Show that under this condition a graph with n vertices has at most n^2 edges.

    Solution: Let n be the number of vertices of the graph and $e(n)$ be the correesponding number of edges. Then for $n = 1$, it is obvious that the unique vertex we can have is the edge to itself, so $e(1) = 1 = 1^2$. Suppose $e(k) = 2^k$ is true for all graphs with number of vertices $k = 1, 2, ..., n$.

    Let us consider a graph of $n+1$ vertices $V_{n+1} = \{v_1, v_2, ..., v_n, v_{n+1}\} = V_n \cup \{v_{n+1}\}$ where V_n is the subset of V_{n+1} containing n vertices. By putting v_{n+1} to the graph of vertices V_n, we can add at most the $2n$ edges $(v_1, v_{n+1}), ..., (v_n, v_{n+1})$ and $(v_{n+1}, v_n), ..., ., (v_{n+1}, v_1)$ as well as the edge (v_{n+1}, v_{n+1}) in the graph. That means there is at most an increament of $2n+1$ edges from a graph of n vertices. Therefore,

$$e(n+1) = e(n) + (2n+1) = n^2 + (2n+1) = (n+1)^2.$$

36. Show that
$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Solution: For $i = 0$, $2^0 = 1$ and
$$\sum_{i=0}^{0} 2^i = 2^{0+1} - 1 = 2 - 1 = 1.$$

Assume it is true for all $k = 0, 1, ..., n$, then
$$\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^{n} 2^i + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2^{n+2} - 1.$$

This completes the inductive proof.

37. Show that
$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

Solution: It is trivial for n $= 1$. Suppose it is true for all $k = 1, 2, ..., n$, then

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{1}{n} - \frac{1}{(n+1)^2}$$

$$= 2 - \frac{n^2+n+1}{n(n+1)^2} \leq 2 - \frac{n(n+1)}{n(n+1)^2} = 2 - \frac{1}{n+1}.$$

38. Prove that for all $n \geq 4$ the inequality $2^n < n!$ holds.

Solution: $2^4 = 16 < 24 = 4!$, so $2^n < n!$ holds for $n = 4$. Assume it is true for $k = 4, 5, ..., n$, then

$$2^{n+1} = 2(2^n) < 2n! < (n+1)n! = (n+1)!$$

and the inductive proof is completed.

39. ** The *Fibonacci sequence* is defined recursively by

$$f(n+2) = f(n+1) + f(n), n = 1, 2, ...,$$

with $f(1) = 1, f(2) = 1$. Show that

(a) $f(n) = O(2^n)$,

(b) $f(n) = \Omega(1.5^n)$.

Solution:

(a) We conjecture that $f(n) < 2(2^n)$. Since $f(1) = 1 < 2(2^1) = 4$, so $f(n) < 2(2^n)$ holds for $n = 1$. Assume it is true for $k = 1, 2, ..., n, n+1$, then

$$f(n+2) = f(n+1) + f(n) < 2(2^{n+1}) + 2(2^n) < 2^{n+2} + 2^{n+1} < 2(2^{n+2}).$$

(b) We conjectue that $f(n) > 1.5^n/10$. It is trivial for $n = 1$. Assume it is true for $k = 1, 2, ..., n+1$, then

$$f(n+2) = f(n+1) + f(n) > 1.5^{n+1}/10 + 1.5^n/10$$

$$= 1.5^{n+1}(1 + \frac{1}{1.5})/10 > 1.5^{n+1}(1 + 0.5)/10 = 1.5^{n+2}/10.$$

40. Show that $\sqrt{8}$ is not a rational number.

Solution: Assume that $\sqrt{8}$ is a rational number so that we can write it as

$$\sqrt{8} = \frac{n}{m},$$

where n and m are integers without a common factor. Rearrange the equation and take square on both sides of the equation, we have $8m^2 = n^2$. Therefore n must be even and has a factor of 4, so we write $n = 4k$ and have

$$8m^2 = n^2 = (4k)^2 = 16k^2$$

and

$$m^2 = 2k^2.$$

Therefore, m must be an even number. But this contradicts the assumption that n and m have no common factor which means $\sqrt{8}$ is not a rational number.